# CHROMATIC NUMBERS OF GRAPHS - LARGE GAPS

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Abstract. We say that a graph $G$ is $(\aleph_0,\kappa)$-chromatic if $\mathrm{Chr}(G)=\kappa$, while $\mathrm{Chr}(G')\leq\aleph_0$ for any subgraph $G'$ of $G$ of size $<|G|$.

The main result of this paper reads as follows. If $\square_\lambda+\mathrm{CH}_\lambda$ holds for a given uncountable cardinal $\lambda$, then for every cardinal $\kappa\leq\lambda$, there exists an $(\aleph_0,\kappa)$-chromatic graph of size $\lambda^+$.

We also study $(\aleph_0,\lambda^+)$-chromatic graphs of size $\lambda^+$. In particular, it is proved that if $0^\sharp$ does not exist, then for every singular strong limit cardinal $\lambda$, there exists an $(\aleph_0,\lambda^+)$-chromatic graph of size $\lambda^+$.

## 1. Introduction

1.1. **Background.** A graph $G$ is a pair $(V,E)$, where $E\subseteq[V]^2=\{\{x,y\}\mid x,y\in V\ \&\ x\neq y\}$. The chromatic number of $G$, denoted $\mathrm{Chr}(G)$, is the least cardinal $\kappa$ such that $V$ is the union of $\kappa$ many $E$-independent sets. Equivalently, $\mathrm{Chr}(G)$ is the least cardinal $\kappa$, for which there exists a coloring $\chi:V\rightarrow\kappa$ that satisfies $\chi(x)\neq\chi(y)$ whenever $xEy$.

*Motivating Questions.* Given a graph $G$ of chromatic number $\geq\kappa$, what can be said about the chromatic number of its subgraphs? can we expect to find a subgraph of chromatic number precisely $\kappa$? and if so, can we moreover find one whose size is $\kappa$?

These questions were first considered in the 50's by de Bruijn and Erdös [2], though only in the case $\kappa\leq\omega$. For an uncountable $\kappa$, these questions were re-raised by Erdös and Hajnal [3] in the 60's, and by Galvin [6] in the 70's. To mention a few results in this vein:

- (Erdös-Hajnal, [4]) if $2^{\aleph_0}=\aleph_1$, then there exists an $(\aleph_0,\aleph_1)$-chromatic graph of size $\aleph_2$;
- (Komjáth, [7]) it is consistent with $2^{\aleph_0}=\aleph_3$ that there exists an $(\aleph_0,\aleph_2)$-chromatic graph of size $\aleph_2$;
- (Baumgartner, [1]) it is consistent with GCH that there exists an $(\aleph_0,\aleph_2)$-chromatic graph of size $\aleph_2$;

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- (Foreman-Laver, [5]) relative to a large cardinal hypothesis, it is consistent with GCH that there does not exist an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$.[1]

As for graphs of size $\geq \aleph_3$, let us mention the following:

- (Komjáth, [7]) it is consistent with $2^{\aleph_0} = \aleph_{\omega_1+1}$ that there exists an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_{\omega_1}$;
- (Shelah, [13]) it is consistent with GCH that there exists an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_{\omega_1}$;
- (Soukup, [14]) for any aleph $\kappa$, it is consistent that $2^{\aleph_0} \geq \kappa$, and there exists an $(\aleph_0, (2^{\aleph_0})^+)$-chromatic graph of size $(2^{\aleph_0})^+$;
- (Shelah, [13]) if $V = L$, then (GCH holds, and) for every regular non-weakly compact cardinal $\lambda$, there exists an $(\aleph_0, \lambda)$-chromatic graph of size $\lambda$.[2]

More recent works include: (1) a ZFC theorem of Todorčević [15] that there exists a graph of size $2^{\aleph_2}$ of uncountable chromatic number, that has no subgraph of size $\aleph_1$ whose chromatic number is $\aleph_1$, (2) a systemic study due to Komjáth [8] of the spectrum of chromatic numbers of all subgraphs of a given graph, and (3) Shelah's paper [12] in which fairly weak hypotheses are utilized to construct $(\aleph_0, \geq \aleph_1)$-graphs of arbitrary large size.

1.2. **Results.** In this paper, we shall mostly be interested in extreme counterexamples to the above-mentioned questions. We shall introduce a simple method for defining $(\aleph_0, \kappa)$-chromatic graphs, and identify a property that controls the value of $\kappa$. In effect, one gets some of the known consequences of $V = L$ to hold in additional scenarios. To exemplify:

**Corollary 3.5.** *If $0^\sharp$ does not exist, then there exists an $(\aleph_0, \lambda^+)$-graph of size $\lambda^+$ for every singular strong limit cardinal $\lambda$.*

We now describe *the simple method*. Given a cardinal $\theta$, fix a sequence $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ in which $C_\alpha$ is a closed and unbounded subset of $\alpha$, for all limit $\alpha < \theta$. Then, to any set $G \subseteq \theta$ of limit ordinals, the graph $G(\overrightarrow{C})$ is defined as follows:

- the set of vertices is $G$;
- $\alpha < \delta$ in $G$ form an edge iff $\alpha \in C_\delta$ and $\min(C_\alpha) > \sup(C_\delta \cap \alpha)$.

Isn't that simple?

It turns out that if $\overrightarrow{C}$ is *coherent*, and all the (proper) initial segments of $G$ are nonstationary, then $G(\overrightarrow{C})$ is $(\aleph_0, \kappa)$-chromatic for some cardinal $\kappa$.

[1] A combination of this with the above-mentioned result of Erdös and Hajnal, reveals that the existence of an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_2$ does not imply the existence of an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$.

[2] The construction in [13] builds on a form of square-with-built-in-diamond that holds in $L$, but may be killed by a cofinality-preserving set-forcing that preserves the GCH.

Once we have that, we shall put some effort on establishing the consistency of existence of such $\overrightarrow{C}$-sequences and sets $G$, for which the corresponding cardinal $\kappa$ would be as we wish.

Now, the main result of this paper reads as follows.

**Main Result.** *Suppose that $\square_\lambda$ holds for a given infinite cardinal $\lambda$.*[3]

- *If $2^\lambda = \lambda^+$, then for every cardinal $\kappa \leq \lambda$, there exists an $(\aleph_0, \kappa)$-chromatic graph of size $\lambda^+$;*
- *If $2^\lambda = \lambda^+$ and $\lambda$ is a singular cardinal, then there exists an $(\aleph_0, \lambda^+)$-chromatic graph of size $\lambda^+$;*
- *if $\lambda^{<\lambda} = \lambda$, then in the forcing extension by adding a single Cohen subset to $\lambda$, there exists an $(\aleph_0, \lambda^+)$-chromatic graph of size $\lambda^+$.*

*Moreover, the coloring number of the constructed graphs would be equal to their chromatic number.*

1.3. **Preliminaries.** We say that a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ is *coherent*, if for every limit ordinals $\beta < \alpha < \lambda$, we have $C_\beta = C_\alpha \cap \beta$ iff $\beta = \sup(C_\alpha \cap \beta)$. Jensen's axiom $\square_\lambda$ asserts the existence of a coherent sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that $C_\alpha$ is a club in $\alpha$ of order-type $\leq \lambda$, for all limit $\alpha < \lambda^+$.

Given a set of ordinals $C$, we define $\mathrm{acc}^+(C) := \{\alpha < \sup(C) \mid \sup(C \cap \alpha) = \alpha\}$, $\mathrm{acc}(C) := C \cap \mathrm{acc}^+(C)$, and $\mathrm{nacc}(C) := C \setminus \mathrm{acc}(C)$. We write $C(i)$ for the $i_{th}$-element of $C$, that is, the unique $\beta \in C$ such that $\mathrm{otp}(C \cap \beta) = i$. For sets of ordinals $C$ and $D$, we write $C \sqsubseteq D$ in the case that $C \subseteq D$ and $C \cap \beta = D \cap \beta$ for all $\beta \in C$. For functions $f : \alpha \to A$ and $g : \beta \to B$, we write $f \sqsubseteq g$ in the case $f \subseteq g$. Cohen's notion of forcing for adding a function $g : \lambda \to 2$ via initial approximations is denoted by $\mathrm{Add}(\lambda, 1)$. We write $\mathrm{CH}_\lambda$ for the arithmetic assertion: $2^\lambda = \lambda^+$. Finally, GCH stands for the assertion that $\mathrm{CH}_\lambda$ holds for every infinite cardinal $\lambda$.

1.4. **Organization of this paper.** In Section 2, we introduce a couple of constructions of $\overrightarrow{C}$-sequences, and adjacent sets $G$. In Section 3, we prove that the associated graphs $G(\overrightarrow{C})$ of $G$ and $\overrightarrow{C}$ from Section 2, serve as counterexamples to the motivating questions.

*Remark.* Some readers may prefer to commence with reading Section 3, gaining motivation to the results of Section 2.

## 2. End-segment-coherent sequences

In this section, we shall be concerned with coherent-type $\overrightarrow{C}$-sequences that have built-in *hitting* features. We commence with recalling a concept that has already been considered in the literature.

[3]For missing definitions, see the next subsection.

