A GUESSING PRINCIPLE FROM A SOUSLIN TREE, WITH APPLICATIONS TO TOPOLOGY

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This paper is dedicated to the memory of Kenneth Kunen (1943–2020)

Abstract. We introduce a new combinatorial principle which we call $\clubsuit_{\mathrm{AD}}$. This principle asserts the existence of a certain multi-ladder system with guessing and almost-disjointness features, and is shown to be sufficient for carrying out de Caux type constructions of topological spaces.

Our main result states that strong instances of $\clubsuit_{\mathrm{AD}}$ follow from the existence of a Souslin tree. It is also shown that the weakest instance of $\clubsuit_{\mathrm{AD}}$ does not follow from the existence of an almost Souslin tree.

As an application, we obtain a simple, de Caux type proof of Rudin's result that if there is a Souslin tree, then there is an S-space which is Dowker.

1. Introduction

All topological spaces under consideration are assumed to be T_1 and Hausdorff.

A *Dowker space* is a normal topological space whose product with the unit interval is not normal. Dowker [Dow51] raised the question of their very existence, and gave a useful characterization of these spaces. The first consistent example of such a space was given by Rudin [Rud55], who constructed a Dowker space of size $\aleph_1$, assuming the existence of a Souslin tree. Later on, in [Rud72], Rudin constructed another Dowker space, this time in ZFC, and of cardinality $(\aleph_\omega)^{\aleph_0}$. Two decades later, Balogh [Bal96] gave a ZFC construction of a Dowker space of size $2^{\aleph_0}$, and Kojman and Shelah [KS98] gave a ZFC construction of a Dowker space of size $\aleph_{\omega+1}$. A question remaining of focal interest ever since is whether ZFC proves the existence of a *small* Dowker space. One of the sleekest consistent constructions of a Dowker space of size $\aleph_1$ may be found in de Caux's paper [dC77], assuming the combinatorial principle $\clubsuit$. Whether $\clubsuit$ implies the existence of a Souslin tree was asked by Juhász around 1987 and remains open to this date. For a comprehensive survey on Dowker spaces, we refer the reader to [Rud84], [SW93] and [Sze07].

An *S-space* is a regular topological space which is hereditary separable but not hereditary Lindelöf. Whether such a space exists was asked at the late 1960's by Hajnal and Juhász and independently by Countryman. Along the years many consistent constructions of S-spaces were found, many of which are due to Kunen and his co-authors [JKR76, vDK82, DK93, dlVK04, HK09, HK18, HK20]. Rudin showed that the existence of a Souslin tree yields an S-space [Rud72], and even an S-space which is Dowker [Rud74a]. Juhász, Kunen, and Rudin [JKR76] gave an example from CH of a first countable, locally compact S-space, known as the *Kunen Line*, as well as an example from CH of a first countable, S-space which is Dowker.

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In the other direction, Kunen [Kun77] proved that, assuming $\mathsf{MA}_{\aleph_1}$, there are no strong S-spaces (that is, spaces all of whose finite powers are S-spaces), Szentmiklóssy [Sze80] proved that, assuming $\mathsf{MA}_{\aleph_1}$, there are no compact S-spaces, and Todorčević [Tod83] proved that, assuming PFA, there are no S-spaces whatsoever. For a comprehensive survey on S-spaces, we refer the reader to [Juh80], [Roi84] and [And09].

An *O-space* is an uncountable regular topological space all of whose uncountable open sets are co-countable. Note that any O-space is an S-space of size $\aleph_1$ all of whose closed sets are G_δ (i.e., the space is *perfect*), and hence not Dowker (cf. [Wei78, pp. 248], [ER99, §2] and [Juh02, §5]). This class of spaces is named after Adam Ostaszewski who constructed in [Ost76], assuming $\clubsuit$, a normal, locally compact, non-Lindelöf O-space. He also showed that, assuming CH, the space can be made countably compact. A few years later, Mohammed Dahroug, who was a Ph.D. student of William Weiss at the University of Toronto, constructed a first-countable, locally compact, non-Lindelöf O-space from a Souslin tree. He also showed that, assuming CH, the space can be made countably compact and normal. Dahroug's work was never typed down.

The purpose of this paper is to formulate a combinatorial principle that follows both from $\clubsuit$ and from the existence of a Souslin tree, and is still strong enough to yield an S-space which is Dowker, as well as a normal O-space. We call it $\clubsuit_{\mathrm{AD}}$. The exact definition may be found in Definition 2.4 below. The main results of this paper read as follows.

Theorem A. *(1) For every infinite cardinal λ, if there exists a $\mathrm{cf}(\lambda)$-complete λ^+-Souslin tree, then for every partition $\mathcal{S}$ of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ into stationary sets, $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\mathrm{cf}(\lambda))$ holds.*

(2) For every regular uncountable cardinal κ, if there exists a regressive κ-Souslin tree, then for every partition $\mathcal{S}$ of E^κ_ω into stationary sets, $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\omega)$ holds.

Theorem B. *Suppose that $\mathcal{S}$ is an infinite partition of some non-reflecting stationary subset of a regular uncountable cardinal κ. If $\clubsuit_{\mathrm{AD}}(\mathcal{S},2)$ holds, then there exists a Dowker space of cardinality κ.*

Note that for every infinite regular cardinal λ, $E^{\lambda^+}_\lambda$ is a non-reflecting stationary subset of λ^+.

Theorem C. *If $\clubsuit_{\mathrm{AD}}(\{\omega_1\},1)$ holds, then there exists a collectionwise normal non-Lindelöf O-space.*

Theorem D. *If $\clubsuit_{\mathrm{AD}}(\{E^{\lambda^+}_\lambda\},1)$ holds for an infinite regular cardinal λ, then there exists a collectionwise normal Dowker space of cardinality λ^+, having hereditary density λ and Lindelöf degree λ^+.*

1.1. **Organization of this paper.** In Section 2, we formulate the guessing principle $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\theta)$ and its refinement $\clubsuit_{\mathrm{AD}}(\mathcal{S},\mu,{<}\theta)$, prove that $\clubsuit(S)$ entails a strong instance of $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\omega)$, and that it is consistent that $\clubsuit_{\mathrm{AD}}(\{\omega_1\},{<}\omega)$ holds, but $\clubsuit(\omega_1)$ fails. It is also shown that the weakest instance $\clubsuit_{\mathrm{AD}}(\{\kappa\},1,1)$ fails for κ weakly compact, and may consistently fail for $\kappa:=\omega_1$. The proof of Theorem A will be found there.

In Section 3, we present a $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$-based construction of a Dowker space which is moreover a ladder-system space. This covers scenarios previously considered by Good, Rudin and Weiss, in which the Dowker spaces constructed were not ladder-system spaces. The proof of Theorem B will be found there.

In Section 4, we present two $\clubsuit_{\mathrm{AD}}(\{E^{\lambda^+}_{\lambda}\},\lambda,1)$-based constructions of collectionwise normal spaces of small hereditary density and large Lindelöf degree. These spaces are not ladder-system spaces, rather, they are de Caux type spaces. The proof of Theorems C and D will be found there.

In Section 5, we comment on a construction of Szeptycki of an $\aleph_2$-sized Dowker space with a normal square, assuming that $\diamondsuit^*(S)$ holds for some stationary $S \subseteq E^{\omega_2}_{\omega_1}$. Here, it is demonstrated that the construction may be carried out from a weaker assumption which is known to be consistent with the failure of $\clubsuit(E^{\omega_2}_{\omega_1})$.

1.2. **Notation.** The *hereditary density number* of a topological space $\mathbb{X}$, denoted $\mathrm{hd}(\mathbb{X})$, is the least infinite cardinal λ such that each subspace of $\mathbb{X}$ contains a dense subset of cardinality at most λ. The *Lindelöf degree* of $\mathbb{X}$, denoted $\mathrm{L}(\mathbb{X})$ is the least infinite cardinal λ such that every open cover of $\mathbb{X}$ has a subcover of cardinality at most λ. For an accessible cardinal κ and a cardinal $\lambda<\kappa$, we write $\log_\lambda(\kappa) := \min\{\chi \mid \lambda^\chi \geq \kappa\}$. $\mathrm{Reg}(\kappa)$ denotes the set of all infinite regular cardinal below κ. For a set of ordinals C, we write $\mathrm{acc}^+(C) := \{\alpha < \sup(C) \mid \sup(C\cap\alpha)=\alpha>0\}$, $\mathrm{acc}(C) := \{\alpha \in C \mid \sup(C\cap\alpha)=\alpha>0\}$ and $\mathrm{nacc}(C) := C\setminus \mathrm{acc}(C)$. For ordinals $\alpha<\gamma$, denote $E^\gamma_\alpha := \{\beta<\gamma \mid \mathrm{cf}(\beta)=\alpha\}$ and define $E^\gamma_{\neq\alpha}, E^\gamma_{<\alpha}, E^\gamma_{\leq\alpha}, E^\gamma_{>\alpha}, E^\gamma_{\geq\alpha}$ similarly. For a set A and a cardinal θ, write $[A]^\theta := \{B\subseteq A \mid |B|=\theta\}$ and define $[A]^{<\theta}$ similarly. For two sets A and B, we write $A\subseteq^* B$ to express that either $A=\emptyset$ or $A\setminus\alpha\subseteq B$ for some $\alpha\in A$. For a poset $(P,\lhd)$ and an element $x\in P$, we write $x_\downarrow := \{y\in P \mid y\lhd x\}$ and $x^\uparrow := \{y \in P \mid x \lhd y\}$. For a family $\mathcal{A}$ of subsets of some ordinal, we let $\mathrm{mup}(\mathcal{A}) := \sup\{\min(a) \mid a\in\mathcal{A},\ a\neq\emptyset\}$.

1.3. **Conventions.** Throughout the paper, κ stands for a regular uncountable cardinal, and λ stands for an infinite cardinal.

2. A new guessing principle

We commence by recalling some classic guessing principles.

Definition 2.1. For a stationary subset $S\subseteq\kappa$:

(1) $\diamondsuit^*(S)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha\in S\rangle$ such that:
- for all $\alpha\in S$, $\mathcal{A}_\alpha\subseteq\mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha|\leq|\alpha|$;
- for every $B\subseteq\kappa$, there exists a club $C\subseteq\kappa$ such that $C\cap S\subseteq\{\alpha\in S \mid B\cap\alpha\in\mathcal{A}_\alpha\}$.

(2) $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha\in S\rangle$ such that:
- for all $\alpha\in S$, $A_\alpha\subseteq\alpha$;
- for every $B\subseteq\kappa$, the set $\{\alpha\in S \mid B\cap\alpha = A_\alpha\}$ is stationary.

(3) $\clubsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha\in S\rangle$ such that:
- for all $\alpha\in S\cap\mathrm{acc}(\kappa)$, A_α is a cofinal subset of α of order type $\mathrm{cf}(\alpha)$;
- for every cofinal subset $B\subseteq\kappa$, the set $\{\alpha\in S \mid A_\alpha\subseteq B\}$ is stationary.

(4) $\clubsuit_J(S)$ asserts the existence of a matrix $\langle A_{\alpha,i} \mid \alpha\in S,\ i<\mathrm{cf}(\alpha)\rangle$ such that:
- For all $\alpha\in S\cap\mathrm{acc}(\kappa)$, $\langle A_{\alpha,i} \mid i<\mathrm{cf}(\alpha)\rangle$ is a sequence of pairwise disjoint cofinal subsets of α, each of order-type $\mathrm{cf}(\alpha)$;

- For every cofinal subset $B \subseteq \kappa$, the following set is stationary:

$$\{\alpha \in S \mid \forall i < \mathrm{cf}(\alpha)[\sup(B \cap A_{\alpha,i}) = \alpha]\}.$$

Remark 2.2. The principle $\diamondsuit^*$ was introduced by Kunen and Jensen in [JK69], the principle $\diamondsuit$ was introduced by Jensen in [Jen72], the principle $\clubsuit$ was introduced by Ostaszewski in [Ost76], and the principle $\clubsuit_J$ was introduced by Juhász in [Juh88] (under the name (t)). It is not hard to see that for stationary $S' \subseteq S \subseteq \kappa$, $\diamondsuit^*(S') \implies \diamondsuit(S) \implies \clubsuit(S) \implies \clubsuit_J(S)$. Devlin (see [Ost76, p. 507]) proved that $\diamondsuit(S) \iff \clubsuit(S) + \kappa^{<\kappa} = \kappa$. In [Juh88], Juhász proved that $\clubsuit_J(\omega_1)$ is adjoined by the forcing to add a Cohen real, and proved that the former suffices for the construction of an Ostaszewski space.

To present our new guessing principle, we shall first need the following definition.

Definition 2.3. For a set of ordinals S:

(1) A sequence $\langle A_\alpha \mid \alpha \in S\rangle$ is said to be an AD-*ladder system* iff the two hold:
   - For all $\alpha \in S \cap \mathrm{acc}(\kappa)$, A_α is a cofinal subset of α;
   - For all two distinct $\alpha, \alpha' \in S$, $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

(2) A sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ is said to be an AD-*multi-ladder system* iff the two hold:
   - For all $\alpha \in S \cap \mathrm{acc}(\kappa)$, $\mathcal{A}_\alpha$ is a nonempty family consisting of pairwise disjoint cofinal subsets of α;
   - For all two distinct $A, A' \in \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Now, we are ready to present the new guessing principle.

Definition 2.4. For a family $\mathcal{S}$ of stationary subsets of κ, $\clubsuit_{\mathrm{AD}}(\mathcal{S}, {<}\theta)$ asserts the existence of an AD-multi-ladder system $\vec{\mathcal{A}} = \langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S}\rangle$ such that:

(1) For every $\alpha \in \bigcup \mathcal{S}$, $|\mathcal{A}_\alpha| = \mathrm{cf}(\alpha)$;
(2) For every $\mathcal{B} \subseteq [\kappa]^\kappa$ with $|\mathcal{B}| < \theta$, and every $S \in \mathcal{S}$, the following set is stationary:

$$G(S, \mathcal{B}) := \{\alpha \in S \mid \forall (A, B) \in \mathcal{A}_\alpha \times \mathcal{B}\ [\sup(A \cap B) = \alpha]\}.$$

Convention 2.5. We write $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \theta)$ for $\clubsuit_{\mathrm{AD}}(\mathcal{S}, {<}(\theta{+}1))$, and $\clubsuit_{\mathrm{AD}}(S)$ for $\clubsuit_{\mathrm{AD}}(\{S\}, 1)$.

