Forcing as a tool to prove theorems

$$(\Vdash_{\mathbb{P}} \varphi) \Longrightarrow (V \models \varphi)$$

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BIU

These slides are be available at:
http://www.assafrinot.com/talk/biu013

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Is CH true? Hilbert put this question on top of his famous 1900 list of major open problems.

A surprising answer (Cohen, 1963)

CH is independent of the usual axioms of set theory (ZFC).
To prove this, he invented the method of <u>forcing</u>.

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Today's talk will consist of two parts:

1. What is forcing?
   1.1 Recasting the theory of algebraic fields
   1.2 A (semi-)formal description of forcing
2. A sample of theorems proved using forcing

Part 1

Recasting field theory

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A statement in the language of field theory is

A (well-formed) formula that uses symbols of first order-logic $(\wedge, \vee, \exists, \forall, =, \ldots)$ together with $+, \cdot, 0, 1$.

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A statement in the language of field theory is

said to be <u>valid</u>, if it is true in any field. For example:

$$(\forall a \forall b \forall c)((+(a, b) = +(a, c)) \to (b = c)).$$

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A statement in the language of field theory is

said to be independent, if both the statement and its negation are consistent. For example:

$$\exists x(\cdot(x, x) = +(1, 1))$$

Fields

Summary

For every statement φ in the language of field theory, exactly one of following holds:

- φ is valid;
- $\neg\varphi$ is valid;
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- φ is independent. That is, both φ and $\neg\varphi$ are consistent.

The independence of a statement φ is sometime seen by passing to a subfield or to a field extension.

Passing to a subfield

Proposition

The statement $\exists x(+(\cdot(x, x), 1) = 0)$ is independent of the axioms of fields.

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The statement $\exists x(+(\cdot(x,x),1)=0)$ is independent of the axioms of fields.

Proof.

Let $\mathbb{C}$ denote the field of complex numbers. We have

$$\mathbb{C} \models i^2+1=0.$$

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In particular, $\mathbb{C} \models \exists x(+(\cdot(x,x),1)=0)$.
On the other hand, the subfield $\mathbb{R}$ of real numbers satisfies the negation: $\mathbb{R} \models \neg(\exists x(+(\cdot(x,x),1)=0))$. $\square$

Passing to a field extension

Recall..

If $\mathbb{F}$ is a field, then the objects of the polynomial ring $\mathbb{F}[X]$ are obtained via the following recursive process:

- $\mathbb{F}_0[X] = \{0\}$;
- $\mathbb{F}_{n+1}[X] := \{\alpha X^n + q \mid \alpha \in \mathbb{F}, q \in \mathbb{F}_n[X]\}$.

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Fact

If $\mathbb{F}$ is a field, and $\mathfrak{I}$ is a maximal ideal in the polynomial ring $\mathbb{F}[X]$, then the quotient $\mathbb{F}[X]/\mathfrak{I}$ is a field extending $\mathbb{F}$.

Passing to a field extension, cont'

Application

The statement $\exists x(\cdot(x, x) = +(1, 1))$ is independent of the axioms of fields.

Passing to a field extension, cont'

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The statement is not valid in the field of rational numbers $\mathbb{Q}$.

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Proof.

The statement is not valid in the field of rational numbers $\mathbb{Q}$.
In particular, the polynomial $X^2 - 2$ is irreducible in $\mathbb{Q}[X]$, and hence the generated ideal $(X^2 - 2)$ is maximal.
Consequently, $\mathbb{Q}[X]/(X^2 - 2)$ is a field (extending $\mathbb{Q}$).

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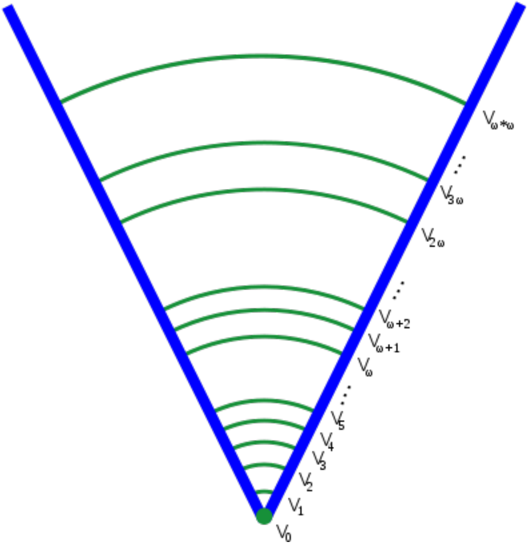
Finally, note that $\exists x(\cdot(x,x) = +(1,1))$ is valid in $\mathbb{Q}[X]/(X^2 - 2)$, as witnessed by the residue class of X. $\square$