**Definition 2.1** ([10]). $\clubsuit_\lambda$ asserts the existence of a $\square_\lambda$-sequence $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+\rangle$ with the additional property that for every limit ordinal $\theta < \lambda$, every sequence $\langle A_i \mid i < \theta\rangle$ of cofinal subsets of $\lambda^+$, and every club $D \subseteq \lambda^+$, there exists some $\alpha < \lambda^+$ with $\mathrm{otp}(C_\alpha) = \theta$, such that for all $i < \theta$:

- $C_\alpha(i+1) \in A_i$;
- $C_\alpha(i) < \delta < C_\alpha(i+1)$ for some $\delta \in D$;
- $C_\alpha(0) \in D$.[4]

Here, we shall work on getting $\clubsuit_\lambda$-like sequences, in which the above restriction "$\theta < \lambda$" is waived.

## 2.1. Singular cardinals.

**Theorem 2.2.** *Suppose that $\square_\lambda + \mathrm{CH}_\lambda$ holds for a given singular cardinal $\lambda$. Then, there exists a sequence $\langle (c_\alpha, C_\alpha) \mid \alpha < \lambda^+\rangle$, and subset $G \subseteq \mathrm{acc}(\lambda^+)$, that satisfy the following:*

1. *$c_\alpha$ and $C_\alpha$ are clubs in $\alpha$ of order-type $\leq \lambda$ for all limit $\alpha < \lambda^+$;*
2. *if $\beta \in \mathrm{acc}(c_\alpha)$, then $c_\beta = c_\alpha \cap \beta$ and $C_\beta = C_\alpha \cap \beta$;*
3. *if $\beta \in \mathrm{acc}(C_\alpha)$, then $C_\beta = C_\alpha \cap [\min(C_\beta), \beta)$;*
4. *$c_\alpha \cap G = \mathrm{acc}(C_\alpha) \cap G = c_{\alpha+1} = C_{\alpha+1} = \emptyset$ for all $\alpha < \lambda^+$;*
5. *for every sequence $\langle A_i \mid i < \lambda\rangle$ of cofinal subsets of $\lambda^+$, and every club $D \subseteq \lambda^+$, there exists some $\alpha \in G$ such that:*
   - *for every $i < \lambda$, there exists $\delta \in D$ such that $C_\alpha(i) \in \delta \in C_\alpha(i+1) \in A_i$.*
   - *$\min(C_\alpha) \in D$;*
   - *$\mathrm{otp}(C_\alpha) = \lambda$.*

*Proof.* By [10], we know that $\square_\lambda + \mathrm{CH}_\lambda$ is equivalent to $\clubsuit_\lambda$. Thus, let us fix a $\square_\lambda$-sequence $\overrightarrow{D} = \langle D_\delta \mid \delta < \lambda^+\rangle$ with the additional property that for every sequence $\langle A_i \mid i < \lambda\rangle$ of cofinal subsets of $\lambda^+$, every club $C \subseteq \lambda^+$, and every limit ordinal $\theta < \lambda$, there exists some $\delta < \lambda^+$ for which $\mathrm{otp}(D_\delta) = \theta$, and such that for all $i < \theta$:

- $D_\delta(i+1) \in A_i$;
- $D_\delta(i) < \beta < D_\delta(i+1)$ for some $\beta \in C$;
- $D_\delta(0) \in C$.

For any $\delta < \lambda^+$, let $\pi_\delta : \mathrm{otp}(D_\delta) \to D_\delta$ denote the order-preserving bijection. Fix an increasing and continuous sequence of cardinals $\langle \lambda_j \mid j < \mathrm{cf}(\lambda)\rangle$ that converges to $\lambda$, with $\lambda_0 := \mathrm{cf}(\lambda)$, and $\mathrm{cf}(\lambda_{j+1}) = \lambda_{j+1}$ for all

[4] Strictly speaking, the requirement that $C_\alpha(0)$ be in $D$ does not appear in the original definition [10], but note that if $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ is a $\clubsuit_\lambda$-sequence, then $\langle C_\alpha \setminus \{\min(C_\alpha)\} \mid \alpha < \lambda^+\rangle$ would be a $\clubsuit_\lambda$-sequence with this additional feature.

$j < \mathrm{cf}(\lambda)$. Denote $\Gamma := \{\lambda_j \mid j < \mathrm{cf}(\lambda)\}$. Then, for every limit $\epsilon \leq \lambda$, let

$$E_\epsilon := \begin{cases} \epsilon, & \epsilon \leq \mathrm{cf}(\lambda) \\ \epsilon \setminus (\lambda_j + 1), & \epsilon \in (\lambda_j, \lambda_{j+1}] \\ \Gamma \cap \epsilon, & \text{otherwise} \end{cases}.$$

It is easy to see that for every limit $\epsilon \leq \lambda$, $E_\epsilon$ is a club in $\epsilon$, and hence, for every limit $\delta < \lambda^+$, $D'_\delta := \pi_\delta[E_{\mathrm{otp}(D_\delta)}]$ is a club subset of $D_\delta$. Write:

- $\Theta := \{\alpha < \lambda^+ \mid \exists j < \mathrm{cf}(\lambda)(\mathrm{otp}(D_\alpha) = \lambda_{j+1})\}$;
- $\star(\delta) := \sup\{\alpha \in D_\delta \mid \mathrm{nacc}(D_\delta) \cap \alpha \subseteq \Theta\}$;
- $G := \{\delta < \lambda^+ \mid \mathrm{otp}(D_\delta) = \mathrm{cf}(\lambda)\}$;
- $T_0 := \{\delta < \lambda^+ \mid \mathrm{otp}(D_\delta) \leq \mathrm{cf}(\lambda), \star(\delta) = \delta\}$;
- $T_1 := \{\delta < \lambda^+ \mid \mathrm{otp}(D_\delta) \leq \mathrm{cf}(\lambda), \star(\delta) < \delta\}$;
- $T_2 := \{\delta < \lambda^+ \mid \mathrm{otp}(D_\delta) > \mathrm{cf}(\lambda)\}$.

Put $C_{\delta+1} := \emptyset$ for all $\delta < \lambda^+$. Then, for every limit $\delta < \lambda^+$, let

$$C_\delta := \begin{cases} D_\delta \cup \bigcup\{D'_\alpha \mid \alpha \in \mathrm{nacc}(D_\delta), \min(D'_\alpha) > \sup(D_\delta \cap \alpha)\}, & \delta \in T_0 \\ D_\delta \setminus \star(\delta), & \delta \in T_1 \\ D'_\delta, & \delta \in T_2 \end{cases}.$$

**Claim 2.2.1.** $\mathrm{acc}(C_\delta) \cap G = \emptyset$ *for all* $\delta < \lambda^+$.

*Proof.* Suppose that $\gamma \in \mathrm{acc}(C_\delta)$, and let us verify that $\gamma \notin G$. As $G \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$, we shall avoid trivialities and assume that $\mathrm{cf}(\gamma) = \mathrm{cf}(\lambda)$. In particular, we may assume that $\mathrm{otp}(C_\delta) > \mathrm{cf}(\lambda)$. So $\delta \in T_0 \cup T_2$.

▶ If $\delta \in T_0$, then $\mathrm{acc}(D_\delta) \subseteq E^{\lambda^+}_{<\mathrm{cf}(\lambda)}$ and $\mathrm{nacc}(D_\delta) \subseteq \Theta \subseteq E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$, so $\gamma \in \mathrm{acc}(D'_\alpha)$ for some $\alpha \in \mathrm{nacc}(D_\delta)$. By $\star(\delta) = \delta$, we have $\mathrm{otp}(D_\alpha) = \lambda_{j+1}$ for some $j < \mathrm{cf}(\lambda)$, and hence $D'_\alpha = D_\alpha \setminus D_\alpha(\lambda_j + 1)$. Recalling that $\gamma \in \mathrm{acc}(D_\alpha)$, we conclude that $\mathrm{otp}(D_\gamma) = \mathrm{otp}(D_\alpha \cap \gamma) \geq \lambda_j + 1 > \lambda_0 = \mathrm{cf}(\lambda)$. So $\gamma \notin G$.

▶ If $\delta \in T_2$, then by $C_\delta = \pi_\delta[E_{\mathrm{otp}(D_\delta)}]$ and the assumption that $\mathrm{otp}(C_\delta) > \mathrm{cf}(\lambda) = \mathrm{otp}(\Gamma)$, we infer that $\mathrm{otp}(D_\delta) \in (\lambda_j, \lambda_{j+1}]$ for some $j < \mathrm{cf}(\lambda)$, and so $D'_\delta = D_\delta \setminus D_\delta(\lambda_j + 1)$. As in the previous clause, this entails that $\mathrm{otp}(D_\gamma) > \mathrm{cf}(\lambda)$, and $\gamma \notin G$. □

Let us point out that $\mathrm{otp}(C_\delta) \leq \lambda$ for all $\delta < \lambda^+$. Fix a limit $\delta < \lambda^+$. If $\mathrm{otp}(C_\delta) \leq \mathrm{otp}(D_\delta)$, then clearly $\mathrm{otp}(C_\delta) \leq \lambda$. If $\mathrm{otp}(C_\delta) > \mathrm{otp}(D_\delta)$, then $\delta \in T_0$ and $\mathrm{otp}(C_\delta) = \sum_{\alpha \in A} \alpha$ for some subset $A \subseteq \Theta \subseteq \lambda$ of order-type $\leq \mathrm{cf}(\lambda)$, and hence $\mathrm{otp}(C_\delta) \leq \lambda$.