Remark 2.6. For any $\chi \in \mathrm{Reg}(\kappa)$ and any stationary $S \subseteq E^\kappa_\chi$, $\clubsuit_J(S) \implies \clubsuit_{\mathrm{AD}}(S)$.

$$\begin{tikzcd}
\diamondsuit(S) \arrow[rr] \arrow[d] & & \diamondsuit(\omega_1) \arrow[dd] \\
\clubsuit(S) \arrow[d] & \text{Cohen real} \arrow[dl] \arrow[dr] & \\
\clubsuit_J(S) \arrow[d] & & \exists\aleph_1\text{-Souslin Tree} \arrow[d] \\
\clubsuit_{\mathrm{AD}}(S) & & \forall\mathcal{S}\ \clubsuit_{\mathrm{AD}}(\mathcal{S}, {<}\omega) \arrow[ll]
\end{tikzcd}$$

Figure 1. Diagram of implications between the combinatorial principles under discussion, at the level of ω_1.

To motivate Definition 2.3, let us point out two easy facts concerning disjointifying AD systems.

Proposition 2.7. *Suppose that S is a non-reflecting stationary subset of κ, and $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ is an* AD*-ladder system. Then there exists a sequence of functions $\langle f_\xi \mid \xi < \kappa\rangle$ such that, for every $\xi < \kappa$:*

(1) f_ξ is a regressive function from $S \cap \xi$ to ξ;
(2) the sets in $\langle A_\alpha \setminus f_\xi(\alpha) \mid \alpha \in S \cap \xi\rangle$ are pairwise disjoint.

Proof. We recursively construct a sequence $\langle f_\xi \mid \xi < \kappa\rangle$ such that for every $\xi < \kappa$, Clauses (1) and (2) above hold true.

▶ The cases where either $\xi = 0$ or $\xi = \beta + 1$ for $\beta \notin S$ are straightforward.

▶ Suppose $\xi = \beta + 1$ with $\beta \in S$ for which f_β has already been defined. Define $g : S \cap \beta \to \beta$ via $g(\alpha) := \sup(A_\beta \cap A_\alpha)$. As $\vec{A}$ is an AD-ladder system, g is regressive. Consequently, we may define a regressive function $f_\xi : S \cap \xi \to \xi$ via $f_\xi(\alpha) := \max\{f_\beta(\alpha), g(\alpha)\}$ for $\alpha \in S \cap \beta$, and $f_\xi(\beta) := 0$. Notice that by the recursive assumption the function f_ξ is as sought.

▶ Suppose $\xi \in \mathrm{acc}(\kappa)$ for which $\langle f_\beta \mid \beta < \xi\rangle$ has already been defined. As S is non-reflecting we may fix a club $C \subseteq \xi$ disjoint from S. For every $\alpha < \xi$, set $\alpha^- := \sup(C \cap \alpha)$ and $\alpha^+ := \min(C \setminus \alpha)$. Note that, for every nonzero $\alpha \in S \cap \xi$, $\alpha^- < \alpha < \alpha^+$. Define a regressive function $f_\xi : S \cap \xi \to \xi$ via $f_\xi(\alpha) := \max\{f_{\alpha^+}(\alpha), \alpha^-\}$. Notice that by the recursive hypothesis, the function f_ξ is as sought. □

Proposition 2.8. *Suppose that S is a subset of $E^\kappa_{\geq\lambda}$ and $\vec{\mathcal{A}} = \langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ is an* AD*-multi-ladder system. For any $\mathcal{B} \subseteq \bigcup_{\alpha\in S} \mathcal{A}_\alpha$ with $|\mathcal{B}| \leq \lambda$, there exists a function $f : \mathcal{B} \to \kappa$ such that:*

(1) for every $B \in \mathcal{B}$, $f(B) \in B$;
(2) the sets in $\langle B \setminus f(B) \mid B \in \mathcal{B}\rangle$ are pairwise disjoint.

Proof. Given $\mathcal{B} \subseteq \bigcup_{\alpha\in S} \mathcal{A}_\alpha$ with $|\mathcal{B}| \leq \lambda$, fix an injective enumeration $\langle B_\xi \mid \xi < |\mathcal{B}|\rangle$ of $\mathcal{B}$. By the hypothesis on $\vec{\mathcal{A}}$, for every pair $\zeta < \xi < |\mathcal{B}|$,

$$\sup(B_\xi \cap B_\zeta) < \sup(B_\xi) \in E^\kappa_{\geq\lambda} \subseteq E^\kappa_{>\xi},$$

so that $\sup_{\zeta<\xi} \sup(B_\xi \cap B_\zeta) < \sup(B_\xi)$.

It follows that we may define a function $f : \mathcal{B} \to \kappa$ via:

$$f(B_\xi) := \begin{cases} \min(B_\xi), & \text{if } \xi = 0;\\ \min\{\beta \in B_\xi \mid \sup\{\sup(B_\xi \cap B_\zeta) \mid \zeta < \xi\} < \beta\}, & \text{otherwise.}\end{cases}$$

Evidently, f is as sought. □

Our next lemma shows, in particular, that for any $\chi \in \mathrm{Reg}(\kappa)$ and any stationary $S \subseteq E^\kappa_\chi$, $\clubsuit(S) \implies \clubsuit_{\mathrm{AD}}(\{S\}, {<}\omega)$. The reverse implication does not hold in general, as established by Corollary 2.27 below. The proof of the lemma will make use of the following fact.

Fact 2.9 (Brodsky-Rinot, [BR21, §3])**.** *For any stationary $S \subseteq \kappa$, all of the following are equivalent:*

(1) $\clubsuit(S)$;
(2) there exists a partition $\langle S_i \mid i < \kappa\rangle$ of S into pairwise disjoint stationary sets such that $\clubsuit(S_i)$ holds for each $i < \kappa$;

(3) *for any (possibly finite) cardinal θ such that $\kappa^\theta=\kappa$, there exists a matrix $\langle A_{\alpha,\tau}\mid \alpha\in S,\ \tau\le\theta\rangle$ such that, for every sequence $\langle A_\tau\mid\tau\le\theta\rangle$ of cofinal subsets of κ, the following set is stationary in κ:*

$$\{\alpha\in S\mid \forall\tau\le\theta\ [A_{\alpha,\tau}\subseteq A_\tau\cap\alpha\ \&\ \sup(A_{\alpha,\tau})=\alpha]\}.$$

(4) *there exists a sequence $\langle X_\alpha\mid\alpha\in S\rangle$ such that:*
   - *for every $\alpha\in S\cap\operatorname{acc}(\kappa)$, $X_\alpha\subseteq[\alpha]^{<\omega}$ with $\operatorname{mup}(X_\alpha)=\alpha$;*[1]
   - *for every $X\subseteq[\kappa]^{<\omega}$ with $\operatorname{mup}(X)=\kappa$, the following set is stationary:*

$$\{\alpha\in S\mid X_\alpha\subseteq X\}.$$

Lemma 2.10. *Suppose that $\clubsuit(S)$ holds for some stationary $S\subseteq E^\kappa_\chi$ with $\chi\in\operatorname{Reg}(\kappa)$. Then there exists a partition $\mathcal S$ of S into κ many stationary sets for which $\clubsuit_{\mathrm{AD}}(\mathcal S,{<}\omega)$ holds as witnessed by a sequence $\vec{\mathcal A}=\langle\mathcal A_\alpha\mid\alpha\in S\rangle$ with $\operatorname{otp}(A)=\chi$ for all $A\in\bigcup_{\alpha\in S}\mathcal A_\alpha$.*

Proof. By Fact 2.9(2), fix a partition $\langle S_i\mid i<\kappa\rangle$ of S into pairwise disjoint stationary sets such that $\clubsuit(S_i)$ holds for each $i<\kappa$. Set $\mathcal S:=\{S_i\mid i<\kappa\}$. Next, for each $i<\kappa$, let $\langle X_\alpha\mid\alpha\in S_i\rangle$ be a sequence as in Fact 2.9(4). Fix a surjection $h:\chi\to\chi$ such that $|h^{-1}\{j\}|=\chi$ for all $j<\chi$.

To simplify the upcoming argument let us agree to write, for any two nonempty sets of ordinals a,b, "$a<b$" iff $\alpha<\beta$ for all $(\alpha,\beta)\in a\times b$.

Let $\alpha\in S\cap\operatorname{acc}(\kappa)$. Recall that $X_\alpha\subseteq[\alpha]^{<\omega}$ and $\operatorname{mup}(X_\alpha)=\alpha$. Fix a strictly increasing sequence of ordinals $\langle\alpha_\zeta\mid\zeta<\chi\rangle$ that converges to α. Now, by recursion on $\zeta<\chi$, we construct a sequence $\langle x_\zeta\mid\zeta<\chi\rangle$ such that, for every $\zeta<\chi$:

(1) $x_\zeta\in X_\alpha$, and
(2) for every $\xi<\zeta$, $(x_\xi\cup\{\alpha_\zeta\})<x_\zeta$.

Suppose $\zeta<\chi$ and that $\langle x_\xi\mid\xi<\zeta\rangle$ has already been defined. Evidently, $\eta:=\sup(\{\max(x_\xi)\mid\xi<\zeta\}\cup\{\alpha_\zeta\})$ is $<\alpha$. So, as $\operatorname{mup}(X_\alpha)=\alpha$, we may let $x_\zeta:=x$ for some $x\in X_\alpha$ with $\min(x)>\eta$.

This completes the construction. By Clause (2), $\langle x_\zeta\mid\zeta<\chi\rangle$ is $<$-increasing, and $\operatorname{mup}\{x_\zeta\mid\zeta<\chi\}=\alpha$. Finally, for every $j<\chi$, let $X^j_\alpha:=\{x_\zeta\mid\zeta<\chi,h(\zeta)=j\}$, and $A^j_\alpha:=\biguplus X^j_\alpha$.

Claim 2.10.1. *$\langle A^j_\alpha\mid j<\chi\rangle$ is a sequence of pairwise disjoint cofinal subsets of α, each of order-type χ.*

Proof. Let $j<\chi$. As $\operatorname{mup}\{x_\zeta\mid\zeta<\chi,h(\zeta)=j\}=\alpha$, and as $\langle x_\zeta\mid\zeta<\chi,h(\zeta)=j\rangle$ is a $<$-increasing $\operatorname{cf}(\alpha)$-sequence of finite sets, we infer that $\sup(A^j_\alpha)=\alpha$, and that, for every $\beta<\alpha$, $\operatorname{otp}(A^j_\alpha\cap\beta)<\chi$. Altogether, $\operatorname{otp}(A^j_\alpha)=\chi$.

Also, since $\langle x_\zeta\mid\zeta<\chi\rangle$ consists of pairwise disjoint sets, the elements of $\langle A^j_\alpha\mid j<\chi\rangle$ are pairwise disjoint. □

For every $\alpha\in S$, set $\mathcal A_\alpha:=\{A^j_\alpha\mid j<\chi\}$. It immediately follows from the preceding claim that $\vec{\mathcal A}:=\langle\mathcal A_\alpha\mid\alpha\in S\rangle$ is an AD-multi-ladder system.

Claim 2.10.2. *For every finite $\mathcal B\subseteq[\kappa]^\kappa$ and every $i<\kappa$, the following set is stationary:*

$$G(S_i,\mathcal B):=\{\alpha\in S_i\mid\forall(A,B)\in\mathcal A_\alpha\times\mathcal B\ [\sup(A\cap B)=\alpha]\}.$$

[1] Recall that mup was defined in Subsection 1.2.

Proof. Suppose that $\langle B_n \mid n < m\rangle$ is a finite sequence of cofinal subsets of κ. For each $n < m$, fix an injective enumeration $\langle b_{n,\iota} \mid \iota < \kappa\rangle$ of B_n. Set $X := \{\{b_{n,\iota} \mid n < m\} \mid \iota < \kappa\}$ and notice that $X \subseteq [\kappa]^{<\omega}$ with $\mathrm{mup}(X) = \kappa$. Consequently, for every $i < \kappa$, the set $T_i := \{\alpha \in S_i \mid X_\alpha \subseteq X\}$ is stationary. Let $i < \kappa$. We claim that $T_i \subseteq G(S_i, \mathcal{B})$. To see this, let $\alpha \in T_i$ and $(A, B) \in \mathcal{A}_\alpha \times \mathcal{B}$ be arbitrary. Fix $j < \chi$ and $n < m$ such that $A = A^j_\alpha$ and $B = B_n$.

As $X^j_\alpha \subseteq X_\alpha \subseteq X$, by the definition of X, for every $x \in X^j_\alpha$, $x \cap B_n \neq \emptyset$. As $\mathrm{mup}(X^j_\alpha) = \alpha$, it follows that $\sup(A^j_\alpha \cap B_n) = \alpha$. □

This completes the proof. □

We conclude this subsection by formulating a three-cardinal variant of $\clubsuit_{\mathrm{AD}}$:

Definition 2.11. $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, {<}\theta)$ asserts the existence of a system $\vec{\mathcal{A}} = \langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S}\rangle$ as in Definition 2.4, but in which Clause (1) is replaced by the requirement that, for every $\alpha \in \bigcup S$, $|\mathcal{A}_\alpha| = \mu$. We write $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \theta)$ for $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, {<}(\theta + 1))$.

It is clear that $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \omega, {<}\theta)$ follows from $\clubsuit_{\mathrm{AD}}(\mathcal{S}, {<}\theta)$. Also, the following lemma is obvious.

Lemma 2.12. *For a family $\mathcal{S}$ of stationary subsets of κ, $\clubsuit_{\mathrm{AD}}(\mathcal{S}, 1, 2)$ holds iff there exists an* AD*-ladder system $\langle A_\alpha \mid \alpha \in \bigcup \mathcal{S}\rangle$ such that, for all $B_0, B_1 \in [\kappa]^\kappa$ and $S \in \mathcal{S}$, the set $\{\alpha \in S \mid \sup(A_\alpha \cap B_0) = \sup(A_\alpha \cap B_1) = \alpha\}$ is stationary.* □

2.1. **Interlude on Souslin trees.** Recall that a poset $\mathbf{T} = (T, <_T)$ is a *κ-Souslin tree* iff all of the following hold:

- $|T| = \kappa$;
- $(T, <_T)$ has no chains or antichains of size κ;
- for every $x \in T$, $(x_\downarrow, <_T)$ is well-ordered.

For every $x \in T$, denote $\mathrm{ht}(x) := \mathrm{otp}(x_\downarrow, <_T)$. For every $A \subseteq \kappa$, let $T \restriction A := \{x \in T \mid \mathrm{ht}(x) \in A\}$. Note that, for every $\alpha < \kappa$, $T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}$ and $T \restriction \alpha$ have size $<\kappa$.