Set Theory


$V_{\omega*\omega}$
...
$V_{3\omega}$
$V_{2\omega}$
...
$V_{\omega+2}$
$V_{\omega+1}$
V_{ω}
...
V_5
V_4
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(pairing) $\forall x \forall y \exists z (x \in z \wedge y \in z)$
(union) $\forall \mathcal{F} \exists u \forall Y \forall x ((x \in Y \wedge Y \in \mathcal{F}) \to x \in u)$
(power set) $\forall x \exists P \forall y ((\forall z ((z \in y) \to (z \in x))) \to z \in P)$
(extensionality) $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to (x = y))$

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A (well-formed) formula that uses symbols of first order-logic together with $\in$.

Hierarchies of sets: the well-founded hierarchy

Notation

Write $A \subseteq B$ whenever $\forall x(x \in A \to x \in B)$.
The power set $\mathcal{P}(B) := \{A \mid A \subseteq B\}$.

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Fact

Suppose that $(V, \in) \models$ *ZFC.*
In V*, recursively define:*

- $V_0 := \emptyset$;
- $V_{\alpha+1} := \mathcal{P}(V_\alpha)$ *for every ordinal* α;
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Then an object x *is in* V *iff* x *is in* V_α *for some ordinal* α.

Hierarchies of sets: the names hierarchy

Suppose that $(V, \in) \models$ ZFC, and $\mathbb{P} = \langle P, \leq \rangle$ is a poset in V.

Definition

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Say that x is a $\underline{\mathbb{P}\text{-name}}$ iff x is in $V_\alpha^{\mathbb{P}}$ for some ordinal α.

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Definition

Let $V^{\mathbb{P}}$ denote the collection of all $\mathbb{P}$-names. Given a subset $G \subseteq P$, define for every $\mathbb{P}$-name x, the interpretation x/G by recursion on the least α with $x \in V^{\mathbb{P}}_\alpha$:

$$x/G = \{y/G \mid \exists p \in G \ (p, y) \in x\}.$$

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Theorem (Cohen, 1963)

If $G \subseteq P$ is a particular form of a maximal ideal (called "generic"), then:

1. $(V[G], \in) \models$ZFC;
2. $V \subseteq V[G]$, and $G \in V[G]$;
3. V and $V[G]$ have the same ordinals.

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1. $(V[G], \in) \models$ZFC;
2. $V \subseteq V[G]$, and $G \in V[G]$;
3. V and $V[G]$ have the same cardinals, provided that $\mathbb{P}$ does not have uncountable antichains.

Forcing

Summary

Starting with a model V of ZFC, and a partial order $\mathbb{P}$ from V, the method of forcing allows to pass to a $\mathbb{P}$-generic extension $V[G]$, which is again a model of ZFC.
The statements which are valid in $V[G]$ are tightly related to the combinatorial properties of the ground model V, and the chosen poset $\mathbb{P}$.

Exercise

Definition

- $D \subseteq P$ is said to be dominating if for all $p \in P$, there exists $d \in D$ with $p \leq d$.
- $I \subseteq P$ is an ideal over $\mathbb{P}$ if:
  1. for every $p, q \in I$, there exists $r \in I$ with $p \leq r$ and $q \leq r$;
  2. for every $p \in I$ and $q \leq p$, we have $q \in I$;
- the ideal $I \subseteq P$ is generic, if $I \cap D \neq \emptyset$ for each dominating D.

Prove!

Suppose that V is a model of ZFC. In V, define $\mathbb{P} := (P, \subseteq)$, where P is the set of all functions of the form $f : x \to 2$, for finite subsets x of ω_2.
Let $V[G]$ denote the $\mathbb{P}$-generic extension of V; then

$$V[G] \models \mathrm{ZFC} + \neg\text{Continuum hypothesis}.$$

Hierarchies of sets: the constructible hierarchy

Definition

The definable power set $\mathcal{D}(B) := \{A \mid A \text{ is definable over } (B, \in)\}$.

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Say that A is <u>definable</u> over $(B, \in)$ if there exists a formula φ in the language of set theory and parameters $b_1, \ldots, b_n$ in B such that

$$A = \{z \in B \mid (B, \in) \models \varphi[z, b_1, \ldots, b_n]\}.$$

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Say that x is a constructible set iff x is in L_α for some ordinal α.