**Claim 2.2.2.** *Suppose that* $\gamma \in \mathrm{acc}(C_\delta)$.

- *if* $\gamma \in \mathrm{acc}(D_\delta)$, *then* $C_\gamma = C_\delta \cap \gamma$;
- *if* $\gamma \notin \mathrm{acc}(D_\delta)$, *then* $C_\gamma = C_\delta \cap [\min(C_\gamma), \gamma)$.

*Proof.* ▶ If $\delta \in T_2$ and $\mathrm{otp}(D_\delta) \in (\lambda_j, \lambda_{j+1}]$ for some fixed $j < \mathrm{cf}(\lambda)$, then $C_\delta = D'_\delta = D_\delta \setminus D_\delta(\lambda_j + 1)$. So, by $\gamma \in \mathrm{acc}(D'_\delta) \subseteq \mathrm{acc}(D_\delta)$, we get that $\lambda_{j+1} \geq \mathrm{otp}(D_\delta) > \mathrm{otp}(D_\gamma) = \mathrm{otp}(D_\delta \cap \gamma) > \lambda_j + 1 > \mathrm{cf}(\lambda)$. So $\gamma \in T_2$, and $C_\gamma = D'_\gamma = D_\gamma \setminus D_\gamma(\lambda_j + 1) = D_\delta \cap \gamma \setminus D_\delta(\lambda_j + 1) = D'_\delta \cap \gamma = C_\delta \cap \gamma$.

▶ If $\delta \in T_2$ and $\mathrm{otp}(D_\delta) = \lambda_j$ for some fixed limit $j < \mathrm{cf}(\lambda)$, then $C_\delta = \pi_\delta[\Gamma \cap \lambda_j]$ and $\gamma = \pi_\delta(\lambda_i)$ for some nonzero limit $i < j$. It follows that $\mathrm{otp}(D_\gamma) = \mathrm{otp}(D_\delta \cap \gamma) = \lambda_i$, and hence $\gamma \in T_2$, and $C_\gamma = \pi_\gamma[\Gamma \cap \lambda_i] = \pi_\delta[\Gamma \cap \lambda_i] = C_\delta \cap \gamma$.

▶ If $\delta \in T_1$, then by $\gamma \in \mathrm{acc}(C_\delta)$, we get that $\gamma \in \mathrm{acc}(D_\delta)$ and $\gamma > \star(\delta)$. In particular, $\star(\gamma) = \star(\delta)$, and $C_\gamma = D_\gamma \setminus \star(\gamma) = (D_\delta \setminus \star(\delta)) \cap \gamma = C_\delta \cap \gamma$.

▶ If $\delta \in T_0$ and $\gamma \in \mathrm{acc}(D_\delta)$, then $D_\gamma = D_\delta \cap \gamma$, $\mathrm{otp}(D_\gamma) < \mathrm{cf}(\lambda)$, $\star(\gamma) = \gamma$, and

$$\begin{gathered} C_\gamma = D_\gamma \cup \bigcup \{D'_\alpha \mid \alpha \in \mathrm{nacc}(D_\gamma), \min(D'_\alpha) > \sup(D_\gamma \cap \alpha)\} = \\ (D_\delta \cap \gamma) \cup \bigcup \{D'_\alpha \mid \alpha \in \mathrm{nacc}(D_\delta \cap \gamma), \min(D'_\alpha) > \sup(D_\delta \cap \alpha)\} = \\ \gamma \cap (D_\delta \cup \bigcup \{D'_\alpha \mid \alpha \in \mathrm{nacc}(D_\delta), \min(D'_\alpha) > \sup(D_\delta \cap \alpha)\}) = C_\delta \cap \gamma. \end{gathered}$$

▶ If $\delta \in T_0$ and $\gamma \notin \mathrm{acc}(D_\delta)$, put $\alpha := \min(D_\delta \setminus \gamma)$ and $\beta = \sup(D_\delta \cap \gamma)$. Then $\beta < \gamma \leq \alpha$, $\alpha \in \mathrm{nacc}(D_\delta)$. Since $(\beta, \alpha] \cap D_\delta$ is a singleton, while $\gamma \in \mathrm{acc}(C_\delta)$, we infer that $\min(D'_\alpha) > \beta$ and $C_\delta \cap (\beta, \alpha) = D'_\alpha$.

By $\star(\delta) = \delta$, we have $\mathrm{otp}(D_\alpha) \in \Theta$, and then $C_\alpha = D'_\alpha = D_\alpha \setminus D_\alpha(\lambda_j + 1)$ for the unique $j < \mathrm{cf}(\lambda)$ such that $\mathrm{otp}(D_\alpha) = \lambda_{j+1}$. As in previous considerations, it follows that $\lambda_{j+1} > \mathrm{otp}(D_\gamma) > \lambda_j + 1 > \mathrm{cf}(\lambda)$, and so $C_\gamma = D_\gamma \setminus D_\gamma(\lambda_j + 1) = C_\alpha \cap \gamma$. Recalling that $C_\delta \cap (\beta, \alpha) = C_\alpha$, we conclude that $C_\delta \cap (\beta, \gamma) = C_\gamma$. In particular, $C_\delta \cap [\min(C_\gamma), \gamma) = C_\gamma$. □

It follows that clause (4) holds. Write $c_{\delta+1} := \emptyset$. For all limit $\delta < \lambda^+$, let $e_\delta$ be some club in $\delta$ of minimal order-type, and then put

$$c_\delta := \begin{cases} \mathrm{acc}(C_\delta) \cap D_\delta, & \sup(\mathrm{acc}(C_\delta) \cap D_\delta) = \delta \\ \{\beta + 1 \mid \beta \in e_\delta\}, & \text{otherwise} \end{cases}.$$

Clearly, $c_\delta$ is a club in $\delta$ of order-type $\leq \mathrm{otp}(D_\delta) \leq \lambda$. By Claim 2.2.1, we know that $c_\delta \cap G = \emptyset$. If $\gamma \in \mathrm{acc}(c_\delta)$, then $c_\delta = \mathrm{acc}(C_\delta) \cap D_\delta$, and $\gamma \in \mathrm{acc}(C_\delta) \cap \mathrm{acc}(D_\delta)$, so $D_\gamma = D_\delta \cap \gamma$, and by Claim 2.2.2, also $C_\gamma = C_\delta \cap \gamma$. Altogether

$$c_\delta \cap \gamma = \mathrm{acc}(C_\delta) \cap D_\delta \cap \gamma = \mathrm{acc}(C_\gamma) \cap D_\gamma = c_\gamma.$$

Thus, we have established clause (3), and are left with proving the following.

**Claim 2.2.3.** *For every sequence $\langle A_i \mid i < \lambda \rangle$ of cofinal subsets of $\lambda^+$, and every club $D \subseteq \lambda^+$, there exists some $\alpha \in G$ such that $\min(D_\delta) \in D$ and for all $i < \lambda$:*

- *$D_\delta(i + 1) \in A_i$;*
- *$D_\delta(i) < \beta < D_\delta(i + 1)$ for some $\beta \in D$.*

*Proof.* Suppose that we are given a club $D \subseteq \lambda^+$, and a sequence $\langle A_i \mid i < \lambda\rangle$, as above. Let $\varphi : \lambda \setminus \lambda_0 \to \lambda$ be the order-preserving isomorphism. For all $i < \lambda$, put

$$B_i := \begin{cases} \lambda^+, & i < \lambda_0 \\ A_{\varphi(i)}, & \text{otherwise} \end{cases}$$

Fix $j < \mathrm{cf}(\lambda)$. For every club $C \subseteq \lambda^+$, let $S^C_j$ denote the collection of all $\alpha < \lambda^+$ for which $\mathrm{otp}(D_\alpha) = \lambda_{j+1}$, and such that for all $i < \lambda_{j+1}$:

- $D_\alpha(i + 1) \in B_i$;
- $D_\alpha(i) < \beta < D_\alpha(i + 1)$ for some $\beta \in C$.

Since $\overrightarrow{D}$ is a $\boxed{\clubsuit}_\lambda$-sequence, $S^C_j$ is nonempty for every club $C \subseteq \lambda^+$. In particular, $\{D_\alpha(\mathrm{cf}(\lambda)) \mid \alpha \in S^D_j\}$ is stationary in $\lambda^+$. Thus, we may define a function $f_j : \lambda^+ \to \lambda^+$, by letting for all $\tau < \lambda^+$:

$$f_j(\tau) := \min\{\alpha \in S^D_j \mid D_\alpha(\mathrm{cf}(\lambda)) > \tau\}.$$

Put $S_j := f_j[\lambda^+]$. Then $S_j$ is a cofinal subset of $S^D_j$. Put $E := \{\eta \in D \mid \forall j < \mathrm{cf}(\lambda)(f_j[\eta] \subseteq \eta)\}$. Then $E$ is a club. Thus, appealing again to the fact that $\overrightarrow{D}$ is a $\boxed{\clubsuit}_\lambda$-sequence, let us pick some $\delta < \lambda^+$, such that $\mathrm{otp}(D_\delta) = \mathrm{cf}(\lambda)$, $\min(D_\delta) \in E$, and for all $j < \mathrm{cf}(\lambda)$:

- $D_\delta(j + 1) \in S_j$;
- $D_\delta(j) < \eta < D_\delta(j + 1)$ for some $\eta \in E$.