The next well-known lemma shows that Souslin trees are similar to Luzin spaces in the sense that every large subset of a Souslin tree is *somewhere dense.*

Lemma 2.13 (folklore)**.** *Suppose $\mathbf{T} = (T, <_T)$ is a κ-Souslin tree and $B \subseteq T$ is a subset with $|B| = \kappa$. Then there exists $w \in T$ such that $w^\uparrow \cap B$ is cofinal in $w^\uparrow$.*

Proof. Let X denote the collection of all $x \in T$ such that $x^\uparrow \cap B$ is empty. Let $A \subseteq X$ be a maximal antichain in X. As $\mathbf{T}$ is a κ-Souslin tree, $|A| < \kappa$, so we may find a large enough $\delta < \kappa$ such that $A \subseteq T \restriction \delta$. As $|B| = \kappa > |T \restriction (\delta+1)|$, let us fix $b \in B$ with $\mathrm{ht}(b) > \delta$. Finally, let w denote the unique element of T_δ with $w <_T b$.

Claim 2.13.1. *$w^\uparrow \cap B$ is cofinal in $w^\uparrow$.*

Proof. Suppose not. Then there must exist some $x \in X$ with $w \leq_T x$. As A is a maximal antichain in X, we may find $\bar{x} \in A$ which is comparable with x. As $\mathrm{ht}(\bar{x}) < \delta \leq \mathrm{ht}(x)$, it follows that $\bar{x} <_T x$. As $\mathrm{ht}(w) = \delta$ and $w \leq_T x$, it follows that $\bar{x} <_T w \leq_T x$. In particular, $\bar{x} <_T w <_T b$, so that $b \in \bar{x}^\uparrow \cap B$, contradicting the fact that $\bar{x} \in X$. □

This completes the proof. □

Definition 2.14. A κ-Souslin tree $\mathbf{T} = (T, <_T)$ is said to be:

- *normal* iff for any $x \in T$ and $\alpha < \kappa$ with $\mathrm{ht}(x) < \alpha$, there exists $y \in T_\alpha$ with $x <_T y$;
- *μ-splitting* iff every node in T admits at least μ-many immediate successors, that is, for every $x \in T$, $|\{y \in T \mid x <_T y, \mathrm{ht}(y) = \mathrm{ht}(x) + 1\}| \geq \mu$;
- *prolific* iff every $x \in T$ admits at least $\mathrm{ht}(x)$-many immediate successors;
- *χ-complete* iff any $<_T$-increasing sequence of elements from T, and of length $<\chi$, has an upper bound in T;
- *regressive* iff there exists a map $\rho : T \restriction \mathrm{acc}(\kappa) \to T$ satisfying the following:
  - for every $x \in T \restriction \mathrm{acc}(\kappa)$, $\rho(x) <_T x$;
  - for all $\alpha \in \mathrm{acc}(\kappa)$ and $x, y \in T_\alpha$, if $\rho(x) <_T y$ and $\rho(y) <_T x$, then $x = y$;
- *ordinal-based* iff $T = \kappa$ and, for all $x, y \in T$, if $\mathrm{ht}(x) < \mathrm{ht}(y)$, then $x \in y$.

A subset $B \subseteq T$ is said to be an *α-branch* iff $(B, <_T)$ is linearly ordered and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$; it is said to be *vanishing* iff it has no upper bound in T.

Definition 2.15. A λ^+-Souslin tree is said to be *maximally-complete* iff it is χ-complete for $\chi := \log_\lambda(\lambda^+)$.

Note that the existence of a $\mathrm{cf}(\lambda)$-complete λ^+-Souslin tree is equivalent to the conjunction of "$\lambda^{<\mathrm{cf}(\lambda)} = \lambda$" and "there is a maximally-complete λ^+-Souslin tree".

Proposition 2.16 (folklore)**.** *For cardinals $\chi, \mu < \mathrm{cf}(\kappa) = \kappa$, if there exists a κ-Souslin tree which is χ-complete (resp. regressive), then there exists an ordinal-based μ-splitting, normal, prolific κ-Souslin tree which is χ-complete (resp. regressive).*

Proof. Suppose $\mathbf{T} = (T, <_T)$ is a κ-Souslin tree. By a standard fact (see [BR17b, Lemma 2.4]), we may fix a club $E \subseteq \kappa$ such that $(T \restriction E, <_T)$ is normal and splitting. Consider the set $D := \{\alpha < \kappa \mid \mathrm{otp}(E \cap \alpha) = \mu^\alpha\}$ which is a subclub of E. It is clear that $\mathbf{T}' := (T \restriction D, <_T)$ is a normal κ-Souslin tree.

Claim 2.16.1. *(1) $\mathbf{T}'$ is prolific and μ-splitting;*
(2) if $\mathbf{T}$ is χ-complete, then so is $\mathbf{T}'$;
(3) if $\mathbf{T}$ is regressive, then so is $\mathbf{T}'$.

Proof. (1) Fix an arbitrary node x of $\mathbf{T}'$, so that $x \in T \restriction D$. Let $\delta := \min(D \setminus (\mathrm{ht}_{\mathbf{T}}(x) + 1))$. As $\delta \in D \subseteq E$ and $(T \restriction E, <_T)$ is normal, let us fix $z \in T_\delta$ with $x <_T z$. Let $e := \{\varepsilon \in E \mid \mathrm{ht}(x) < \varepsilon < \delta\}$. Note that from $\mathrm{otp}(E \cap \delta) = \mu^\delta$, it follows that $\mathrm{otp}(e) = \delta$ and $|e| \geq \mu$.

For every $\varepsilon \in e$, let y_ε denote the unique element of T_ε satisfying $y_\varepsilon <_T z$, and denote $\varepsilon^+ := \min(e \setminus (\varepsilon + 1))$. Then, using the fact that $(T \restriction E, <_T)$ is normal and splitting, for every $\varepsilon \in e$, pick $\hat{y}_\varepsilon \in T_{\varepsilon^+}$ such that $y_\varepsilon <_T \hat{y}_\varepsilon$ and $\hat{y}_\varepsilon \neq y_{\varepsilon^+}$, and then pick $z_\varepsilon \in T_\delta$ with $\hat{y}_\varepsilon <_T z_\varepsilon$. Then $\{z_\varepsilon \mid \varepsilon \in e\}$ consists of $|e|$-many immediate successors of x in $\mathbf{T}'$.

(2) Since D is closed.

(3) Suppose $\rho : T \restriction \mathrm{acc}(\kappa) \to T$ witnesses that $\mathbf{T}$ is regressive. Define $\rho' : T \restriction \mathrm{acc}(D) \to T \restriction D$ as follows. Given $\alpha \in \mathrm{acc}(D)$ and $x \in T_\alpha$, let $\delta := \min(D \setminus (\mathrm{ht}_{\mathbf{T}}(\rho(x)) + 1))$, and then let $\rho'(x)$ be the unique $y <_T x$ with $\mathrm{ht}_{\mathbf{T}}(y) = \delta$. It is clear that ρ' witnesses that $\mathbf{T}'$ is regressive. □

Recursively define a sequence of injections $\langle \pi_\alpha : T_\alpha \to \kappa \mid \alpha \in D\rangle$ such that for, every $\alpha \in D$:

- For every $\alpha' \in D \cap \alpha$, $\mathrm{Im}(\pi_{\alpha'}) \cap \mathrm{Im}(\pi_\alpha) = \emptyset$;

- $\biguplus\{\mathrm{Im}(\pi_{\alpha'}) \mid \alpha' \in D \cap (\alpha+1)\}$ is an ordinal.

Evidently, $\pi := \bigcup_{\alpha\in D} \pi_\alpha$ is an injection from $T \restriction D$ onto κ. Let $\lhd := \{(\pi(x), \pi(y)) \mid (x,y) \in <_T\}$. Then $(\kappa, \lhd)$ is an ordinal-based κ-Souslin tree order-isomorphic to $\mathbf{T}'$. □

A richer introduction to abstract transfinite trees, Aronszajn trees and Souslin trees may be found in Section 2 of [BR21], and a comprehensive treatment of the consistency of existence of Souslin trees may be found in Section 6 of the same paper. For our purpose, it suffices to mention the following fact:

Fact 2.17. *For an infinite cardinal λ satisfying $\diamondsuit(\lambda^+)$:*[2]

1. *(1)* [Jen72] *Assuming $\lambda^{<\lambda} = \lambda$, if $\diamondsuit(E^{\lambda^+}_\lambda)$ holds, then there exists a λ-complete λ^+-Souslin tree;*
2. *(2)* [Rin19] *Assuming $\lambda^{<\lambda} = \lambda$, if $\square(\lambda^+, {<}\lambda)$ holds, then there exists a λ-complete λ^+-Souslin tree;*
3. *(3)* [Rin17, Rin22] *Assuming $\lambda^{\aleph_0} = \lambda$ or $\lambda \geq \beth_\omega$ or $\mathfrak{b} \leq \lambda < \aleph_\omega$, if $\square(\lambda^+)$ holds, then there exists a maximally-complete λ^+-Souslin tree and there exists a regressive λ^+-Souslin tree;*
4. *(4)* [LR19] *If $\square_{\lambda^+}$ holds, then there exists a λ^+-complete λ^{++}-Souslin tree.*

We now introduce a new characteristic of Souslin trees.

Definition 2.18 (The levels of vanishing branches)**.** For a κ-Souslin tree $\mathbf{T} = (T, <_T)$, let $V(\mathbf{T})$ denote the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that, for every $x \in T \restriction \alpha$, there exists a vanishing α-branch containing x.

It follows from a theorem of Shelah [She99] that it is consistent that for some Mahlo cardinal κ, there exists a κ-Souslin tree $\mathbf{T}$ for which $V(\mathbf{T}) = \emptyset$.

Lemma 2.19. *Suppose that $\mathbf{T} = (T, <_T)$ is a normal 2-splitting regressive κ-Souslin tree. Then $V(\mathbf{T}) \supseteq E^\kappa_\omega$.*

Proof. Towards a contradiction, suppose that $\alpha \in E^\kappa_\omega \setminus V(\mathbf{T})$. Fix $x \in T \restriction \alpha$ such that every α-branch B with $x \in B$ has an upper bound in T. Fix a strictly increasing sequence of ordinals $\langle \alpha_n \mid n < \omega\rangle$ that converges to α, and $\alpha_0 := \mathrm{ht}(x)$. We shall recursively construct an array $\langle x_t \mid t \in {}^{<\omega}2\rangle$ in such a way that $x_t \in T_{\alpha_{|t|}}$. Set $x_\emptyset := x$. Now, for every $t \in {}^{<\omega}2$ such that x_t has already been defined, since $\mathbf{T}$ is 2-splitting and normal, we may find $y \neq z$ in $T_{\alpha_{|t|+1}}$ with $x <_T y, z$; then, let $x_{t^\frown\langle 0\rangle} := y$ and $x_{t^\frown\langle 1\rangle} := z$. Next, given $t \in {}^\omega 2$, let $B_t := \{y \in T \restriction \alpha \mid \exists n < \omega(y <_T x_{t\restriction n})\}$. As B_t is an α-branch containing x, it must have a bound $b_t \in T$. Clearly, $\mathrm{ht}(b_t) \geq \alpha$, and we may moreover assume that $\mathrm{ht}(b_t) = \alpha$. Note that the construction secures that, for all $t \neq t'$ in ${}^\omega 2$, $b_t \neq b_{t'}$.

Let $\rho : T \restriction \mathrm{acc}(\kappa) \to T$ be a witness to the fact that $\mathbf{T}$ is regressive. Next, for every $t \in {}^\omega 2$, fix a large enough $n_t < \omega$ such that $\rho(b_t) <_T x_{t\restriction n_t}$. By the pigeonhole principle, we may now fix $s \in {}^{<\omega}2$ such that $\{t \in {}^\omega 2 \mid t \restriction n_t = s\}$ is uncountable. Pick $t \neq t'$ in ${}^\omega 2$ such that $t \restriction n_t = s = t' \restriction n_{t'}$. Then, $\rho(b_t) <_T x_s <_T b_{t'}$ and $\rho(b_{t'}) <_T x_s <_T b_t$, contradicting the fact that $b_t \neq b_{t'}$. □

Remark 2.20. It follows that if $\mathbf{T}$ is a normal 2-splitting coherent κ-Souslin tree, then $V(\mathbf{T}) = E^\kappa_\omega$. A consistent construction of such a tree may be found in [BR17a, Proposition 2.5].

[2] Note that, by [She10], for any *uncountable* cardinal λ, $\diamondsuit(\lambda^+)$ holds iff $2^\lambda = \lambda^+$.

Lemma 2.21. *Suppose that $\mathbf{T} = (T, <_T)$ is a normal maximally-complete λ-splitting λ^+-Souslin tree. Then $V(\mathbf{T}) \supseteq E^{\lambda^+}_{\chi}$ for the (regular) cardinal $\chi := \log_\lambda(\lambda^+)$.*

Proof. Towards a contradiction, suppose that $\alpha \in E^{\lambda^+}_{\chi} \setminus V(\mathbf{T})$. Fix $x \in T \restriction \alpha$ such that every α-branch B with $x \in B$ has an upper bound in T. Fix a strictly increasing and continuous sequence of ordinals $\langle \alpha_\epsilon \mid \epsilon < \chi\rangle$ that converges to α, and $\alpha_0 := \mathrm{ht}(x)$. Very much like the proof of Lemma 2.19, we may recursively construct an array $\langle x_t \mid t \in {}^{<\chi}\lambda\rangle$ in such a way that:

- $x_\emptyset = x$;
- for all $t \in {}^{<\chi}\lambda$, $x_t \in T_{\alpha_{\mathrm{dom}(t)}}$;
- for all $t, s \in {}^{<\chi}\lambda$, if $t \subseteq s$, then $x_t <_T x_s$;
- for all $t \in {}^{<\chi}\lambda$ and $i < j < \lambda$, $x_{t^\frown\langle i\rangle} \neq x_{t^\frown\langle j\rangle}$.

For each $t \in {}^{\chi}\lambda$, find $b_t \in T_\alpha$ such that, for every $\epsilon < \chi$, $x_{t\restriction\epsilon} <_T b_t$. Then, $\{b_t \mid t \in {}^{\chi}\lambda\}$ is an antichain of size $\geq \lambda^+$ in $\mathbf{T}$. This is a contradiction. □

Remark 2.22. It follows that if $\mathbf{T}$ is a normal λ-complete λ-splitting λ^+-Souslin tree, then $V(\mathbf{T}) = E^{\lambda^+}_{\lambda}$.