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Suppose that $(V, \in) \models$ *ZFC.*
Let L *denote the (sub)collection of all constructible sets.*
Then $(L, \in) \models$*ZFC+GCH.*

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Application of passing to an inner model

If ZFC is consistent, then so is ZFC + Continuum Hypothesis.

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Theorem (Gödel, 1936)

Suppose that $(V, \in) \models ZFC$.
Let L *denote the (sub)collection of all constructible sets.*
Then $(L, \in) \models ZFC + GCH$.

> *Given a conjecture, the best thing is to prove it. The second best thing is to disprove it. The third best thing is to prove that it is not possible to disprove it, since it will tell you not to waste your time trying to disprove it. That's what Gödel did for the Continuum Hypothesis.*
>
> Saharon Shelah

Part 2: theorems proved via forcing

Ramsey-type theorems

Theorem (Ramsey, 1929)

For every coloring $c : [\mathbb{N}]^2 \to \{0, 1\}$, *there exists an infinite* $A \subseteq \mathbb{N}$, *such that* $c \restriction [A]^2$ *is constant.*

Ramsey-type theorems

Theorem (Sierpinski, 1933)

There exists a coloring $c : [\omega_1]^2 \to \{0, 1\}$ *such that* $c \restriction [A]^2$ *is non-constant for all uncountable* $A \subseteq \omega_1$.

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Theorem (Baumgartner-Hajnal, 1973)

For every coloring $c : [\omega_1]^2 \to \{0,1\}$, *end every ordinal* $\alpha < \omega_1$, *there exists a subset* $A \subseteq \omega_1$ *of order-type* α *such that* $c \restriction [A]^2$ *is constant.*

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They prove that the monochromatic set of order-type α exists in a forcing extension $V[G]$, and then pull (a copy of) it back to V, using absoluteness reasoning.

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Shortly afterwards, Galvin found a direct combinatorial proof.

Cardinal Arithmetic

Theorem (Silver, 1975)

If $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *for all* $\alpha < \aleph_{\omega_1}$, *then* $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

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Afterwards, Baumgartner and Prikry carefully analyzed Silver's arguments, and devised a direct, forcing-free proof.

Covering the plane

Identify a function $f : \mathbb{R} \to \mathbb{R}$ with its graph $\{(x, f(x)) \mid x \in \mathbb{R}\}$.
Denote $f^{-1} := \{(f(x), x) \mid x \in \mathbb{R}\}$.

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Theorem (Sierpinski, 1919)

For every $\aleph_1$-sized sets of reals X, there exists a countable collection of functions $\{f_n : \mathbb{R} \to \mathbb{R} \mid n < \omega\}$ such that $\bigcup_{n<\omega}(f_n \cup f_n^{-1})$ covers X^2.

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Theorem (Kubiś-Vejnar, 2012)

There exists a countable collection of continuous functions $\{f_n : \mathbb{R} \to \mathbb{R} \mid n < \omega\}$ such that $\bigcup_{n<\omega}(f_n \cup f_n^{-1})$ covers X^2, for some uncountable set $X \subseteq \mathbb{R}$.

Covering the plane

They introduced the functions by forcing, and then appealed to the work of Keisler on the logic $L^\omega(Q)$, to obtain the functions in the ground model.

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Covering the plane

Kunen found a forcing-free proof, of an even stronger statement.

Theorem (Kunen, 2012)

There exists a countable collection of C^∞ functions $\{f_n : \mathbb{R} \to \mathbb{R} \mid n < \omega\}$ such that $\bigcup_{n<\omega}(f_n \cup f_n^{-1})$ covers X^2, for some uncountable set $X \subseteq \mathbb{R}$.

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Rational distances

Theorem (Komjáth, 1994)

$\mathbb{R}^n$ is the union of countably many sets, none containing two points a rational distance apart.

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That is, letting $E^n := \{\{\bar{x}, \bar{y}\} \in [\mathbb{R}^n]^2 \mid |\bar{x} - \bar{y}| \in \mathbb{Q}\}$, the graph $(\mathbb{R}^n, E^n)$ is countably chromatic!

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Theorem (Kumar, 2012)

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A Forcing-free proof of the Gitik-Shelah theorem was given by Burke and Fremlin.

Compact subsets of the first Baire class

Suppose that X is a Polish space. Let $B_1(X)$ denote the space of all Baire class-1 real-valued functions on X, endowed with the topology of pointwise convergence.

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Todorčević's proof is involved and uses the forcing machinery in a deep way. A forcing-free proof is unknown.