Evidently, $\delta \in G$ and $\star(\delta) = \delta$. So $\delta \in T_0$, and

$$C_\delta = D_\delta \cup \bigcup\{D'_\alpha \mid \alpha \in \mathrm{nacc}(D_\delta), \min(D'_\alpha) > \sup(D_\delta \cap \alpha)\}.$$

Fix $\alpha \in \mathrm{nacc}(D_\delta)$. Let $j < \mathrm{cf}(\lambda)$ be such that $\alpha = D_\delta(j + 1)$. Then $\alpha \in S_j = f_j[\lambda^+]$. Let $\tau < \lambda^+$ be such that $\alpha = f_j(\tau)$, and let $\eta \in E$ be such that $D_\delta(j) < \eta < D_\delta(j + 1)$. Since $f_j[\eta] \subseteq \eta < D_\delta(j + 1) = f_j(\tau)$, we infer that $\tau > \eta$ and so by $\alpha \in \Theta$ and $\alpha = f_j(\tau)$, we have:

$$\min(D'_\alpha) > D_\alpha(\mathrm{cf}(\lambda)) > \tau > \eta > D_\delta(j) = \sup(D_\delta \cap \alpha).$$

Thus, we have shown that $\min(D'_\alpha) > \sup(D_\delta \cap \alpha)$ for all $\alpha \in \mathrm{nacc}(D_\delta)$. In particular, $C_\delta = D_\delta \cup \bigcup\{D'_\alpha \mid \alpha \in \mathrm{nacc}(D_\delta)\}$, $D_\delta \subseteq \mathrm{acc}(C_\delta)$, $(\beta, \beta') \cap D \neq \emptyset$ for all $\beta < \beta'$ in $C_\delta$, and $\min(C_\delta) = \min(D_\delta) \in E \subseteq D$.

Let $\{\delta_j \mid j < \mathrm{cf}(\lambda)\}$ denote the increasing enumeration of $D_\delta$. For all $j < \mathrm{cf}(\lambda)$, $\delta_{j+1} \in S^D_j$ and hence $\mathrm{otp}(D_{\delta_{j+1}}) = \lambda_{j+1}$, and $D'_{\delta_{j+1}} = \pi_{\delta_{j+1}}[\lambda_{j+1} \setminus \lambda_j + 1]$. Since $\langle \lambda_j \mid j < \mathrm{cf}(\lambda)\rangle$ is increasing and continuous, we infer that $\mathrm{otp}(C_\delta \cap \delta_{j+1}) = \lambda_{j+1}$ for all $j < \mathrm{cf}(\lambda)$. So:

$$\{\beta \in C_\delta \mid 0 < \mathrm{otp}(C_\delta \cap \beta) < \lambda_1\} = \{\beta \in D_{\delta_1} \mid \lambda_0 < \mathrm{otp}(D_{\delta_1} \cap \beta) < \lambda_1\},$$

and for all nonzero $j < \mathrm{cf}(\lambda)$:

$$\{\beta \in C_\delta \mid \lambda_j < \mathrm{otp}(C_\delta \cap \beta) < \lambda_{j+1}\} = \{\beta \in D_{\delta_{j+1}} \mid \lambda_j < \mathrm{otp}(D_{\delta_{j+1}} \cap \beta) < \lambda_{j+1}\}.$$

As $\varphi``(\lambda_0,\lambda_1)=(0,\lambda_1)$ and $\varphi``(\lambda_j,\lambda_{j+1})=(\lambda_j,\lambda_{j+1})$ for all nonzero $j<\mathrm{cf}(\lambda)$, we conclude that if $\beta\in\mathrm{nacc}(C_\delta)$ and $j<\mathrm{cf}(\lambda)$ is the least such that $\beta<\delta_{j+1}$, then $\beta\in D_{\delta_{j+1}}$ and $\mathrm{otp}(C_\delta\cap\beta)=\varphi(\mathrm{otp}(D_{\delta_{j+1}}\cap\beta))$. So, if $i<\lambda$ is such that $\beta=C_\delta(i+1)$, then

$$\beta=D_{\delta_{j+1}}(\varphi^{-1}(i+1))=D_{\delta_{j+1}}(\varphi^{-1}(i)+1)\in B_{\varphi^{-1}(i)}=A_{\varphi(\varphi^{-1}(i))}=A_i,$$

as desired. □

□

### 2.2. **Regular cardinals.**

**Theorem 2.3.** *Suppose that $\square_\lambda$ holds for a given infinite cardinal $\lambda=\lambda^{<\lambda}$.*

*Then in $V^{\mathrm{Add}(\lambda,1)}$, there exists a $\square_\lambda$-sequence $\langle C_\alpha\mid\alpha<\lambda^+\rangle$, with the property that for every sequence $\langle A_i\mid i<\lambda^+\rangle$ of cofinal subsets of $\lambda^+$, and every club $D\subseteq\lambda^+$, there exists some $\alpha\in E^{\lambda^+}_\lambda$ such that $\min(C_\alpha)>\min(D)$, and for all $i<\alpha$:*

$$\mathrm{otp}(\{\beta\in\mathrm{nacc}(C_\alpha)\cap A_i\mid(\sup(C_\alpha\cap\beta),\beta)\cap D\neq\emptyset\})=\lambda.$$

*Proof.* Denote $\mathbb{P}:=\mathrm{Add}(\lambda,1)$. That is, $p\in\mathbb{P}$ iff $p\in{}^{<\lambda}\lambda$. In particular $|\mathbb{P}|=\lambda^{<\lambda}=\lambda$, and hence every cofinal/stationary/club subset of $\lambda^+$ in $V^{\mathbb{P}}$ contains a cofinal/stationary/club subset from $V$, respectively. Also note that $\mathbb{P}$ preserves the cardinals structure, as it is $(<\lambda)$-closed.

Work in $V$. Let $\langle C_\alpha\mid\alpha<\lambda^+\rangle$ be a $\square_\lambda$-sequence. For every $\alpha<\lambda^+$, let $\pi_\alpha:\mathrm{otp}(C_\alpha)\to C_\alpha$ denote the inverse collapse, and let $\psi_\alpha:\lambda\setminus\{0\}\to\alpha$ be an arbitrary surjection. For every function $p\in{}^{\leq\lambda}\lambda$, and every $\alpha<\lambda^+$, denote

$$1^p_\alpha:=\{j\in\mathrm{dom}(p)\cap\mathrm{dom}(\pi_\alpha)\mid p(j)\neq 0\}.$$

For every $j\in 1^p_\alpha$, we write $p_\alpha(j):=\psi_{\pi_\alpha(j)}(p(j))$. Note that $p_\alpha(j)<\pi_\alpha(j)$. Then, put

$$C^p_\alpha:=\{\pi_\alpha(j)\mid j\in\mathrm{acc}^+(1^p_\alpha)\}\cup\{\max\{p_\alpha(j),\pi_\alpha(\sup(1^p_\alpha\cap j))\}\mid j\in\mathrm{nacc}(1^p_\alpha)\}.$$

Notice that if $q$ extends $p$, then $C^q_\alpha$ is an end-extension of $C^p_\alpha$.

**Definition 2.3.1.** *For every $D,A\in\mathcal{P}(\lambda^+)$, $i<\lambda$, and $\alpha<\lambda^+$, let*

$$\mathbb{D}(D,A,i,\alpha):=\{q\in\mathbb{P}\mid\{\gamma\in\mathrm{nacc}(C^q_\alpha)\cap A\setminus\pi_\alpha(i)\mid(\sup(C^q_\alpha\cap\gamma),\gamma)\cap D\neq\emptyset\}\neq\emptyset\}.$$

**Claim 2.3.1.** *For every club $D\subseteq\lambda^+$, cofinal subset $A\subseteq\lambda^+$, and $i<\lambda$, there exist a club $E\subseteq\lambda^+$ such that $\mathbb{D}(D,A,i,\alpha)$ is dense for all $\alpha\in E\cap E^{\lambda^+}_\lambda$.*

*Proof.* Suppose that we are given $D,A,i$ as above. Put $E:=\mathrm{acc}^+(A)\cap\mathrm{acc}(D)$. Then $E$ is a club in $\lambda^+$. Fix $\alpha\in E\cap E^{\lambda^+}_\lambda$. Given a condition $p\in\mathbb{P}$, put $i':=\max\{\mathrm{dom}(p),i\}$. As $\sup(A\cap\alpha)=\sup(D\cap\alpha)=\alpha$, let $j<\lambda$ be the least ordinal for which there exists $\delta\in D$ and $\gamma\in A$ such that

$\pi_\alpha(i') < \delta < \gamma < \pi_\alpha(j)$. Clearly, $j$ is a successor ordinal. Pick $k \in \lambda \setminus \{0\}$ such that $\psi_{\pi_\alpha(j)}(k) = \gamma$, and then define $q : j+1 \to \lambda$ as follows:

$$q(l) := \begin{cases} k, & l = j \\ p(l), & l \in \mathrm{dom}(p) \\ 0, & \text{otherweise} \end{cases}.$$