2.2. **Deriving our guessing principle from a Souslin tree.**

Theorem 2.23. *Suppose that $\mathbf{T} = (T, <_T)$ is an ordinal-based χ-splitting κ-Souslin tree, and $\theta \leq \chi$ is a cardinal satisfying $\kappa^{<\theta} = \kappa$. Then, for every collection $\mathcal{S}$ of pairwise disjoint stationary subsets of $V(\mathbf{T}) \cap E^{\kappa}_{\chi}$, there exists an* AD*-multi-ladder system $\langle \mathcal{A}_\alpha \mid \alpha \in E^\kappa_\chi\rangle$ satisfying the following. For every $\mathcal{B} \subseteq [\kappa]^\kappa$ with $|\mathcal{B}| < \theta$, every $S \in \mathcal{S}$, and every cardinal $\mu < \kappa$, the following set is stationary:*

$$G_{\geq\mu}(S, \mathcal{B}) := \{\alpha \in S \mid |\mathcal{A}_\alpha| \geq \mu \ \&\ \forall (A, B) \in \mathcal{A}_\alpha \times \mathcal{B}\ [\sup(A \cap B) = \alpha]\}.$$

Proof. As $\mathbf{T}$ is χ-splitting and prolific, for each $w \in T$, we may fix an injective sequence $\langle w_i \mid i < \max\{\chi, \mathrm{ht}(w)\}\rangle$ consisting of immediate successors of w.

As $\kappa^{<\theta} = \kappa$, we may fix an injective enumeration $\langle W_\eta \mid \eta < \kappa\rangle$ of all subsets W of T such that:

- $0 < |W| < \theta$, and
- $\mathrm{ht} \restriction W$ is a constant function whose sole value is in $[\chi, \kappa)$.

Claim 2.23.1. *For every $\alpha \in E^\kappa_\chi$, there exists a cofinal subset A_α of α which is an antichain in $\mathbf{T}$.*

Proof. Fix an arbitrary $\alpha \in E^\kappa_\chi$, and let X be an arbitrary cofinal subset of α of order-type χ.

If there exists some $\epsilon < \kappa$ such that $|X \cap T_\epsilon| = \chi$, then we are done by letting $A_\alpha := X \cap T_\epsilon$. Thus, hereafter assume this is not the case, and pick a sequence $\langle x^j \mid j < \chi\rangle$ of elements of X for which $\langle \mathrm{ht}(x^j) \mid j < \chi\rangle$ is strictly increasing. For notational simplicity, let us assume that $\{x^j \mid j < \chi\} = X$. Now, there are two cases to consider:

► Suppose that there exists a node $w \in T$ such that, for every $i < \chi$, there exists $j_i < \chi$ such that x^{j_i} extends w_i. Recalling that $\langle w_i \mid i < \chi\rangle$ is an injective sequence of immediate successors of w, we infer that $A_\alpha := \{x^{j_i} \mid i < \chi\}$ is an antichain as sought.

▶ Suppose not. In particular, for each $j<\chi$ (using $w:=x^j$), we may fix $i_j<\chi$ such that $(x^j)_{i_j}$ is not extended by any element of X. We claim that $A_\alpha:=\{(x^j)_{i_j}\mid j<\chi\}$ is as sought.

For every $j<\chi$, we have $\mathrm{ht}(x^j)<\mathrm{ht}((x^j)_{i_j})<\mathrm{ht}(x^{j+2})$, so that $x^j\in(x^j)_{i_j}\in x^{j+2}$, and hence A_α is yet another cofinal subset of α. To see that A_α is an antichain, suppose that there exists a pair $j<j'$ such that $(x^j)_{i_j}$ is comparable with $(x^{j'})_{i_{j'}}$. As $\mathrm{ht}((x^j)_{i_j})=\mathrm{ht}(x^j)+1<\mathrm{ht}(x^{j'})+1=\mathrm{ht}((x^{j'})_{i_{j'}})$, it follows that $(x^j)_{i_j}$ is extended by $(x^{j'})_{i_{j'}}$, and in particular, $(x^j)_{i_j}$ is extended by $x^{j'}$ which is an element of X, contradicting the choice of i_j. □

Next, suppose that we are given a collection $\mathcal{S}$ of pairwise disjoint stationary subsets of $V(\mathbf{T})\cap E^\kappa_\chi$. As $T=\kappa$ and $|T\restriction\alpha|<\kappa$ for all $\alpha<\kappa$, $C:=\{\alpha<\kappa\mid\alpha=T\restriction\alpha\}$ is a club in κ. Let $\langle S_\eta\mid\eta<\kappa\rangle$ be a sequence of pairwise disjoint subsets of $E^\kappa_\chi\cap\mathrm{acc}(C)$ satisfying:

- For every $S\in\mathcal{S}$, $S_\eta\cap S$ is stationary;
- For every $\eta<\kappa$, $\min(S_\eta)$ is greater than the unique element of $\{\mathrm{ht}(w)\mid w\in W_\eta\}$, which we hereafter denote by ϵ_η.

Let $R:=E^\kappa_\chi\setminus\biguplus_{\eta<\kappa}S_\eta$.

For every $\alpha\in R$, we appeal to Claim 2.23.1 and pick a cofinal subset $A_\alpha\subseteq\alpha$ which is an antichain in $\mathbf{T}$. By possibly thinning out, we may also assume that $\mathrm{otp}(A_\alpha)=\mathrm{cf}(\alpha)$. Then, we set $\mathcal{A}_\alpha:=\{A_\alpha\}$.

Next, let $\alpha\in\biguplus_{\eta<\kappa}S_\eta$ be arbitrary. Let $\eta<\kappa$ be the unique ordinal such that $\alpha\in S_\eta$. As $\alpha\in V(\mathbf{T})$, for every $w\in W_\eta$ and every $i<\epsilon_\eta$, we may find a subset $A^w_{\alpha,i}$ of T such that:

(1) $A^w_{\alpha,i}$ is a chain with minimal element w_i;
(2) $\{\mathrm{ht}(x)\mid x\in A^w_{\alpha,i}\}=\alpha\setminus\epsilon_\eta+1$;
(3) there exists no $z\in T_\alpha$ such that, for all $x\in A^w_{\alpha,i}$, $x<_T z$.

Finally, let $\mathcal{A}_\alpha:=\{A_{\alpha,i}\mid i<\epsilon_\eta\}$, where $A_{\alpha,i}:=\bigcup_{w\in W_\eta}A^w_{\alpha,i}$.

Claim 2.23.2. $\langle A_{\alpha,i}\mid i<\epsilon_\eta\rangle$ *is a sequence of pairwise disjoint cofinal subsets of* α.

Proof. Let $w,u\in W_\eta$ and $i,j<\epsilon_\eta$ and suppose that $x\in A^w_{\alpha,i}\cap A^u_{\alpha,j}$. By Clause (1) above, x extends both w_i and u_j. But w_i and u_j are predecessors of x at the same level $T_{\epsilon_\eta+1}$, so that $w_i=u_j$ and it easily follows that $(w,i)=(u,j)$.

For all $w\in W_\eta$ and $i<\epsilon_\eta$, it follows from $\alpha\in C$ and Clause (2) above that $A^w_{\alpha,i}\subseteq\alpha$, so that $A_{\alpha,i}\subseteq\alpha$. In addition, as $\alpha\in\mathrm{acc}(C)$, if we pick any $w\in W_\eta$, then it follows from Clause (2) above that $\sup\{\beta\in C\mid A^w_{\alpha,i}\cap T_\beta\neq\emptyset\}=\alpha$, and hence $A_{\alpha,i}$ is cofinal in α. □

Claim 2.23.3. *Let* $A,A'\in\bigcup_{\alpha\in E^\kappa_\chi}\mathcal{A}_\alpha$ *with* $A\neq A'$. *Then* $\sup(A\cap A')<\sup(A)$.

Proof. Let α,α' be such that $A\in\mathcal{A}_\alpha$ and $A'\in\mathcal{A}_{\alpha'}$. As the elements of $\mathcal{A}_\alpha$ are pairwise disjoint, we may assume that $\alpha\neq\alpha'$. If $\alpha'<\alpha$, then $\sup(A\cap A')\le\alpha'<\alpha$, so assume that $\alpha'>\alpha$.

▶ If $\alpha'\in R$, then $\mathrm{otp}(A\cap A')\le\mathrm{otp}(\alpha\cap A')<\mathrm{otp}(A')=\mathrm{cf}(\alpha)$, so that $\sup(A\cap A')<\alpha$.

▶ If $\alpha'\notin R$ and $\alpha\in R$, then A is antichain, while A' is the union of $<\theta$ many chains, so that $|A\cap A'|<\theta\le\chi=\mathrm{cf}(\alpha)$, and again $\sup(A\cap A')<\alpha$.

► If $\alpha, \alpha' \notin R$, then let η, ζ, i, j be such that $A = \bigcup_{w \in W_\eta} A^w_{\alpha,i}$ and $A' = \bigcup_{u \in W_\zeta} A^u_{\alpha,j}$. Since $\max\{|W_\eta|, |W_\zeta|\} < \theta \leq \chi = \mathrm{cf}(\alpha)$, it suffices to show that for each $w \in W_\eta$ and $u \in W_\zeta$, $\sup(A^w_{\alpha,i} \cap A^u_{\beta,j}) < \alpha$. But the latter follows from Clause (3) above together with the fact that $\alpha \in C$. □

Thus, we are left with verifying the following.

Claim 2.23.4. *Suppose $\mathcal{B} = \{B_\tau \mid \tau < \theta'\}$ is a family of cofinal subsets of κ, with $\theta' < \theta$. Suppose $S \in \mathcal{S}$ and $\mu < \kappa$ is some cardinal. Then the set $G_{\geq\mu}(S, \mathcal{B})$ is stationary.*

Proof. Recalling Lemma 2.13, for each $\tau < \theta'$, we may fix $w^\tau \in T$ such that $(w^\tau)^\uparrow \cap B_\tau$ is cofinal in $(w^\tau)^\uparrow$. Since $\mathbf{T}$ is normal, we may extend the said elements to ensure that $\mathrm{ht} \restriction \{w^\tau \mid \tau < \theta'\}$ is a constant function whose sole value is some $\epsilon < \kappa$ with $\epsilon \geq \max\{\chi, \mu\}$. It follows that there exists (a unique) $\eta < \kappa$ such that $W_\eta = \{w^\tau \mid \tau < \theta'\}$. Next, since $\mathbf{T}$ has no antichains of size κ, we may fix a sparse enough club $C' \subseteq C$ with $\min(C') > \epsilon_\eta$ such that, for every pair of ordinals $\gamma < \beta$ from C' and every $w^\tau \in W_\eta$, the set $B_\tau \cap ((w^\tau)^\uparrow) \setminus (T \restriction \gamma)$ contains a maximal antichain in itself which is a subset of $T \restriction \beta$.

Consider the stationary set $\Gamma := S_\eta \cap \mathrm{acc}(C')$. Now, let $\alpha \in \Gamma$ be arbitrary. By Claim 2.23.2, $|\mathcal{A}_\alpha| = |\epsilon_\eta| \geq \mu$. Next, let $\tau < \theta'$ and $A \in \mathcal{A}_\alpha$ be arbitrary. Find $i < \epsilon_\eta$ such that $A = A_{\alpha,i} = \bigcup_{w \in W_\eta} A^w_{\alpha,i}$. In particular, $A \supseteq A^{w_\tau}_{\alpha,i}$. As $\sup(C' \cap \alpha) = \alpha$, it thus suffices to show that for every $\gamma \in C' \cap \alpha$, $\sup(A^{w^\tau}_{\alpha,i} \cap B_\tau) \geq \gamma$. For this, let $\gamma \in C' \cap \alpha$ be arbitrary. Let $\beta := \min(C' \setminus (\gamma + 1))$. Let x denote the unique element of $A^{w^\tau}_{\alpha,i} \cap T_\beta$. As $w^\tau <_T (w^\tau)_i \leq_T x$, we may find $a \in B_\tau$ with $x \leq_T a$. As $\gamma < \beta$ is a pair of elements of C', it follows that there exists $a' \in B_\tau$ with $a' \leq_T a$ such that $\gamma \leq \mathrm{ht}(a') < \beta$. As $x, a' \leq_T a$ and $\mathrm{ht}(a') > \epsilon_\eta = \mathrm{ht}(w^\tau)$, it follows that $a' \in A^{w^\tau}_{\alpha,i} \setminus (T \restriction \gamma)$, as sought. □

This completes the proof of Theorem 2.23. □

Corollary 2.24. *Suppose that $\mathbf{T}$ is a κ-Souslin tree. For every $\chi \in \mathrm{Reg}(\kappa)$ and every collection $\mathcal{S}$ of pairwise disjoint stationary subsets of E^κ_χ, any of the following implies that $\clubsuit_{\mathrm{AD}}(\mathcal{S}, {<}\chi)$ holds:*

(1) $\kappa = \lambda^+$, $\lambda^{\aleph_0} > \lambda$ and $\chi = \aleph_0$;
(2) $\kappa = \lambda^+$, $\chi = \log_\lambda(\lambda^+)$ and $\mathbf{T}$ is maximally-complete;
(3) $\chi = \aleph_0$ and $\mathbf{T}$ is regressive.

Proof. (1) This follows from Clause (2).

(2) Appeal to Proposition 2.16, Lemma 2.21 and Theorem 2.23 using $(\kappa, \mu, \theta) := (\lambda^+, \chi, \chi)$ to get a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in E^{\lambda^+}_\chi \rangle$. The only thing that possibly does not fit is that there may be $\alpha \in \bigcup \mathcal{S}$ for which $|\mathcal{A}_\alpha| \neq \chi$. But this is easy to fix:

- ► For any $\alpha \in \bigcup \mathcal{S}$ such that $|\mathcal{A}_\alpha| > \chi$, replace $\mathcal{A}_\alpha$ by some subset of it of size χ.
- ► For any $\alpha \in \bigcup \mathcal{S}$ such that $|\mathcal{A}_\alpha| < \chi$, pick $A \in \mathcal{A}_\alpha$ and replace $\mathcal{A}_\alpha$ by some partition of A into χ many sets.

(3) Appeal to Proposition 2.16, Lemma 2.19 and Theorem 2.23 using $(\mu, \theta) := (\omega, \omega)$ to get a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in E^\kappa_\omega \rangle$. The only thing that possibly does not fit is that there may be $\alpha \in \bigcup \mathcal{S}$ for which $|\mathcal{A}_\alpha| \neq \omega$. But we may fix it as in the previous case. □

Theorem A now follows immediately.