Uniformization

Definition

Suppose that $\varphi(\bar{x}, \bar{y})$ is a formula for which $\forall \bar{x} \exists \bar{y}(\varphi(\bar{x}, \bar{y}))$ is valid. A formula $\psi(\bar{x}, \bar{y})$ is a <u>uniformization</u> of $\varphi(\bar{x}, \bar{y})$ provided that:

- $\forall \bar{x} \forall \bar{y}(\psi(\bar{x}, \bar{y}) \to \varphi(\bar{x}, \bar{y}))$;
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The proof uses a forcing argument, and then appeals to an absolute decision procedure for the monadic second-order theory of the full binary tree T, due to Rabin.

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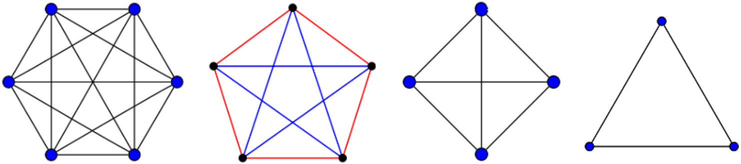
In a paper from 2010, Löding, Niwiński, and Walukiwicz provide a simpler forcing-free proof that only uses basic tools from automata theory.

Partition relations for graphs

For every coloring $c : [6]^2 \to \{0, 1\}$, there exists a monochromatic triangle Δ. That is, $|\Delta| = 3$ such that $c \restriction [\Delta]^2$ is constant. One cannot replace 6 with 5.

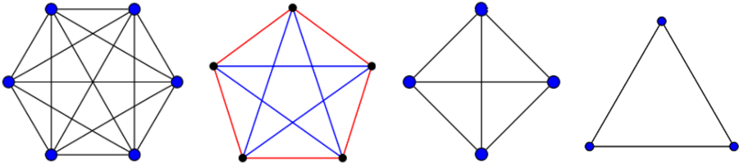
Erdös and Hajnal asked: could there be a graph (G, E) that does not embed a copy of $[4]^2$, yet for any coloring $c : E \to \{0, 1\}$, there would be a monochromatic triangle?



Partition relations for graphs

Theorem (Shelah, 1987)

There exists a K_4-free graph (G, E), such that for every coloring $c : E \to \{0, 1\}$, there exists a monochromatic triangle $\Delta \subseteq G$. That is, $|\Delta| = 3$, $[\Delta]^2 \subseteq E$ and $c \restriction [\Delta]^2$ is constant.



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Shelah constructs a forcing extension which adds a graph $\mathcal{H}$ with the same partition property, even for $\aleph_0$ colors. In particular, $\mathcal{H}$ has the edge-coloring property for 2 colors. By compactness of first-order logic, $\mathcal{H}$ must contain a finite subgraph $\mathcal{G}$ with the same property. As forcing cannot create new finite graphs, $\mathcal{G}$ is already present in the ground model!

The tensor product of graphs

Conjecture (Hedetniemi, 1966)

For every graphs $\mathcal{G}, \mathcal{H}$:

$$\mathsf{Chr}(\mathcal{G} \times \mathcal{H}) = \min\{\mathsf{Chr}(\mathcal{G}), \mathsf{Chr}(\mathcal{H})\}.$$

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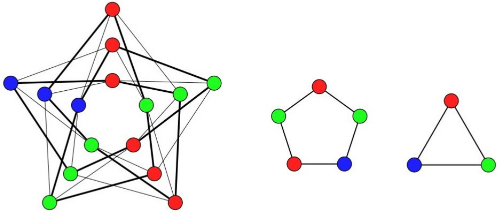

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Theorem (Hajnal, 1985)

For every infinite cardinal κ, there exist graphs $\mathcal{G}, \mathcal{H}$ such that:

1. $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \kappa^+$;
2. $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \kappa$.

Theorem (Soukup, 1988)

If ZFC is consistent, then so is ZFC+GCH+there exist graphs $\mathcal{G}, \mathcal{H}$ of size $\aleph_2$ such that:

1. $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \aleph_2$;
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Theorem (2013)

In the constructible universe, for every infinite cardinal κ, there exist graphs $\mathcal{G}, \mathcal{H}$ of size κ^+ such that:

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2. $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_0$.

Aspects of forcing are built into the very definition of the graphs, and items (1),(2) above are established through an inspection of $\mathcal{G}, \mathcal{H}$ in different forcing extensions. Forcing seems crucial here, and I do not know of a forcing-free proof.