Then $q \in \mathbb{P}$, $j \in \mathrm{nacc}(1^q_\alpha)$, $\sup(1^q_\alpha \cap j) = \sup(1^p_\alpha) \leq i'$, and hence $\pi_\alpha(\sup(1^q_\alpha \cap j)) \leq \pi_\alpha(i') < \delta < \gamma = \psi_{\pi_\alpha(j)}(k) = q_\alpha(j)$. So $\gamma \in (\mathrm{nacc}(C^q_\alpha) \cap A) \setminus \pi_\alpha(i)$ and $\delta \in (\sup(C^q_\alpha \cap \gamma), \gamma) \cap D$. It follows that $q$ is an extension of $p$ that belongs to $\mathbb{D}(D, A, i, \alpha)$. □

Let $G \subseteq \mathbb{P}$ be generic over $V$, and work in $V[G]$. Put $g := \bigcup G$. Then, define for every limit $\alpha < \lambda^+$:

$$D_\alpha := \begin{cases} C^g_\alpha, & \sup(C^g_\alpha) = \alpha \\ C_\alpha \setminus \sup(C^g_\alpha), & \text{otherwise} \end{cases}.$$

Note that since $g$ is Cohen, the set $\{j < \lambda \mid g(j) \neq 0\}$ is cofinal in $\lambda$, and hence $D_\alpha = C^g_\alpha$ for all $\alpha \in E^{\lambda^+}_\lambda$.

**Claim 2.3.2.** *$\langle D_\alpha \mid \alpha < \lambda^+ \rangle$ is a $\square_\lambda$-sequence.*

*Proof.* Fix a limit $\alpha < \lambda^+$. It is clear that $D_\alpha$ is unbounded in $\alpha$, so let us verify that it is closed. Suppose that $\beta \in \mathrm{acc}^+(D_\alpha)$. Since $C_\alpha$ is closed, we may assume that $D_\alpha = C^g_\alpha$. By $\beta \in \mathrm{acc}(C^g_\alpha)$, one of the following must hold:

- there exists $J \subseteq \mathrm{acc}^+(1^g_\alpha)$, such that $\pi_\alpha[J]$ is a cofinal subset of $\beta$;
- there exists $J \subseteq \mathrm{nacc}(1^g_\alpha)$ such that $\beta = \sup\{g_\alpha(j) \mid j \in J\}$, where $\pi_\alpha(\sup(1^g_\alpha \cap j)) < g_\alpha(j) < \pi_\alpha(j)$ for all $j \in J$;
- there exists $J \subseteq \mathrm{nacc}(1^g_\alpha)$ such that $\beta = \sup\{\pi_\alpha(\sup(1^g_\alpha \cap j)) \mid j \in J\}$.

In any of the above three cases, we get that $\sup(J) \in \mathrm{acc}^+(1^g_\alpha)$, and $\beta = \sup(\pi_\delta[J]) = \pi_\delta(\sup(J)) \in C^g_\alpha \subseteq D_\alpha$.

Notice that the preceding analysis also shows that $\mathrm{acc}(D_\alpha) \subseteq \mathrm{acc}(C_\alpha)$ for all limit $\alpha < \lambda^+$. In particular, if $\beta \in \mathrm{acc}(D_\alpha)$, then $C_\beta = C_\alpha \cap \beta$, $\pi_\beta \sqsubseteq \pi_\alpha$, $1^g_\beta = 1^g_\alpha \cap \mathrm{otp}(C_\beta)$, $g_\beta(j) = g_\alpha(j)$ for all $j \in 1^g_\beta$, and $C^g_\beta = C^g_\alpha \cap \beta$.

Next, let us verify coherence. Suppose that $\beta \in \mathrm{acc}(D_\alpha)$.

► if $\beta > \sup(C^g_\alpha)$, then by $\beta \in \mathrm{acc}(C_\alpha)$, we have $C_\beta = C_\alpha \cap \beta$ and $C^g_\alpha \cap \beta = C^g_\beta$. So $\beta > \sup(C^g_\beta)$ and $D_\beta = C_\beta \setminus \sup(C^g_\beta) = C_\alpha \cap \beta \setminus \sup(C^g_\alpha) = D_\alpha \cap \beta$.

► if $\beta \leq \sup(C^g_\alpha)$, then by $\beta \in \mathrm{acc}(D_\alpha)$, we have $D_\alpha = C^g_\alpha$. In particular, $\beta \in \mathrm{acc}(C^g_\alpha)$, and hence $\sup(C^g_\beta) = \beta$. Altogether, $D_\beta = C^g_\beta = C^g_\alpha \cap \beta = D_\alpha \cap \beta$.

To conclude the proof, notice that for every $\alpha < \lambda^+$ with $\sup(C^g_\alpha) = \alpha$, there is a naturally-arising injection from $C^g_\alpha$ to $C_\alpha$. It follows that $\mathrm{otp}(D_\alpha) \leq \lambda$ for all limit $\alpha < \lambda^+$. $\square$

**Claim 2.3.3.** *For every club $D \subseteq \lambda^+$ and every cofinal subset $A \subseteq \lambda^+$, there exists a club $E(D,A) \subseteq \lambda^+$ such that for every $\alpha \in E(D,A) \cap E^{\lambda^+}_\lambda$, we have*

$$\sup(\{\beta \in \mathrm{nacc}(D_\alpha) \cap A \mid (\sup(D_\alpha \cap \beta), \beta) \cap D \neq \emptyset\}) = \alpha.$$

*Proof.* Suppose that in $V[G]$, $D$ is a club, and $A$ is a cofinal subset of $\lambda^+$. Fix a club $D' \subseteq D$ together with a cofinal subset $A' \subseteq A$, such that $D', A'$ lie in $V$. By Claim 2.3.1, we may pick a sequence of clubs $\langle E_i \mid i < \lambda\rangle$ such that for every $i < \lambda$, and every $\alpha \in E_i \cap E^{\lambda^+}_\lambda$, the set $\mathbb{D}(D', A', i, \alpha)$ is dense. Put $E(D,A) := \bigcap_{i<\lambda} E_i$. Now, if $\alpha \in E(D,A) \cap E^{\lambda^+}_\lambda$ and $i < \lambda$, then we may find some $q_i \in G \cap \mathbb{D}(D', A', i, \alpha)$. So $q_i \sqsubseteq g$, $C^{q_i}_\alpha \sqsubseteq C^g_\alpha = D_\alpha$, and

$$\sup\{\gamma \in \mathrm{nacc}(C^{q_i}_\alpha) \cap A' \mid (\sup(C^{q_i}_\alpha \cap \gamma), \gamma) \cap D' \neq \emptyset\} \geq \pi_\alpha(i).$$

In particular

$$\sup(\{\gamma \in \mathrm{nacc}(D_\alpha) \cap A \mid (\sup(D_\alpha \cap \gamma), \gamma) \cap D \neq \emptyset\}) = \alpha.$$

$\square$

For every $j < \lambda$, and $\alpha < \lambda^+$, let

$$D^j_\alpha := \begin{cases} D_\alpha, & \mathrm{otp}(D_\alpha) \leq j \\ \{\beta \in D_\alpha \mid \mathrm{otp}(D_\alpha \cap \beta) \geq j\}, & \text{otherwise} \end{cases}.$$

Clearly, $\langle D^j_\alpha \mid \alpha < \lambda^+\rangle$ is a $\square_\lambda$-sequence, for all $j < \lambda$.

**Claim 2.3.4.** *There exists some $j < \lambda$ such that for every sequence $\langle A_i \mid i < \lambda^+\rangle$ of cofinal subsets of $\lambda^+$, and every club $D \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_\lambda$ such that $\min(D^j_\alpha) > \min(D)$, and for all $i < \alpha$:*

$$\mathrm{otp}(\{\beta \in \mathrm{nacc}(D^j_\alpha) \cap A_i \mid (\sup(D^j_\alpha \cap \beta), \beta) \cap D \neq \emptyset\}) = \lambda.$$

*Proof.* Suppose not. For all $j < \lambda$, fix a counterexample $(D^j, \langle A^j_i \mid i < \lambda^+\rangle)$. Put $D := \bigcap_{j<\lambda} D^j$. For all $i < \lambda^+$ and $j < \lambda$, let $E(D, A^j_i)$ be the corresponding club given by Claim 2.3.3. Write $E_i := \bigcap_{j<\lambda} E(D, A^j_i)$. Pick $\alpha \in E^{\lambda^+}_\lambda \cap \Delta_{i<\lambda^+} E_i$. Then for all $i < \alpha$ and $j < \lambda$, we have $\alpha \in E(D, A^j_i)$ and hence

$$\sup(\{\beta \in \mathrm{nacc}(D_\alpha) \cap A^j_i \mid (\sup(D_\alpha \cap \beta), \beta) \cap D \neq \emptyset\}) = \alpha.$$

Fix $\beta \in D_\alpha$ such that $\beta > \min(D)$. Put $j := \mathrm{otp}(D_\alpha \cap \beta)$. Then $\min(D^j_\alpha) = \beta > \min(D) \geq \min(D^j)$, and since $D^j_\alpha$ is a final segment of $D_\alpha$, we also have for all $i < \alpha$:

$$\sup(\{\beta \in \mathrm{nacc}(D^j_\alpha) \cap A^j_i \mid (\sup(D^j_\alpha \cap \beta), \beta) \cap D^j \neq \emptyset\}) = \alpha.$$

As $\mathrm{otp}(D_\alpha)=\lambda=\mathrm{cf}(\alpha)$, we then get:

$$\mathrm{otp}(\{\beta\in\mathrm{nacc}(D^j_\alpha)\cap A^j_i \mid (\sup(D^j_\alpha\cap\beta),\beta)\cap D^j\neq\emptyset\})=\lambda,$$

contradicting the choice of $(D^j,\langle A^j_i \mid i<\lambda^+\rangle)$. $\square$

Let $j<\lambda$ be given by the previous claim. Then $\langle D^j_\alpha \mid \alpha<\lambda^+\rangle$ has all the desired properties. $\square$

2.3. **Discussion.** Notice that the converse of Theorem 2.2 is true, as well. Specifically, items (1)+(2) there entail that $\square_\lambda$ holds, and item (5) entails that $\mathrm{CH}_\lambda$ holds. In fact, item (5) entails $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$.