Corollary 2.25. *(1) If there exists a* $\mathrm{cf}(\lambda)$*-complete* λ^+*-Souslin tree, then for every partition* $\mathcal{S}$ *of* $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *into stationary sets,* $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\mathrm{cf}(\lambda))$ *holds.*

(2) If there exists a regressive κ*-Souslin tree, then for every partition* $\mathcal{S}$ *of* E^κ_ω *into stationary sets,* $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\omega)$ *holds.* □

Corollary 2.26. *If there exists an* ω_1*-Souslin tree, then* $\clubsuit_{\mathrm{AD}}(\{\omega_1\},{<}\omega)$ *holds.* □

Corollary 2.27. *It is consistent with* CH *that* $\clubsuit_{\mathrm{AD}}(\{\omega_1\},{<}\omega)$ *holds, but* $\clubsuit(\omega_1)$ *fails.*

Proof. Start with a model of $\mathrm{GCH}+\neg\diamondsuit(\omega_1)$ (e.g., Jensen's model [DJ74] of GCH with no $\aleph_1$-Souslin trees). Now, force to add a single Cohen real and work in the corresponding extension. As this is a countable forcing, CH still holds and $\diamondsuit(\omega_1)$ still fails, so that by Devlin's theorem (see Remark 2.2), $\clubsuit(\omega_1)$ fails as well. Finally, by a theorem of Shelah [She84], the forcing to add a Cohen real introduces an ω_1-Souslin tree, so that, by the preceding corollary, $\clubsuit_{\mathrm{AD}}(\{\omega_1\},{<}\omega)$ holds. □

Corollary 2.28. *The assertion that* $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\omega)$ *holds for every partition* $\mathcal{S}$ *of* ω_1 *into stationary sets is consistent with any of the two:*

(1) $\clubsuit_J(\omega_1)$ *fails;*

(2) $\clubsuit_J(S)$ *fails for some stationary* $S\subseteq\omega_1$*, and* CH *holds.*

Proof. For a stationary subset S of ω_1, let $\mathrm{Unif}(S,2)$ assert that for every sequence of functions $\vec f=\langle f_\alpha : A_\alpha\rightarrow 2\mid \alpha\in\mathrm{acc}(\omega_1)\cap S\rangle$ where each A_α is a cofinal subset of α of order-type ω, there exists a function $f:\omega_1\rightarrow 2$ that uniformizes $\vec f$, i.e., for every $\alpha\in S\cap\mathrm{acc}(\omega_1)$, $\Delta(f,f_\alpha):=\{\beta\in A_\alpha\mid f(\beta)\neq f_\alpha(\beta)\}$ is finite.

Claim 2.28.1. $\clubsuit_J(S)$ *refutes* $\mathrm{Unif}(S,2)$.

Proof. Suppose that $\langle A_{\alpha,i}\mid \alpha\in S,\ i<\omega\rangle$ is as in Definition 2.1(4). In particular, for every $\alpha\in S\cap\mathrm{acc}(\omega_1)$, $A_\alpha:=A_{\alpha,0}\uplus A_{\alpha,1}$ is a cofinal subset of α of order-type ω, and we may define a function $f_\alpha:A_\alpha\rightarrow 2$ via $f_\alpha(\beta):=0$ iff $\beta\in A_{\alpha,0}$. Towards a contradiction, suppose that there exists a function $f:\omega_1\rightarrow 2$ that uniformizes $\vec f:=\langle f_\alpha\mid \alpha\in S\cap\mathrm{acc}(\omega_1)\rangle$. By the pigeonhole principle, pick $j<2$ for which $B:=\{\beta<\omega_1\mid f(\beta)=j\}$ is uncountable. Now, fix $\alpha\in S\cap\mathrm{acc}(\omega_1)$ such that $\sup(B\cap A_{\alpha,i})=\alpha$ for all $i<\omega$. Let $i:=1-j$. Pick $\beta\in B\cap A_{\alpha,i}\setminus\Delta(f,f_\alpha)$. Then $j=f(\beta)=f_\alpha(\beta)=i$. This is a contradiction. □

(1) As made clear by the proof of [DS78, Theorem 5.2], it is possible to force $\mathrm{Unif}(\omega_1,2)$ via a finite support iteration of Knaster posets. In particular, if we start with a ground model with an $\aleph_1$-Souslin tree, then we can force $\mathrm{Unif}(\omega_1,2)$ without killing the tree. Now appeal to Corollary 2.24(1).

(2) In [She77, Theorems 2.1 and 2.4], $\diamondsuit(\omega_1)$ is shown to be consistent with the existence of a stationary subset $S\subseteq\omega_1$ on which $\mathrm{Unif}(S,2)$ holds. Now, appeal to Fact 2.17(1) and Corollary 2.24(1). □

We conclude this subsection by stating an additional result, this time concerning the three-cardinal variant of $\clubsuit_{\mathrm{AD}}$ (recall Definition 2.11):

Theorem 2.29. *Suppose that there exists a* κ*-Souslin tree. Then* $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,1)$ *holds for some collection* $\mathcal{S}$ *of* κ*-many pairwise disjoint stationary subsets of* κ*. If* κ *is a successor cardinal, then moreover* $\bigcup\mathcal{S}=E^\kappa_\chi$ *for some cardinal* $\chi\in\mathrm{Reg}(\kappa)$*.* □

We omit the proof due to the lack of applications of $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,1)$, at present.

2.3. **The consistency of the negation of our guessing principle.** By Lemma 2.10, for a stationary set S consisting of points of some fixed cofinality, $\clubsuit(S)$ entails $\clubsuit_{\mathrm{AD}}(\{S\},{<}\omega)$. The next theorem shows that the restriction to a fixed cofinality is crucial.

Theorem 2.30. *If κ is weakly compact, then for any S with $\mathrm{Reg}(\kappa)\subseteq S\subseteq\kappa$, $\clubsuit_{\mathrm{AD}}(\{S\},1,1)$ fails.*

Proof. Suppose that $\vec{A}=\langle A_\alpha\mid\alpha\in S\rangle$ is a $\clubsuit_{\mathrm{AD}}(\{S\},1,1)$-sequence, with $\mathrm{Reg}(\kappa)\subseteq S\subseteq\kappa$. We define a sequence $\langle C_\alpha\mid\alpha<\kappa\rangle$, as follows:

▶ Let $C_0:=\emptyset$.

▶ For every $\alpha<\kappa$, let $C_{\alpha+1}:=\{\alpha\}$.

▶ For every $\alpha\in\mathrm{acc}(\kappa)\setminus\mathrm{Reg}(\kappa)$, fix a closed and cofinal $C_\alpha\subseteq\alpha$ with $\mathrm{otp}(C_\alpha)=\mathrm{cf}(\alpha)<\min(C_\alpha)$.

▶ For every $\alpha\in\mathrm{Reg}(\kappa)$, let $B_\alpha:=\mathrm{nacc}(A_\alpha)$ and finally let $C_\alpha:=B_\alpha\uplus\mathrm{acc}^+(B_\alpha)$. Note that B_α is a cofinal subset of A_α, and that C_α is a closed and cofinal subset of α.

Towards a contradiction, suppose that κ is weakly compact. So, by [Tod87, Theorem 1.8], we may fix a club $C\subseteq\kappa$ such that, for every $\delta<\kappa$, for some $\alpha(\delta)<\kappa$, $C\cap\delta=C_{\alpha(\delta)}\cap\delta$. Consider the club $D:=\{\delta\in\mathrm{acc}(\kappa)\mid\mathrm{otp}(C\cap\delta)=\delta\}$.

Claim 2.30.1. *For every $\delta\in D$, $\alpha(\delta)\in\mathrm{Reg}(\kappa)$.*

Proof. Let $\delta\in D$. Since $C_{\alpha(\delta)}\cap\delta=C\cap\delta$ is infinite, $\alpha(\delta)\in\mathrm{acc}(\kappa)$. Now, if $\alpha(\delta)\in\mathrm{acc}(\kappa)\setminus\mathrm{Reg}(\kappa)$, then $\delta=\mathrm{otp}(C_{\alpha(\delta)}\cap\delta)\le\mathrm{otp}(C_{\alpha(\delta)})<\min(C_{\alpha(\delta)})$, which is an absurd. □

Evidently, $B:=\mathrm{nacc}(C)$ is a cofinal subset of κ. So, by the choice of $\vec{A}$, $G:=\{\delta\in S\cap D\mid\sup(B\cap A_\delta)=\delta\}$ is stationary. Pick a pair of ordinals $\delta_0<\delta_1$ from G. For each $i<2$, since $\alpha(\delta_i)\in\mathrm{Reg}(\kappa)$,

$$B\cap\delta_i=\mathrm{nacc}(C)\cap\delta_i=\mathrm{nacc}(C_{\alpha(\delta_i)})\cap\delta_i=B_{\alpha(\delta_i)}\cap\delta_i\subseteq A_{\alpha(\delta_i)}.$$

As $\sup(B\cap A_{\delta_i})=\delta_i$ and $B\cap\delta_i\subseteq A_{\alpha(\delta_i)}$, $\sup(A_{\alpha(\delta_i)}\cap A_{\delta_i})=\delta_i$, so, since $\vec{A}$ is an AD-ladder system, it must be the case that $\alpha(\delta_i)=\delta_i$. Altogether, $B\cap\delta_0\subseteq A_{\delta_0}\cap A_{\delta_1}$, so that $\delta_0>\sup(A_{\delta_0}\cap A_{\delta_1})\ge\sup(B\cap\delta_0)=\delta_0$. This is a contradiction. □

An ideal $\mathcal{I}$ consisting of countable sets is said to be a *P-ideal* iff every countable family of sets in the ideal admits a pseudo-union in the ideal. That is, for every sequence $\langle X_n\mid n<\omega\rangle$ of elements of $\mathcal{I}$, there exists $X\in\mathcal{I}$ such that $X_n\setminus X$ is finite for all $n<\omega$.

Definition 2.31 (Todorčević, [Tod00])**.** The P-ideal dichotomy (PID) asserts that for every P-ideal $\mathcal{I}$ consisting of countable subsets of some set Z, either:

1. there is an uncountable $B\subseteq Z$ such that $[B]^{\aleph_0}\subseteq\mathcal{I}$, or
2. there is a sequence $\langle B_n\mid n<\omega\rangle$ such that $\bigcup_{n<\omega}B_n=Z$ and, for each $n<\omega$, $[B_n]^{\aleph_0}\cap\mathcal{I}=\emptyset$.

We denote by $\mathsf{PID}_{\aleph_1}$ the restriction of the above principle to $Z:=\aleph_1$. This special case was first introduced and studied by Abraham and Todorčević in [AT97], and was denoted there by $(*)$.

Lemma 2.32. *Suppose that $\mathsf{PID}_{\aleph_1}$ holds and $\mathfrak{b} > \omega_1$. Then, for any stationary $S \subseteq \omega_1$, $\clubsuit_{\mathrm{AD}}(\{S\}, 1, 1)$ fails.*

Proof. Towards a contradiction, suppose that $S \subseteq \omega_1$ is stationary, and that $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ is a $\clubsuit_{\mathrm{AD}}(\{S\}, 1, 1)$-sequence. Let

$$\mathcal{I} := \{X \in [\omega_1]^{\le\aleph_0} \mid \forall\alpha \in \mathrm{acc}(\omega_1) \cap S[A_\alpha \cap X \text{ is finite}]\}.$$

It is clear that $\mathcal{I}$ is an ideal.

Claim 2.32.1. *$\mathcal{I}$ is a P-ideal.*

Proof. Let $\vec{X} = \langle X_n \mid n < \omega\rangle$ be a sequence of elements of $\mathcal{I}$. We need to find a pseudo-union of $\vec{X}$ that lies in $\mathcal{I}$. As $\mathcal{I}$ is downward closed, we may assume that $\langle X_n \mid n < \omega\rangle$ consists of pairwise disjoint sets.

Fix a bijection $e : \omega \leftrightarrow \biguplus_{n<\omega} X_n$. Then, for all $\alpha \in S$, define a function $f_\alpha : \omega \to \omega$ via

$$f_\alpha(n) := \min\{m < \omega \mid X_n \cap A_\alpha \subseteq e``m\}.$$

As $\mathfrak{b} > \omega_1$, let us fix a function $f : \omega \to \omega$ such that $f_\alpha <^* f$ for all $\alpha \in S$. Set $X := \biguplus\{X_n \setminus e[f(n)] \mid n < \omega\}$. Clearly, for every $n < \omega$, $X_n \setminus X$ is a subset of $e[f(n)]$, and, in particular, it is finite.

Towards a contradiction, suppose that $X \notin \mathcal{I}$. Fix $\alpha \in \mathrm{acc}(\omega_1) \cap S$ such that $A_\alpha \cap X$ is infinite. Since $X \subseteq \biguplus_{n<\omega} X_n$, but $A_\alpha \cap X_n$ is finite for all $n < \omega$, we may find a large enough $n < \omega$ such that $A_\alpha \cap X \cap X_n \neq \emptyset$ and $f_\alpha(n) < f(n)$. Pick $\beta \in A_\alpha \cap X \cap X_n$. By the definition of f_α, $\beta \in e[f_\alpha(n)]$. But $f_\alpha(n) < f(n)$, so that $\beta \in e[f(n)]$, contradicting the fact that $\beta \in X$. □

Claim 2.32.2. *Let $B \subseteq \omega_1$ be uncountable.*

1. *There exists $X \in [B]^{\aleph_0}$ with $X \notin \mathcal{I}$;*
2. *There exists $X \in [B]^{\aleph_0}$ with $X \in \mathcal{I}$.*

Proof. As B is uncountable, $G := \{\alpha \in \mathrm{acc}(\omega_1)\cap S \mid \sup(A_\alpha\cap B) = \alpha\}$ is stationary.

(1) Fix arbitrary $\alpha \in G$. Then $X := A_\alpha \cap B$ is an element of $[B]^{\aleph_0} \setminus \mathcal{I}$.

(2) Let $\langle \alpha_n \mid n < \omega\rangle$ be some increasing sequence of elements of G. For every $n < \omega$, let $\langle \alpha_n^m \mid m < \omega\rangle$ be the increasing enumeration of some cofinal subset of $A_{\alpha_n} \cap B$. Furthermore, we require that, for all $n < \omega$, $\alpha_n < \alpha_{n+1}^0$.

Set $\beta := \sup_{n<\omega} \alpha_n$. As $\vec{A}$ is an AD-ladder system, for every $\alpha \in S \cap \mathrm{acc}(\omega_1) \setminus \beta$, we may define a function $f_\alpha : \omega \to \omega$ via:

$$f_\alpha(n) := \min\{m < \omega \mid A_{\alpha_n} \cap A_\alpha \subseteq \alpha_n^m\}.$$

As $\mathfrak{b} > \omega_1$, let us fix a function $f : \omega \to \omega$ such that $f_\alpha <^* f$ for all $\alpha \in S$. Set $X := \{\alpha_n^{f(n)} \mid 0 < n < \omega\}$. For every $n < \omega$, the interval (α_n, α_{n+1}) contains a single element of X, so that X is a cofinal subset of β with $\mathrm{otp}(X) = \omega$. In particular, $X \in [B]^{\aleph_0}$.

Towards a contradiction, suppose that $X \notin \mathcal{I}$. Fix $\alpha \in \mathrm{acc}(\omega_1) \cap S$ such that $A_\alpha \cap X$ is infinite. Clearly, $\alpha \ge \beta$. So, we may find $k < \omega$ such that, for every integer $n \ge k$, $f_\alpha(n) < f(n)$. As $A_\alpha \cap X$ is infinite, let us now pick a positive integer $n \ge k$ such that $\alpha_n^{f(n)} \in A_\alpha$. Recalling that $\{\alpha_n^m \mid m < \omega\} \subseteq A_{\alpha_n}$, we altogether infer that $\alpha_n^{f(n)} \in A_{\alpha_n} \cap A_\alpha \subseteq \alpha_n^{f_\alpha(n)}$. In particular, $\alpha_n^{f(n)} < \alpha_n^{f_\alpha(n)}$, contradicting the fact that $f_\alpha(n) < f(n)$. □

Altogether, $\mathcal{I}$ is a P-ideal for which the two alternatives of $\mathsf{PID}_{\aleph_1}$ fail. This is a contradiction. $\square$

Corollary 2.33. *(1)* PFA *implies that* $\clubsuit_{\mathrm{AD}}(\{\omega_1\},1,1)$ *fails;*
(2) $\clubsuit_{\mathrm{AD}}(\{\omega_1\},1,1)$ *does not follow from the existence of an almost Souslin tree.*

Proof. (1) It is well-known that PFA implies $\mathsf{PID}+\mathsf{MA}_{\aleph_1}$. In particular, it implies $\mathsf{PID}_{\aleph_1}$ together with $\mathfrak{b}>\omega_1$.

(2) Almost Souslin trees were defined in [DS79, §3]. In [KT20] and [Kru20] one can find models of PID with $\mathfrak{p}>\omega_1$ (in particular, $\mathfrak{b}>\omega_1$), in which there exists an Aronszajn tree which is almost Souslin. $\square$

Question 2.34. *Does* $\mathsf{MA}_{\aleph_1}$ *imply that* $\clubsuit_{\mathrm{AD}}(\omega_1)$ *fails?*

Question 2.35. *Is* CH *consistent with the failure of* $\clubsuit_{\mathrm{AD}}(\omega_1)$*?*

Note that a combination of the main results of [Juh88, ER99] implies that CH is consistent with the failure of $\clubsuit_J(\omega_1)$.

Update (January 2022). In an upcoming paper, we answer Question 2.35 in the negative.

3. A Ladder-system Dowker space

In [Goo95b], Good constructed a Dowker space of size κ^+ using $\clubsuit(S,2)$, where S is a non-reflecting stationary subset of $E^{\kappa^+}_\omega$. We won't define the principle $\clubsuit(S,2)$, but only mention that, by Fact 2.9(3), it is no stronger than $\clubsuit(S)$. In this section, a ladder-system Dowker space of size κ is constructed under the assumption of $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$, where $\mathcal{S}$ is an infinite partition of some non-reflecting stationary subset of κ. By Lemma 2.10, $\clubsuit(S)$ implies $\clubsuit_{\mathrm{AD}}(\mathcal{S},{<}\omega)$, which surely implies $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$, so, our construction in particular gives a ladder-system Dowker space in Good's scenario. It also gives ladder-system Dowker spaces in scenarios considered by Rudin [Rud74b] and Weiss [Wei81], as explained at the end of this section.

The constructions in this and in the next section are motivated by the following lemma.

Lemma 3.1. *Suppose that* $\mathbb{X}=(X,\tau)$ *is a normal Hausdorff topological space of size* κ*, having no two disjoint closed sets of size* κ*. If there exists a* $\subseteq$*-decreasing sequence* $\langle D_n\mid n<\omega\rangle$ *of closed sets of cardinality* κ *such that* $\bigcap_{n<\omega}D_n=\emptyset$*, then* $\mathbb{X}$ *is Dowker.*

Proof. Recall that, by [Dow51], a space is Dowker iff it is Hausdorff, normal and there is a $\subseteq$-decreasing sequence $\langle D_n\mid n<\omega\rangle$ of closed sets with $\bigcap_{n<\omega}D_n=\emptyset$, such that, for every sequence $\langle U_n\mid n<\omega\rangle$ of open sets, if $D_n\subseteq U_n$ for every $n<\omega$, then $\bigcap_{n<\omega}U_n\neq\emptyset$.

Now suppose that there exists a $\subseteq$-decreasing sequence $\langle D_n\mid n<\omega\rangle$ of closed sets of cardinality κ such that $\bigcap_{n<\omega}D_n=\emptyset$. Suppose that $\langle U_n\mid n<\omega\rangle$ is a sequence of open sets such that $D_n\subseteq U_n$ for every $n<\omega$.

For every $n<\omega$, $F_n:=X\setminus U_n$ is a closed set disjoint from D_n, and hence of cardinality $<\kappa$. So, as $\omega<\operatorname{cf}(\kappa)=\kappa$, $\bigcup_{n<\omega}F_n$ has size less than κ. In particular, $X\setminus\bigcup_{n<\omega}F_n\neq\emptyset$. Altogether, $\bigcap_{n<\omega}U_n=\bigcap_{n<\omega}X\setminus F_n=X\setminus\bigcup_{n<\omega}F_n\neq\emptyset$. Recalling that $\mathbb{X}$ is normal, we altogether infer that $\mathbb{X}$ is Dowker. $\square$

The space. Suppose that S is a stationary subset of $\mathrm{acc}(\kappa)$, $\mathcal{S}$ is a partition of S with $|\mathcal{S}| = \aleph_0$, and $\clubsuit_{\mathrm{AD}}(\mathcal{S}, 1, 2)$ holds. Fix an AD-ladder system $\langle A_\alpha \mid \alpha \in S\rangle$ as in Lemma 2.12. Fix an injective enumeration $\langle S_{n+1} \mid n < \omega\rangle$ of $\mathcal{S}$, and let $S_0 := \kappa \setminus S$. As $\langle S_n \mid n < \omega\rangle$ is a partition of κ, for each $\alpha < \kappa$, we may let $n(\alpha)$ denote the unique $n < \omega$ such that $\alpha \in S_n$. For each $n < \omega$, let $W_n := \bigcup_{i \leq n} S_i$. Finally, define a sequence $\vec{L} = \langle L_\alpha \mid \alpha < \kappa\rangle$ via:

$$L_\alpha := \begin{cases} W_{n(\alpha)-1} \cap A_\alpha, & \text{if } n(\alpha) > 0 \ \&\ \sup(W_{n(\alpha)-1} \cap A_\alpha) = \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 3.2. *(1) For all $n < \omega$ and $\alpha \in S_{n+1}$, $L_\alpha \subseteq W_n$;*

(2) $\bar{S} := \{\alpha \in \mathrm{acc}(\kappa) \mid \sup(L_\alpha) = \alpha\}$ is a stationary subset of S;

(3) For all $\alpha \neq \alpha'$ from $\bar{S}$, $\sup(L_\alpha \cap L_{\alpha'}) < \alpha$;

(4) For all $B_0, B_1 \in [\kappa]^\kappa$, there exists $m < \omega$ such that, for every $n \in \omega \setminus m$, the following set is stationary:

$$\{\alpha \in S_n \mid \sup(L_\alpha \cap B_0) = \sup(L_\alpha \cap B_1) = \alpha\}.$$

Proof. (2) This follows from Clause (4).

(3) For all $\alpha \neq \alpha'$ from $\bar{S}$, $\sup(L_\alpha \cap L_{\alpha'}) \leq \sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

(4) Given two cofinal subsets B_0, B_1 of κ, find $m_0, m_1 < \omega$ be such that $|B_0 \cap S_{m_0}| = |B_1 \cap S_{m_1}| = \kappa$. Evidently, $m := \max\{m_0, m_1\} + 1$ is as sought. $\square$

Now, consider the ladder-system space $\mathbb{X} = (\kappa, \tau)$ which is determined by $\vec{L}$ (equivalently, determined by $\vec{L} \restriction \bar{S}$). This means that a set $U \subseteq \kappa$ is τ-open iff, for every $\alpha \in U$, $L_\alpha \subseteq^* U$. Put differently, if we denote $N_\alpha^\epsilon := (L_\alpha \setminus \epsilon) \cup \{\alpha\}$, then, for every $\alpha < \kappa$, every neighborhood of α covers some element from $\mathcal{N}_\alpha := \{N_\alpha^\epsilon \mid \epsilon < \alpha\}$.

Definition 3.3. For any set of ordinals N, denote $N^- := N \cap \sup(N)$.

Note that, for all $\alpha < \kappa$ and $N \in \mathcal{N}_\alpha$, N^- is either empty or a cofinal subset of α.

Lemma 3.4. *The space $\mathbb{X}$ is T_1.*

Proof. Let x be an element of the space $\mathbb{X}$. To see that $\kappa \setminus \{x\}$ is τ-open, notice that for every $y \in X$, $\mathcal{N}_y$ is a chain such that $\bigcap \mathcal{N}_y = \{y\}$. In particular, for every $y \in X \setminus \{x\}$, there exists $N_y \in \mathcal{N}_y$ with $N_y \subseteq \kappa \setminus \{x\}$. $\square$

Lemma 3.5. *(1) $\kappa \setminus \bar{S}$ is a discrete subspace of size κ;*

(2) For every $\xi < \kappa$, ξ is τ-open;

(3) For every $\xi \in \kappa \setminus \bar{S}$, ξ is τ-closed;

(4) For every $n < \omega$, W_n is τ-open.

Proof. (4) By Lemma 3.2(1). $\square$

Lemma 3.6. *There are no two disjoint τ-closed subsets of cardinality κ.*

Proof. Towards a contradiction, suppose that K_0 and K_1 are two disjoint τ-closed subsets of cardinality κ. Using Lemma 3.2(4), let us fix $n < \omega$ such that $\sup(L_\alpha \cap K_0) = \sup(L_\alpha \cap K_1) = \alpha$. As both K_0 and K_1 are τ-closed, this implies that $\alpha \in K_0$ and $\alpha \in K_1$, contradicting the fact K_0 and K_1 are disjoint. $\square$

Following the terminology coined in [Hig51] and [BEG+04], we introduce the following.

Definition 3.7. The sequence $\vec{L}$ is said to be *almost* $\mathcal{P}_0$ iff, for every $\xi < \kappa$ and every function $c : \bar{S} \cap \xi \to \omega$, there exists a function $c^* : \xi \to \omega$ such that, for every $\alpha \in \bar{S} \cap \xi$, $c^* \restriction L_\alpha$ is eventually constant with value $c(\alpha)$.

Lemma 3.8. *If $\vec{L}$ is almost $\mathcal{P}_0$, then $\mathbb{X}$ is normal and Hausdorff.*

Proof. Suppose that $\vec{L}$ is almost $\mathcal{P}_0$. Let K_0 and K_1 be disjoint τ-closed subsets of κ. By Lemma 3.6, at least one of them is bounded, say K_0. Using Lemma 3.5(1), fix a large enough $\xi \in \kappa \setminus \bar{S}$ such that $K_0 \subseteq \xi$. Note that by Clauses (2) and (3) of Lemma 3.5, ξ is clopen. Now, set $K_0^0 := K_0$ and $K_1^0 := K_1$.

Claim 3.8.1. *Suppose $n < \omega$ and that K_0^n and K_1^n are disjoint τ-closed sets with $K_0^n \subseteq \xi$. Then there exist disjoint τ-closed sets K_0^{n+1} and K_1^{n+1} with $K_0^{n+1} \subseteq \xi$ such that for all $i < 2$:*

(1) $K_i^n \subseteq K_i^{n+1}$;
(2) for every $\alpha \in K_i^n \cap \xi$, $L_\alpha \subseteq^* K_i^{n+1}$.

Proof. For every $i < 2$, define $c_i : \bar{S} \cap \xi \to 2$ via $c_i(\alpha) := 1$ iff $\alpha \in K_i^n$. Now, as $\vec{L}$ is almost $\mathcal{P}_0$, for each $i < 2$, we may fix a function $c_i^* : \xi \to 2$ such that, for every $\alpha \in \bar{S} \cap \xi$, $c_i^* \restriction L_\alpha$ is eventually constant with value $c_i(\alpha)$. For each $i < 2$, set

$$K_i^{n+1} := K_i^n \cup \{\gamma \in \xi \setminus K_{1-i}^n \mid c_i^*(\gamma) = 1 \ \&\ c_{1-i}^*(\gamma) = 0\}.$$

Evidently $K_i^n \subseteq K_i^{n+1}$. It is also easy to see that $K_0^{n+1} \cap K_1^{n+1} = \emptyset$.

Let $i < 2$. To see that K_i^{n+1} is τ-closed, fix an arbitrary nonzero $\alpha < \kappa$ such that $\sup(K_i^{n+1} \cap L_\alpha) = \alpha$, and we shall prove that $\alpha \in K_i^{n+1}$. As $K_i^n \subseteq K_i^{n+1}$, we may avoid trivialities, and assume that $\alpha \notin K_i^n$, so that α belongs to the set

$$\{\gamma \in \xi \setminus K_{1-i}^n \mid c_i^*(\gamma) = 1 \ \&\ c_{1-i}^*(\gamma) = 0\}.$$

Altogether, $\alpha \in (\bar{S} \cap \xi) \setminus K_i^n$, which must mean that $c_i(\alpha) = 0$. Fix a large enough $\epsilon < \alpha$ such that $c_i^* \restriction (L_\alpha \setminus \epsilon)$ is eventually constant with value 0. Then $\sup(K_i^{n+1} \cap L_\alpha) \leq \epsilon < \alpha$, contradicting the choice of α.