For our readers who wonder whether Theorem 2.2 can be generalized to cover the case of $\lambda$ regular, we point out that there is no hope for this, since $\mathrm{GCH}+\square_\lambda$ is consistent with the failure of $\diamondsuit(E^{\lambda^+}_\lambda)$ for a regular (countable, or uncountable) cardinal $\lambda$ (see [11]). On the bright side, we mention that an easy forcing construction shows that the powerful $\square_\lambda$-type principles of Theorem 2.3 may hold for a regular cardinal $\lambda$, even when $\mathrm{CH}_\lambda$ (let alone $\boxed{\diamondsuit}_\lambda$) fails.

## 3. The associated graphs

Recall that the *coloring number* of a graph $G=(W,E)$, denoted $\mathrm{Col}(G)$, is the least cardinal $\kappa$ such that there exists a well-ordering $\trianglelefteq$ of $W$, in which $\{x\in W \mid xEy \ \&\ x\trianglelefteq y\}$ has size $<\kappa$ for all $y\in W$. We say that $\chi:W\rightarrow\kappa$ is a *chromatic* coloring of $(W,E)$ if $xEy$ implies $\chi(x)\neq\chi(y)$ for all $x,y\in W$.

Note that $\mathrm{Chr}(G)\leq\mathrm{Col}(G)$, and that $\mathrm{Chr}(G)$ is the least cardinal $\kappa$ for which there exists a chromatic coloring $\chi:W\rightarrow\kappa$.

### 3.1. Maximal gaps.

**Theorem 3.1.** *Suppose that $\square_\lambda+\mathrm{CH}_\lambda$ holds for a singular cardinal $\lambda$. Then there exists a graph $(G,E)$ such that:*

- $\mathrm{Chr}(G,E)=\mathrm{Col}(G,E)=|G|=\lambda^+$*;*
- $\mathrm{Chr}(G',E')\leq\aleph_0$ *for every* $G'\in[G]^{<\lambda^+}$ *and* $E'\subseteq E\cap[G']^2$.

*Proof.* Let $\langle(c_\alpha,C_\alpha) \mid \alpha<\lambda^+\rangle$ and $G\subseteq\mathrm{acc}(\lambda^+)$ be given by Theorem 2.2. Write $C^\bullet_\delta:=\{\alpha\in\mathrm{nacc}(C_\delta) \mid \min(C_\alpha)>\sup(C_\delta\cap\alpha)\}$, and put:

$$E:=\{\{\alpha,\delta\}\in[G]^2 \mid \alpha\in C^\bullet_\delta\}.$$

**Claim 3.1.1.** $\mathrm{Chr}(G,E)=\mathrm{Col}(G,E)=\lambda^+$.

*Proof.* By $|C_\delta|\leq\delta<\lambda^+$ for all $\delta<\lambda^+$, we have $\mathrm{Col}(G,E)\leq\lambda^+$. So, it suffices to prove that $\mathrm{Chr}(G,E)\geq\lambda^+$. Let $\chi:G\rightarrow\lambda$ be an arbitrary coloring. We shall show that $\chi$ is not chromatic. Fix $i<\lambda$.

Put $H_i:=\{\alpha\in G \mid \chi(\alpha)=i\}$, and $M_i:=\{\min(C_\alpha) \mid \alpha\in H_i\}$.

▶ If $\sup(M_i) = \lambda^+$, define a function $f_i : \lambda^+ \to \lambda^+$ by letting for all $\eta < \lambda^+$,

$$f_i(\eta) := \min\{\alpha \in H_i \mid \min(C_\alpha) > \eta\}.$$

▶ If $\sup(M_i) < \lambda^+$, define a constant function $f_i : \lambda^+ \to \lambda^+$ by letting for all $\eta < \lambda^+$,

$$f_i(\eta) := \sup(M_i).$$

Put

$$A_i := \begin{cases} \mathrm{rng}(f_i), & \sup(M_i) = \lambda^+ \\ \lambda^+, & \sup(M_i) < \lambda^+ \end{cases},$$

and

$$D := \{\beta < \lambda^+ \mid \forall i < \lambda, f_i[\beta] \subseteq \beta\}.$$

Now, fix some $\delta \in G$ such that:

- for every $i < \lambda$, there exists $\beta \in D$ and $\alpha \in \mathrm{nacc}(C_\delta)$ such that $\sup(C_\delta \cap \alpha) \in \beta \in \alpha \in A_i$;
- $\min(C_\delta) \in D$.

Put $i := \chi(\delta)$. Note that $\sup(M_i) = \lambda^+$, because otherwise

$$\sup(M_i) < \min(D) \leq \min(C_\delta),$$

contradicting the fact that $i = \chi(\delta)$ entails $\sup(M_i) \geq \min(C_\delta)$.

Fix $\alpha \in \mathrm{nacc}(C_\delta)$ and $\beta \in D$ such that $\sup(C_\delta \cap \alpha) \in \beta \in \alpha \in A_i$. By $\alpha \in A_i$ and $\sup(M_i) = \lambda^+$, we have $\alpha \in \mathrm{rng}(f_i)$. Fix $\eta < \lambda^+$ such that $f_i(\eta) = \alpha$. Then $\min(C_\alpha) > \eta$. By $f_i[\beta] \subseteq \beta < \alpha = f_i(\eta)$, we have $\eta \geq \beta$, and hence $\min(C_\alpha) > \eta \geq \beta > \sup(C_\delta \cap \alpha)$. It follows that $\{\alpha, \delta\} \in E$. Recalling that $\alpha \in \mathrm{rng}(f_i) \subseteq H_i$, we conclude that $\chi(\alpha) = i = \chi(\delta)$, which means indeed that $\chi$ is not a chromatic coloring. □

Denote $G_j := G \cap j$. Then $G = \bigcup_{j<\lambda^+} G_j$, and $G_{j+1} = G_j$ whenever $j \notin G$.

**Definition 3.1.1.** *We say that $\chi : G_j \to \omega$ is a* suitable coloring *if $\chi$ is a chromatic coloring, and $\chi[C_i^\bullet]$ is finite for all $i \leq \sup(G_j)$.*

**Claim 3.1.2.** *If $\chi : G_i \to \omega$ is a suitable coloring, and $i < \delta < \lambda^+$, then $\chi[C_\delta^\bullet]$ is finite.*

*Proof.* Suppose not. Then, we may find a subset $B \subseteq C_\delta^\bullet \cap i$ of order-type $\omega$ such that $\chi \restriction B$ is injective. Put $\beta := \sup(B)$. Then $\beta \in \mathrm{acc}(C_\delta)$, and hence $C_\delta \cap (\gamma, \beta) = C_\beta \cap (\gamma, \beta)$ for some $\gamma < \beta$. In particular, $C_\delta^\bullet \cap (\gamma, \beta) = C_\beta^\bullet \cap (\gamma, \beta)$ some $\gamma < \beta$. Since $\sup(B) = \beta$ and $\mathrm{otp}(B) = \omega$, this must mean that $C_\beta^\bullet \cap B$ is infinite, and then $\chi[C_\beta^\bullet]$ is infinite, contradicting the fact that $\beta \leq \sup(G_i)$ and that $\chi : G_i \to \omega$ is a suitable coloring. □

The next claim is a mixture of approaches taken in [12] and in [13].

**Claim 3.1.3.** *Suppose that $u \in [\omega]^\omega$. Then, for every $i \leq j < \lambda^+$ with $i \notin G$, and a suitable coloring $\chi : G_i \to \omega$, there exists a suitable coloring $\chi' : G_j \to \omega$ extending $\chi$ such that $\chi'[G_j \setminus G_i] \subseteq u$.*

*Proof.* By induction on $j < \lambda^+$ (for all $i$ and $u$).

▶ The case $j = 0$ is trivial. ◀

▶ if $j = \delta + 1$ is a successor ordinal and $\delta \notin G$, then $G_j = G_\delta$, and hence it suffices to appeal to the induction hypothesis for $j - 1$. ◀

▶ if $j = \delta + 1$ is a successor ordinal, and $\delta \in G$, we shall do the following. Let $A := \chi[C^\bullet_\delta]$. By Claim 3.1.2, we know that $A$ is finite, so let us pick some $\zeta \in u \setminus A$. By $i < \delta + 1$ and $i \notin G$, we have $i < \delta$, thus, we may appeal to the induction hypothesis with $i$ and $u \setminus \{\zeta\}$, and find a suitable coloring $\chi^* : G_\delta \to \omega$ extending $\chi$, with $\chi^*[G_\delta \setminus G_i] \subseteq u \setminus \{\zeta\}$. Finally, by $G_j = G_\delta \cup \{\delta\}$, let $\chi' : G_j \to \omega$ be the unique extension of $\chi^*$ that satisfies $\chi'(\delta) = \zeta$.

Evidently, $\chi'[G_j \setminus G_i] \subseteq u$. By Claim 3.1.2 and the fact that $\chi^*$ is a suitable coloring of $G_j \setminus \{\delta\}$, to see that $\chi'$ is a suitable coloring of $G_j$, it suffices to verify that $\chi'(\alpha) \neq \chi'(\delta)$ for all $\alpha$ with $\{\alpha, \delta\} \in E$. Fix $\alpha \in C^\bullet_\delta$. If $\alpha < i$, then $\chi'(\alpha) \in A$ and hence $c'(\alpha) \neq \zeta$. If $\alpha \geq i$, then $\chi'(\alpha) \in u \setminus \{\zeta\}$, and hence $\chi'(\alpha) \neq \zeta$. So $\chi'$ is chromatic and hence suitable. ◀

▶ if $j$ is a limit ordinal, then we do the following. Put $i^* := \min(C_j \setminus i) + 1$. Then $i < i^* < j$, and so by the induction hypothesis, we may find a suitable coloring $\chi^* : G_{i^*} \to \omega$ extending $\chi$ such that $\chi^*[G_{i^*} \setminus G_i] \subseteq u$.

Put $e := (c_j \setminus i^*) \cup \{i^*\}$. Then $e$ is a club in $j$, $e \cap G = \emptyset$, $\min(e) = i^*$, and $C_n = C_j \cap n$ for all $n \in \operatorname{acc}(e)$. Put $A := \chi^*[C^\bullet_j]$. Then $A$ is finite, and we may fix some $\zeta \in u \setminus A$. Next, we shall build an ascending chain of suitable colorings $\{\chi_n : G_n \to \omega \mid n \in e\}$ with the property that $\chi_n^{-1}\{\zeta\} \subseteq C^\bullet_j \cup i^*$ and $\chi_n[C^\bullet_j \setminus i^*] = \{\zeta\}$ for all $n \in e$.

- for $n = \min(e)$, we have $n = i^*$, so let $\chi_n := \chi^*$;
- for $n \in \operatorname{nacc}(e)$, let $i' := \sup(e \cap n)$, and $j' := n$. Then $i' < j' < j$, and $i' \notin G$, so by the induction hypothesis, we may pick a suitable coloring $d : G_{j'} \to \omega$ that extends $\chi_{i'}$, with $d[G_{j'} \setminus G_{i'}] \subseteq u \setminus \{\zeta\}$. Then, define $\chi_n : G_n \to \omega$ by letting for all $\alpha \in G_n$:

$$\chi_n(\alpha) := \begin{cases} \zeta, & \alpha \in C^\bullet_j \setminus i^* \\ d(\alpha), & \text{otherwise} \end{cases}.$$

Evidently, $\chi_n$ extends $\chi_{i'}$, and $\chi_n[G_n \setminus G_{i'}] \subseteq u$. Since $d[C^\bullet_\beta]$ is finite for all $\beta \leq \sup(G_n)$, and $\chi_n$ differs from $d$ in a single color, $\chi_n[C^\bullet_\beta]$ is finite for all $\beta \leq \sup(G_n)$, as well. Thus, to see that $\chi_n$ is suitable, it suffices to verify that $\chi_n(\alpha) \neq \chi_n(\beta)$ for all $\alpha < \beta < n$ with $\{\alpha, \beta\} \in E$.

Towards a contradiction, assume that $\alpha < \beta < n$ are such that $\{\alpha, \beta\} \in E$ and $\chi_n(\alpha) = \chi_n(\beta)$. Since $d$ is chromatic, the definition of $\chi_n$ makes it clear that $\chi_n(\alpha) = \chi_n(\beta) = \zeta$, and $\{\alpha,\beta\} \setminus i^* \neq \emptyset$. Since $\beta \notin i^*$ while $\chi_n(\beta) = \zeta$, we infer from $d[G_{j'} \setminus G_{i'}] \subseteq u \setminus \{\zeta\}$ and $\chi_{i'}^{-1}\{\zeta\} \subseteq C_j^\bullet \cup i^*$, that $\beta \in C_j^\bullet$. In particular,

$$\min(C_\beta) > \sup(C_j \cap \beta).$$

By $\{\alpha,\beta\} \in E$, we have $\alpha \in \mathrm{nacc}(C_\beta)$ and

$$\min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta).$$

If $\alpha < i^*$, then $\alpha < i^* \leq \beta$, and so recalling that $(i^* - 1) \in C_j$, we get that $\sup(C_j \cap \beta) \geq (i^* - 1) \geq \alpha$. If $\alpha \geq i^*$, then by $\chi_n(\alpha) = \zeta$, we infer that $\alpha \in C_j^\bullet$, which means that $\{\alpha,\beta\} \subseteq C_j$. Thus, in any case

$$\sup(C_j \cap \beta) \geq \alpha.$$

Altogether, we meet the following contradiction:

$$\min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta) > \sup(C_j \cap \beta) \geq \alpha.$$

- for $n \in \mathrm{acc}(e)$, let $\chi_n := \bigcup_{k \in e \cap n} \chi_k$. Then $\chi_n$ is chromatic as the limit of chromatic colorings. Since $\chi_k[C_j^\bullet \setminus i^*] = \{\zeta\}$ for all $k \in e \cap n$, and $C_n^\bullet = C_j^\bullet \cap n$, we get that $\chi_n[C_n^\bullet] = A \cup \{\zeta\}$. So $\chi_n[C_k^\bullet]$ is finite for all $k \leq \sup(G_n)$, and hence $\chi_n$ is suitable.

Finally, put $\chi' := \bigcup_{n \in e} \chi_n$. Then $\chi'$ has all the desired properties. ◀

□

It follows that $\mathrm{Chr}(G_j, E \cap [j]^2) \leq \aleph_0$ for all $j < \lambda^+$. In particular, $(G', E')$ is countably chromatic for all $G' \in [G]^{<\lambda^+}$ and $E' \subseteq [G']^2 \cap E$. □

As for $(\aleph_0, \lambda^+)$-chromatic graph of size $\lambda^+$, for $\lambda$ regular, we have the following.

**Theorem 3.2.** *If $\lambda^{<\lambda} = \lambda$ and $\square_\lambda$ holds, then in the forcing extension by adding a single Cohen subset to $\lambda$, there exists a graph $(G, E)$ such that:*

- $\mathrm{Chr}(G, E) = \mathrm{Col}(G, E) = |G| = \lambda^+$*;*
- $\mathrm{Chr}(G', E) \leq \aleph_0$ *for every* $G' \in [G]^{<\lambda^+}$*.*

*Proof.* To avoid trivialities, assume that $\lambda$ is uncountable. Work in the forcing extension. Let $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ be given by Theorem 2.3. Let $G := E_\lambda^{\lambda^+}$. Let $c_{\alpha+1} := \emptyset$ for all $\alpha < \lambda^+$. For every limit $\alpha < \lambda^+$ such that $\sup(\mathrm{acc}(C_\alpha)) = \alpha$, let $c_\alpha := \mathrm{acc}(C_\alpha)$. For every limit $\alpha < \lambda^+$ such that $\sup(\mathrm{acc}(C_\alpha)) < \alpha$, let $c_\alpha$ be some cofinal subset of $\alpha$ of order-type $\omega$, consisting of successor ordinals. Now, run the proof of Theorem 3.1 using the sequence $\langle (c_\alpha, C_\alpha) \mid \alpha < \lambda^+ \rangle$ and the set $G$. □

3.2. **Arbitrary gaps.** Let us now deal with $(\aleph_0,\mu)$-chromatic graphs of size $\lambda^+$, for an arbitrary $\mu\le\lambda$.

**Theorem 3.3.** *Suppose that $\square_\lambda+\mathrm{CH}_\lambda$ holds for a given infinite cardinal $\lambda$. Then, for every cardinal $\mu\le\lambda$, there exists a graph $(G,E)$ such that:*

- $|G|=\lambda^+$;
- $\mathrm{Chr}(G,E)=\mathrm{Col}(G,E)=\mu$;
- $\mathrm{Chr}(G',E')\le\aleph_0$ *for every* $G'\in[G]^{<\lambda^+}$ *and* $E'\subseteq E\cap[G']^2$.