Finally, to verify Clause (2), fix arbitrary $i < 2$ and $\alpha \in K_i^n \cap \xi \cap \bar{S}$. By the definition of the two functions, $c_i(\alpha) = 1$ and $c_{1-i}(\alpha) = 0$. So, there exists a large enough $\epsilon < \alpha$ such that $c_i^* \restriction (L_\alpha \setminus \epsilon)$ is a constant function with value 1, and $c_{1-i}^* \restriction (L_\alpha \setminus \epsilon)$ is a constant function with value 0. Consequently, $L_\alpha \setminus \epsilon \subseteq K_i^{n+1}$. □

By an iterative application of the preceding claim, we obtain a sequence of pairs $\langle (K_0^n, K_1^n) \mid n < \omega \rangle$. Set $U_0 := \bigcup_{n<\omega} K_0^n$ and $U_1 := (\kappa \setminus \xi) \cup \bigcup_{n<\omega} K_1^n$. By Clause (2) of the preceding claim and the fact that ξ is clopen, U_i is open for each $i < 2$. Thus, we are left with verifying the following.

Claim 3.8.2. *$U_0 \cap U_1 = \emptyset$.*

Proof. Suppose not, and pick $\alpha \in U_0 \cap U_1$. Notice that $U_0 \subseteq \xi$, hence we can find $n_0, n_1 < \omega$ such that $\alpha \in K_0^{n_0} \cap K_1^{n_1}$. Then, for $n := \max\{n_0, n_1\}$, we get that $\alpha \in K_0^n \cap K_1^n$, contradicting the fact that K_0^n and K_1^n are disjoint. □

This completes the proof of normality. Since $\mathbb{X}$ is T_1, it also follows that it is Hausdorff. □

Corollary 3.9. *If $\vec{L}$ is almost $\mathcal{P}_0$, then $\mathbb{X}$ is Dowker.*

Proof. For every $n<\omega$, set $D_n := \kappa \setminus W_n$. Then $\langle D_n \mid n<\omega\rangle$ is a $\subseteq$-decreasing sequence of closed sets of cardinality κ such that $\bigcap_{n<\omega} D_n = \emptyset$. So, by Lemmas 3.8, 3.6 and 3.1, $\mathbb{X}$ is Dowker. $\square$

Lemma 3.10. *Each of the following two imply that $\vec{L}$ is almost $\mathcal{P}_0$:*

(1) $\mathsf{MA}(<\kappa)$ *holds and* $\mathrm{otp}(L_\alpha)=\omega$ *for all* $\alpha\in\bar{S}$;
(2) $\bar{S}$ *is a non-reflecting stationary set.*

Proof. (1) This follows immediately from [She98, §II, Theorem 4.3].

(2) By Lemma 3.2(3) and Proposition 2.7, we may fix a sequence $\langle f_\xi \mid \xi<\kappa\rangle$ such that, for every $\xi<\kappa$:

- f_ξ is a regressive function from $\bar{S}\cap\xi$ to ξ;
- the sets in $\langle L_\alpha \setminus f_\xi(\alpha) \mid \alpha\in\bar{S}\cap\xi\rangle$ are pairwise disjoint.

Now, given a nonzero $\xi<\kappa$, let $f^+_\xi:\xi\rightarrow\xi$ denote the function such that $f^+_\xi(\alpha)=f_\xi(\alpha)$ for all $\alpha\in\bar{S}\cap\xi$, and $f^+_\xi(\alpha)=0$ for all $\alpha\in\xi\setminus\bar{S}$. Evidently, for every $\beta<\xi$, $\{\alpha<\xi \mid \beta\in L_\alpha\setminus f^+_\xi(\alpha)\}$ is a subset of $\bar{S}$ containing at most one element. So, for any function $c:\bar{S}\cap\xi\rightarrow\omega$, we may define a corresponding function $c^*:\xi\rightarrow\omega$ via:

$$c^*(\beta) := \begin{cases} c(\alpha), & \text{if } \beta\in L_\alpha\setminus f^+_\xi(\alpha);\\ 0, & \text{otherwise.}\end{cases}$$

A moment's reflection makes it clear that c^* is as sought. $\square$

Corollary 3.11. *Suppose that $\mathcal{S}$ is an infinite partition of some non-reflecting stationary subset of a regular uncountable cardinal κ. If $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$ holds, then there exists a ladder-system Dowker space of cardinality κ.* $\square$

Remark 3.12. The preceding is the Introduction's Theorem B.

Corollary 3.13. *If* $\mathsf{MA}+\clubsuit(E^{\mathfrak{c}}_\omega)$ *holds, then there exists a ladder-system Dowker space over $\mathfrak{c}$.*

In particular, if MA *holds and $\mathfrak{c}$ is the successor of a cardinal of uncountable cofinality, then there exists a ladder-system Dowker space over $\mathfrak{c}$.*

Proof. The first part follows from Lemmas 2.10 and 3.10(1). For the second part, note that if MA holds and $\mathfrak{c}=\lambda^+$, then $2^\lambda=\lambda^+$, so if, moreover, λ is a cardinal of uncountable cofinality, then by the main result of [She10], $\diamondsuit(E^{\lambda^+}_\omega)$ holds. In particular, in this case, $\clubsuit(E^{\mathfrak{c}}_\omega)$ holds. $\square$

Remark 3.14. In [Wei81], Weiss proved that if MA and $\diamondsuit(E^{\mathfrak{c}}_\omega)$ both hold, then there exists a locally compact, first countable, separable, real compact, Dowker space of size $\mathfrak{c}$.

Corollary 3.15. *If there exists a λ-complete λ^+-Souslin tree, then there exists a ladder-system Dowker space over λ^+.*

Proof. By Corollary 2.25, Lemma 3.10(2), and the fact that $E^{\lambda^+}_\lambda$ is a non-reflecting stationary subset of λ^+. $\square$

Remark 3.16. In [Rud74b], Rudin constructed a Dowker space of size λ^+ from a λ^+-Souslin tree, for λ regular.

Corollary 3.17. *If there exist a regressive κ-Souslin tree and a non-reflecting stationary subset of E^κ_ω, then there exists a ladder-system Dowker space over κ.*

Proof. By Corollary 2.24 and Lemma 3.10(2). $\square$

Remark 3.18. It is well-known (see [Jen72] or [BR17a]) that if $\kappa > \aleph_0$, $\square_\kappa$ holds and $2^\kappa = \kappa^+$, then there exists a regressive κ-Souslin tree and there exists a non-reflecting stationary subset of $E^{\kappa^+}_\omega$. In [Goo95b], Good proved that if $\kappa > \aleph_0$, $\square_\kappa$ holds and $2^\kappa = \kappa^+$, then there exists a Dowker space over κ^+ which is first countable, locally countable, locally compact, zero-dimensional, and collectionwise normal that is of scattered length ω.

4. de Caux type spaces

4.1. **Collectionwise normality.** In this short subsection, we present a sufficient condition for a certain type of topological space to be collectionwise normal. This will be used in verifying that the spaces constructed later in this section are indeed collectionwise normal.

Definition 4.1. Let $\mathbb{X} = (X, \tau)$ be a topological space.

- A sequence $\langle K_i \mid i < \theta\rangle$ of subsets of X is said to be *discrete* iff for every $x \in X$, there is an open neighborhood U of x such that $\{i < \theta \mid U \cap K_i \neq \emptyset\}$ contains at most one element.
- The space $\mathbb{X}$ is said to be *collectionwise normal* iff for every discrete sequence $\langle K_i \mid i < \theta\rangle$ of closed sets, there exists a sequence $\langle U_i \mid i < \theta\rangle$ of pairwise disjoint open sets such that $K_i \subseteq U_i$ for all $i < \theta$.

Remark 4.2. Note that any collectionwise normal space is normal.

Let $\mathbb{X} = (X, \tau)$ be some topological space determined by a sequence of weak neighborhoods, $\langle \mathcal{N}_x \mid x \in X\rangle$. This means that a subset $U \subseteq X$ is τ-open iff for any $x \in U$, there is $N \in \mathcal{N}_x$ with $N \subseteq U$.

Lemma 4.3. *Suppose that θ is some nonzero cardinal and that $\langle K_i \mid i < \theta\rangle$ is a discrete sequence of τ-closed sets, O is a τ-open set covering $\bigcup_{0<i<\theta} K_i$, and there exists a transversal $\langle N_x \mid x \in X\rangle \in \prod_{x\in X} \mathcal{N}_x$ such that:*

(a) for all $x \in O$, $N_x \subseteq O$;
(b) for all $x \in O$ and $x' \in X \setminus \{x\}$, $N_x \cap N_{x'} \subseteq \{x, x'\}$;
(c) for all $x \in X$ and $i < \theta$, if $N_x \cap K_i \neq \emptyset$, then $x \in K_i$.

Then there exists a sequence $\langle U_i \mid i < \theta\rangle$ of pairwise disjoint τ-open sets such that $K_0 \subseteq U_0$, and $K_i \subseteq U_i \subseteq O$ for all nonzero $i < \theta$.

Proof. By recursion on $n < \omega$, we construct a matrix $\langle U^n_i \mid i < \theta,\ n < \omega\rangle$, as follows:

- ▶ For each $i < \theta$, set $U^0_i := K_i$.
- ▶ For every $n < \omega$ such that $\langle U^n_i \mid i < \theta\rangle$ has already been defined, set $U^{n+1}_i := \bigcup\{N_x \mid x \in U^n_i\}$ for each $i < \theta$.

Evidently, $U_i := \bigcup_{n<\omega} U^n_i$ is an open set covering K_i.

Claim 4.3.1. *Let i with $0 < i < \theta$. Then $U_i \subseteq O$.*

Proof. We have $U_{i,0} = K_i \subseteq O$. In addition, for every $n < \omega$ such that $U^n_i \subseteq O$, Clause (a) implies that $N_x \subseteq O$ for every $x \in U^n_i$, and hence $U^{n+1}_i \subseteq O$. $\square$

Claim 4.3.2. *The sets in the sequence $\langle U_i \mid i < \theta\rangle$ are pairwise disjoint.*

Proof. Suppose not. Fix $i \neq i'$ in θ such that $U_i \cap U_{i'} \neq \emptyset$. Let $n := \min\{k < \omega \mid U_i^k \cap U_{i'} \neq \emptyset\}$, and then let $n' := \min\{k < \omega \mid U_i^n \cap U_{i'}^k \neq \emptyset\}$.

Subclaim 4.3.2.1. $\min\{n, n'\} > 0$.

Proof. First, as $U_i^0 = K_i$ is disjoint from $U_{i'}^0 = K_{i'}$, we infer that $(n, n') \neq (0, 0)$.

▶ If $n = 0$ and $n' > 0$, then let $y \in K_i \cap U_{i'}^{n'}$. It follows that there exists some $x \in U_{i'}^{n'-1}$ such that $y \in K_i \cap N_x$. By Clause (c), then, $x \in K_i$. So $x \in U_i^0 \cap U_{i'}^{n'-1}$, contradicting the minimality of n'.

▶ If $n > 0$ and $n' = 0$, then let $y \in U_i^n \cap K_{i'}$. It follows that there exists some $x \in U_i^{n-1}$ such that $N_x \cap K_{i'} \neq \emptyset$. By Clause (c), then, $x \in K_{i'}$. So $x \in U_i^{n-1} \cap U_{i'}^0$, contradicting the minimality of n. □

It follows that, for all $y \in U_i^n$, either $y \in U_i^{n-1}$ or $y \in N_x$ for some $x \in U_i^{n-1} \setminus \{y\}$. Likewise, for all $y \in U_{i'}^{n'}$, either $y \in U_{i'}^{n'-1}$ or $y \in N_x$ for some $x \in U_{i'}^{n'-1} \setminus \{y\}$. Now, by the choice of the pair (n, n'), let us fix $y \in U_i^n \cap U_{i'}^{n'}$. There are four cases to consider:

(1) $y \in U_i^{n-1} \cap U_{i'}^{n'-1}$. In this case, $U_i^{n-1} \cap U_{i'} \neq \emptyset$, contradicting the minimality of n.
(2) $y \in N_x \cap U_{i'}^{n'-1}$ for some $x \in U_i^{n-1} \setminus \{y\}$. In this case, $N_x \subseteq U_i^n$, so $U_i^n \cap U_{i'}^{n'-1} \neq \emptyset$, contradicting the minimality of n'.
(3) $y \in U_i^{n-1} \cap N_x$ for some $x \in U_{i'}^{n'-1} \setminus \{y\}$. In this case, $N_x \subseteq U_{i'}^{n'}$, so $U_i^{n-1} \cap U_{i'} \neq \emptyset$, contradicting the minimality of n.
(4) There exist $x \in U_i^{n-1} \setminus \{y\}$ and $x' \in U_{i'}^{n'-1} \setminus \{y\}$ such that $y \in N_x \cap N_{x'}$. Equivalently, there exist $x \in U_i^{n-1}$ and $x' \in U_{i'}^{n'-1}$ such that $y \in (N_x \cap N_{x'}) \setminus \{x, x'\}$. In this case, there are two subcases:
   ▶ If $x = x'$, then $x \in U_i^{n-1} \cap U_{i'}^{n'-1}$, contradicting the minimality of n.
   ▶ If $x \neq x'$, then, by Claim 4.3.1, either $x \in O$ or $x' \in O$. This is in contradiction with Clause (b).

Altogether, $\{U_i \mid i < \theta\}$ is a family of pairwise disjoint sets as sought. □

This completes the proof. □

4.2. ***O*-space.** This subsection is dedicated to proving Theorem C:

Theorem 4.4. *Suppose that $\clubsuit_{\mathrm{AD}}(\{\omega_1\}, \omega, 1)$ holds. Then there exists a collectionwise normal, non-Lindelöf O-space.*

Remark 4.5. Note that unlike Dahroug's construction that defines a topology over the $\aleph_1$-Souslin tree, here the topology will be defined over ω_1. Thus, when taken together with Corollary 2.26, this appears to yield the first "linear" construction of an O-space from an $\aleph_1$-Souslin tree.

Remark 4.6. The arguments of this subsection immediately generalize to yield a collectionwise normal higher O-space from $\clubsuit_{\mathrm{AD}}(\{E^{\lambda^+}_\lambda\}, \lambda, 1)$. The focus on the case $\lambda = \omega$ is just for simplicity.