*Proof.* To avoid trivialities, assume that $\lambda$ is uncountable. Then, by [10], $\square_\lambda+\mathrm{CH}_\lambda$ entails $\clubsuit_\lambda$, and we may fix a sequence $\langle C_\alpha\mid\alpha<\lambda^+\rangle$ as in Definition 2.1. For a cardinal $\theta$, denote the set of all infinite regular cardinals below $\theta$ by $\mathbb{R}_\theta$. That is:

$$\mathbb{R}_\theta:=\{\kappa<\theta\mid\kappa=\mathrm{cf}(\kappa)>1\}.$$

Given $\kappa\in\mathbb{R}_\lambda$, let:

- $G_\kappa:=\{\alpha<\lambda^+\mid\mathrm{otp}(C_\alpha)=\kappa\}$;
- $E_\kappa:=\{\{\alpha,\delta\}\in[G_\kappa]^2\mid\alpha\in C_\delta\ \&\ \min(C_\alpha)>\sup(C_\delta\cap\alpha)\}$.

For every $\alpha\in E^{\lambda^+}_\omega$, pick a club $e_\alpha$ in $\alpha$, consisting of successor ordinals. For all limit $\alpha<\lambda^+$, let

$$c^\kappa_\alpha:=\begin{cases}\mathrm{acc}(C_\alpha), & \mathrm{otp}(C_\alpha)\le\kappa\ \&\ \sup(\mathrm{acc}(C_\alpha))=\alpha\\ \mathrm{acc}(C_\alpha\setminus C_\alpha(\kappa)), & \mathrm{otp}(C_\alpha)>\kappa\ \&\ \sup(\mathrm{acc}(C_\alpha\setminus C_\alpha(\kappa)))=\alpha\ .\\ e_\alpha & \text{otherwise}\end{cases}$$

Then, we have:

1. $c^\kappa_\alpha$ and $C_\alpha$ are clubs in $\alpha$ of order-type $\le\lambda$ for all limit $\alpha<\lambda^+$;
2. if $\beta\in\mathrm{acc}(c^\kappa_\alpha)$, then $c^\kappa_\beta=c^\kappa_\alpha\cap\beta$ and $C_\beta=C_\alpha\cap\beta$;
3. if $\beta\in\mathrm{acc}(C_\alpha)$, then $C_\beta=C_\alpha\cap\beta$;
4. $c^\kappa_\alpha\cap G_\kappa=\emptyset$ for all limit $\alpha<\lambda^+$;
5. for every sequence $\langle A_i\mid i<\kappa\rangle$ of cofinal subsets of $\lambda^+$, and every club $D\subseteq\lambda^+$, there exists some $\alpha\in G_\kappa$ such that:
   - for every $i<\kappa$, there exists $\delta\in D$ such that $C_\alpha(i)\in\delta\in C_\alpha(i+1)\in A_i$.
   - $\min(C_\alpha)\in D$.

Thus, if we run the proof of Theorem 3.1 using the sequence $\langle(c^\kappa_\alpha,C_\alpha)\mid\alpha<\lambda^+\rangle$ and the set $G_\kappa\subseteq\mathrm{acc}(\lambda^+)$, then we see that $\mathrm{Chr}(G_\kappa,E_\kappa)>\kappa$, while $\mathrm{Chr}(G',E')\le\aleph_0$ for any $G'\in[G_\kappa]^{<\lambda^+}$ and $E'\subseteq[G']^2\cap E_\kappa$. On the other hand, $\mathrm{otp}(\{\alpha<\delta\mid(\alpha,\delta)\in E_\kappa\})\le\mathrm{otp}(C_\delta)=\kappa$ for all $\delta\in G_\kappa$, and hence $\mathrm{Col}(G_\kappa,E_\kappa)\le\kappa^+$. So, altogether, $\mathrm{Chr}(G_\kappa,E_\kappa)=\mathrm{Col}(G_\kappa,E_\kappa)=\kappa^+$.

Note that so far we have established the conclusion of the theorem for every $\mu$ which is a successor infinite cardinal $\le\lambda$. As the case $\mu\le\aleph_0$ is straight-forward, we are left with handling the case in which $\mu\le\lambda$ is a limit uncountable cardinal. So, suppose that $\mu$ is a limit uncountable cardinal, and put:

$$E^\mu := \biguplus\{E_\kappa \mid \kappa \in \mathbb{R}_\mu\}.$$

**Claim 3.3.1.** $\mathrm{Chr}(\lambda^+, E^\mu) = \mathrm{Col}(\lambda^+, E^\mu) = \mu$.

*Proof.* Let us first point out that any coloring $\chi : \lambda^+ \rightarrow \theta$ with $\theta < \mu$ is not chromatic. To see this, let $\kappa := \min(\mathbb{R}_\mu \setminus \theta)$. Since $\mathrm{Chr}(G_\kappa, E_\kappa) > \kappa \geq \theta$, we know that there exists $\{\alpha, \delta\} \in E_\kappa$ such that $\chi(\alpha) = \chi(\delta)$. As $E_\kappa \subseteq E^\mu$, this shows that $\chi$ is not chromatic for $(\lambda^+, E^\mu)$.

Next, since we have already established that $\mathrm{Chr}(\lambda^+, E^\mu) \geq \mu$, it suffices to prove that $\mathrm{Col}(\lambda^+, E^\mu) \leq \mu$.

By definition of $E^\mu$, for all $\delta < \lambda^+$, either $\{\alpha \in \delta \mid \{\alpha, \delta\} \in E^\mu\}$ is empty, or $\mathrm{otp}(\{\alpha \in \delta \mid \{\alpha, \delta\} \in E^\mu\}) \leq \mathrm{otp}(C_\delta) \in \mathbb{R}_\mu$. So in either case,

$$|\{\alpha \in \delta \mid \{\alpha, \delta\} \in E^\mu\}| < \mu,$$

and hence $\in$ is a well-ordering of $\lambda^+$ witnessing that $\mathrm{Col}(\lambda^+, E^\mu) \leq \mu$. $\square$

**Claim 3.3.2.** $\mathrm{Chr}(G', E') \leq \aleph_0$ *for every* $G' \in [\lambda^+]^{<\lambda^+}$ *and* $E' \subseteq [G']^2 \cap E^\mu$.

*Proof.* Fix $G'$ and $E'$ as above. For all $\kappa \in \mathbb{R}_\mu$, pick a witness $\chi_\kappa : G_\kappa \cap G' \rightarrow \omega$ to the fact that $\mathrm{Chr}(G_\kappa \cap G', E_\kappa \cap E') \leq \aleph_0$. Then, define $\chi : G' \rightarrow \omega$ by letting for all $\alpha \in G'$:

$$\chi(\alpha) := \begin{cases} \chi_{\mathrm{otp}(C_\alpha)}(\alpha), & \mathrm{otp}(C_\alpha) \in \mathbb{R}_\mu \\ 0, & \text{otherwise} \end{cases}.$$

To see that $\chi$ is a witness to $\mathrm{Chr}(G', E') \leq \aleph_0$, fix $\{\alpha, \delta\} \in E'$. By $E' \subseteq E^\mu = \biguplus_{\kappa \in \mathbb{R}_\mu} E_\kappa$, there exists a unique $\kappa \in \mathbb{R}_\mu$ such that $\{\alpha, \delta\} \in E_\kappa \cap E'$. So, $\{\alpha, \delta\} \subseteq G_\kappa \cap G'$, and:

$$\chi(\alpha) = \chi_\kappa(\alpha) \neq \chi_\kappa(\delta) = \chi(\delta).$$

$\square$

This completes the proof. $\square$

3.3. **Corollaries.**

**Corollary 3.4.** *Assume* $\square_{\omega_1}$. *If* CH *holds, then there exists a* $\sigma$*-closed notion of forcing* $\mathbb{P}$*, such that* $|\mathbb{P}| = \aleph_1$*, and in* $V^{\mathbb{P}}$*, there exists an* $(\aleph_0, \aleph_2)$*-chromatic graph of size* $\aleph_2$.

*Proof.* By Theorem 3.2, $\mathbb{P} = \mathrm{Add}(\omega_1, 1)$ works. $\square$

Compare the preceding with the starting point of this study:

**Theorem** (Baumgartner, [1]). CH *entails the existence of a* $\sigma$*-closed notion of forcing* $\mathbb{P}$*, such that* $|\mathbb{P}| = \aleph_2$*,* $\mathbb{P}$ *has that* $\aleph_2$*-c.c., and in* $V^{\mathbb{P}}$*, there exists an* $(\aleph_0, \aleph_2)$*-chromatic graph of size* $\aleph_2$.

Note that while Baumgartner's poset $\mathbb{P}$ is rather complicated (indeed, a full page of [1] is dedicated to justifying the definition of $\mathbb{P}$), here $\mathbb{P}$ stands for the simplest example of a $\sigma$-closed notion of forcing.

**Corollary 3.5.** *If $0^\sharp$ does not exist, then for every singular strong limit cardinal $\lambda$, there exists an $(\aleph_0, \lambda^+)$-chromatic graph of size $\lambda^+$.*

*Proof.* By results of Jensen (see [9]), if $0^\sharp$ does not exist, then $\square_\lambda + \mathrm{CH}_\lambda$ holds for every singular strong limit cardinal $\lambda$. Now, invoke Theorem 3.1. $\square$

Compare the preceding with Theorem 8.18 of [16].

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