Let $\langle \mathcal{A}_\alpha \mid \alpha < \omega_1\rangle$ be a $\clubsuit_{\mathrm{AD}}(\{\omega_1\}, \omega, 1)$-sequence. For each $\alpha \in \mathrm{acc}(\omega_1)$, fix an injective enumeration $\{A_{\alpha+i} \mid i < \omega\}$ of the elements of $\mathcal{A}_\alpha$. For every infinite $\xi < \omega_1$, as $\mathcal{B}_\xi := \{A_\beta \mid \omega \leq \beta < \xi + \omega\}$ is a countable subset of $\bigcup_{\alpha \in E^{\omega_1}_\omega} \mathcal{A}_\alpha$, we may appeal to Proposition 2.8 to fix a function $f_\xi : \mathcal{B}_\xi \to \omega_1$ such that:

(1) for every $B \in \mathcal{B}_\xi$, $f_\xi(B) \in B$;
(2) the sets in $\langle B \setminus f_\xi(B) \mid B \in \mathcal{B}_\xi \rangle$ are pairwise disjoint.

Definition 4.7. For every $\beta < \omega_1$, let $\alpha_\beta := \min\{\alpha \leq \beta \mid \exists i < \omega(\beta = \alpha + i)\}$.

We are now ready to define our topological space $\mathbb{X} = (\omega_1, \tau)$. For all $\beta < \omega$, let $\mathcal{N}_\beta := \{\{\beta\}\}$. For all infinite $\beta < \omega_1$ and $\epsilon < \alpha_\beta$, denote $N^\epsilon_\beta := (A_\beta \setminus \epsilon) \cup \{\beta\}$, and then set $\mathcal{N}_\beta := \{N^\epsilon_\beta \mid \epsilon < \alpha_\beta\}$. Now, a subset $U \subseteq \omega_1$ is τ-open iff for any $\beta \in U$, there is $N \in \mathcal{N}_\beta$ with $N \subseteq U$. It is easy to check that $\mathbb{X}$ is a T_1 topological space and that, for every $\xi < \omega_1$, ξ is τ-open.

Definition 4.8. For any set of ordinals N, denote $N^- := N \cap \sup(N)$.

Note that for all $\beta < \omega_1$ and $N \in \mathcal{N}_\beta$, N^- is a cofinal subset of α_β.

Remark 4.9. The topology of the space from the previous section is such that a set U is open iff, for every $\beta \in U$, $L_\beta \subseteq^* U$, and the topology of the space here is such that a set U is open iff, for every $\beta \in U$, $A_\beta \subseteq^* U$. The approach seems identical, but there is a subtle difference: in the previous section, for every ordinal β, we had $\sup(L_\beta) \in \{0, \beta\}$, whereas here, for every ordinal β, we have $\sup(A_\beta) = \alpha_\beta \in \beta + 1$.

Lemma 4.10. *For every $\alpha \in \operatorname{acc}(\omega_1)$, $\operatorname{cl}([\alpha, \alpha + \omega)) = \omega_1 \setminus \alpha$.*

Proof. We omit the proof because a similar verification is given in details in the next subsection. □

Corollary 4.11. *(1) (ω_1, τ) is hereditary separable;*
(2) Every τ-closed set is either countable or co-countable.

Proof. Let B be an arbitrary uncountable subset of ω_1. By Clause (2) of Definition 2.4, we can find an $\alpha \in \operatorname{acc}(\omega_1)$ such that $\sup(A_{\alpha+i} \cap B) = \alpha$ for all $i < \omega$. Let $D := B \cap \alpha$. It follows that, for every $\beta \in [\alpha, \alpha + \omega)$ and every $N \in \mathcal{N}_\beta$, $\sup(N \cap D) = \alpha$. So, $[\alpha, \alpha + \omega) \subseteq \operatorname{cl}(D)$ and hence $\omega_1 \setminus \alpha = \operatorname{cl}([\alpha, \alpha + \omega)) \subseteq \operatorname{cl}(D)$.
(1) As $B \setminus (\omega_1 \setminus \alpha) = D$, it follows that $\operatorname{cl}(B) = \operatorname{cl}(D)$, so that D is a countable dense subset of the subspace B.
(2) If B is moreover closed, then $(\omega_1 \setminus \alpha) \subseteq B$, so that B is co-countable. □

Lemma 4.12. *$\mathbb{X}$ is Hausdorff and collectionwise normal.*

Proof. As $\mathbb{X}$ is T_1, it suffices to verify that it is collectionwise normal. Let $\vec{K} = \langle K_i \mid i < \theta \rangle$ be an arbitrary discrete sequence of closed sets, for some cardinal θ. To avoid trivialities, assume that $\theta \geq 2$.

Claim 4.12.1. *$\theta \leq \omega$.*

Proof. Otherwise, set $B := \{\beta_i \mid i < \omega_1\}$ for some transversal $\langle \beta_i \mid i < \omega_1 \rangle \in \prod_{i<\omega_1} K_i$. As B is necessarily uncountable, using Clause (2) of Definition 2.4, we may fix $\alpha \in \operatorname{acc}(\omega_1)$ such that $\sup(A_\alpha \cap B) = \alpha$. Then any open neighborhood of α meets infinitely many elements of $\vec{K}$, contradicting its discreteness. □

By Corollary 4.11(2), the sequence $\vec{K}$ contains at most one uncountable set. By possibly re-indexing, we may assume that $\{i < \theta \mid |K_i| = \aleph_1\} \subseteq \{0\}$. Now, as θ is countable, we may find a large enough $\xi \in \operatorname{acc}(\omega_1)$ such that $K_i \subseteq \xi$ for all nonzero $i < \theta$.

Claim 4.12.2. *There exists a sequence $\langle N_\beta \mid \beta < \omega_1 \rangle \in \prod_{\beta<\omega_1} \mathcal{N}_\beta$ such that:*

(a) *for all* $\beta < \xi$, $N_\beta \subseteq \xi$;
(b) *for all* $\beta < \xi$ *and* $\beta' \in \omega_1 \setminus \{\beta\}$, $N_\beta^- \cap N_{\beta'}^- = \emptyset$;
(c) *for all* $\beta < \omega_1$ *and* $i < \theta$, *if* $N_\beta \cap K_i \neq \emptyset$ *then* $\beta \in K_i$.

Proof. There are three cases to consider:

▶ For every $\beta < \omega$, just set $N_\beta := \{\beta\}$.

▶ For every $\beta \in \omega_1 \setminus (\xi + \omega)$, as $\alpha_\beta > \xi$, we may let

$$\epsilon := \max(\{\xi, \sup(A_\beta \cap K_0)\} \cap \alpha_\beta),$$

and then set $N_\beta := N_\beta^{\epsilon+1}$.

▶ Suppose β is not of the above form. In particular, $A_\beta \in \mathcal{B}_\xi$ and $f_\xi(A_\beta) < \alpha_\beta$.

As $\vec{K}$ is discrete, let us pick an open neighborhood U of β which for which $I := \{i < \theta \mid U \cap K_i \neq \emptyset\}$ contains at most one element. Find $\varepsilon < \alpha_\beta$ such that $A_\beta \setminus \varepsilon \subseteq U$, and then let

$$\epsilon := \begin{cases} \max(\{f_\xi(A_\beta), \varepsilon\}), & \text{if } I = \emptyset; \\ \max(\{f_\xi(A_\beta), \varepsilon, \sup(A_\beta \cap K_i)\} \cap \alpha_\beta), & \text{if } I = \{i\}. \end{cases}$$

Finally, set $N_\beta := N_\beta^{\epsilon+1}$.

We omit the proof that $\langle N_\beta \mid \beta < \omega_1\rangle$ is as sought, because a similar verification is given in details in the proof of Claim 4.21.1 below. □

It now follows from Lemma 4.3 that there exists a sequence $\langle U_i \mid i < \theta\rangle$ of pairwise open sets such that $K_i \subseteq U_i$ for all $i < \theta$. □

4.3. **A Dowker space with small hereditary density.** In [Rud55], Rudin constructed a Dowker space of size $\aleph_1$ from a Souslin tree. In [Rud72], she constructed an S-space of size $\aleph_1$ from a Souslin tree. In [Rud74a], she constructed an S-space of size $\aleph_1$ which is Dowker from a Souslin tree, and in [Rud74b], she constructed a Dowker space of size λ^+ from a λ^+-Souslin tree, for λ regular.

In [dC77], de Caux constructed a Dowker space of size $\aleph_1$ assuming $\clubsuit(\omega_1)$. Here we roughly follow de Caux's general approach, but using $\clubsuit_{\mathrm{AD}}$, instead. So this gives a simultaneous generalization of de Caux's result and Rudin's result.

Theorem 4.13. *Suppose that* $\clubsuit_{\mathrm{AD}}(\{E^{\lambda^+}_\lambda\}, \lambda, 1)$ *holds for an infinite regular cardinal* λ. *Then there exists a collectionwise normal Dowker space* $\mathbb{X}$ *of cardinality* λ^+ *such that* $\mathrm{hd}(\mathbb{X}) = \lambda$ *and* $\mathrm{L}(\mathbb{X}) = \lambda^+$.

Remark 4.14. The preceding is the Introduction's Theorem D.

Let $\langle \mathcal{A}_\alpha \mid \alpha \in E^{\lambda^+}_\lambda\rangle$ be a $\clubsuit_{\mathrm{AD}}(\{E^{\lambda^+}_\lambda\}, \lambda, 1)$-sequence. For each $\alpha \in E^{\lambda^+}_\lambda$, fix an injective enumeration $\{A^{j,n}_{\alpha+i} \mid i < \lambda, j \leq n < \omega\}$ of the elements of $\mathcal{A}_\alpha$. For every $\xi \in \lambda^+ \setminus \lambda$, as $\mathcal{B}_\xi := \{A^{j,n}_\beta \mid \lambda \leq \beta < \xi + \lambda, j \leq n < \omega\}$ is a subset of $\bigcup_{\alpha \in E^{\lambda^+}_\lambda} \mathcal{A}_\alpha$ of size λ, we may appeal to Proposition 2.8 to fix a function $f_\xi : \mathcal{B}_\xi \to \xi$ such that:

(1) For every $B \in \mathcal{B}_\xi$, $f_\xi(B) \in B$;
(2) The sets in $\langle B \setminus f_\xi(B) \mid B \in \mathcal{B}_\xi\rangle$ are pairwise disjoint.

Definition 4.15. For every $\beta < \lambda^+$, let $\alpha_\beta := \min\{\alpha \leq \beta \mid \exists i < \lambda(\beta = \alpha + i)\}$.

We are now ready to define a topology τ on the set $X := \lambda^+ \times \omega$. For all $x \in \lambda \times \omega$, just let $\mathcal{N}_x := \{\{x\}\}$. For all $x = (\beta, n)$ in X with $\beta \geq \lambda$, denote $N^\epsilon_x := \{x\} \cup \bigcup_{j \leq n}((A^{j,n}_\beta \setminus \epsilon) \times \{j\})$, and then set $\mathcal{N}_x := \{N^\epsilon_x \mid \epsilon < \alpha_\beta\}$. Finally, a subset $U \subseteq \lambda^+ \times \omega$ is τ-open iff for any $x \in U$, there is $N \in \mathcal{N}_x$ with $N \subseteq U$. It is easy to check that $\mathbb{X} := (X, \tau)$ is a T_1 topological space.

Definition 4.16. For any subset $N \subseteq \lambda^+ \times \omega$, denote:

- $N^- := \{(\gamma, j) \in N \mid \exists(\beta, n) \in N\, (\gamma < \beta)\}$;
- $N^j := \{\gamma \mid (\gamma, j) \in N^-\}$ for any $j < \omega$.

The following is obvious.

Lemma 4.17. *For all $x = (\beta, n)$ in X, $N \in \mathcal{N}_x$, and $j \leq n$, N^j is a cofinal subset of α_β and $N \subseteq (\beta + 1) \times (n + 1)$. In particular, for all $\delta < \lambda^+$ and $n < \omega$:*

- *$\delta \times n$ is τ-open;*
- *$\lambda^+ \times (\omega \setminus n)$ is τ-closed.* □

So $\{\delta \times \omega \mid \delta < \lambda^+\}$ witnesses that $\mathrm{L}(\mathbb{X}) = \lambda^+$.

Lemma 4.18. *For every $\alpha \in E^{\lambda^+}_\lambda$ and $k < \omega$,*

$$(\lambda^+ \setminus (\alpha + \lambda)) \times (\omega \setminus k) \subseteq \mathrm{cl}([\alpha, \alpha + \lambda) \times \{k\}).$$

Proof. Denote $I_{\alpha,j} := [\alpha, \alpha + \lambda) \times \{j\}$ and $F_{\alpha,j} := \mathrm{cl}(I_{\alpha,j})$.

Claim 4.18.1. *Let $\alpha \in E^{\lambda^+}_\lambda$ and $j < \omega$. Then $(\lambda^+ \setminus \alpha) \times \{j\} \subseteq F_{\alpha,j}$.*

Proof. We prove by induction on $\delta \in E^{\lambda^+}_\lambda \setminus \alpha$ that $I_{\delta,j} \subseteq F_{\alpha,j}$.

► For $\delta = \alpha$, trivially $I_{\delta,j} \subseteq F_{\alpha,j}$.

► Suppose that $\delta \in E^{\lambda^+}_\lambda \setminus (\alpha + \lambda)$ is an ordinal such that, for every $\gamma \in E^\delta_\lambda \setminus \alpha$, $I_{\gamma,j} \subseteq F_{\alpha,j}$. Therefore, $[\alpha, \delta) \times \{j\} \subseteq F_{\alpha,j}$. Let $x \in I_{\delta,j}$. Since, for all $N \in \mathcal{N}_x$, N^j is a cofinal subset of δ, $N \cap ([\alpha, \delta) \times \{j\}) \neq \emptyset$. Therefore, $x \in \mathrm{cl}([\alpha, \delta) \times \{j\}) \subseteq F_{\alpha,j}$. □

Let $\alpha \in E^{\lambda^+}_\lambda$ and $k < \omega$. Let $j \geq \kappa$ be some integer; we need to prove that $(\lambda^+ \setminus (\alpha + \lambda)) \times \{j\} \subseteq F_{\alpha,k}$. By the preceding claim, it suffices to prove that $F_{\alpha+\lambda,j} \subseteq F_{\alpha,k}$. But the latter is a closed set, so it suffices to prove that $I_{\alpha+\lambda,j} \subseteq F_{\alpha,k}$. Let $x \in I_{\alpha+\lambda,j}$ be arbitrary. For each $N \in \mathcal{N}_x$, as $k \leq j$, N^k is a cofinal subset of $\alpha + \lambda$, and then $N \cap I_{\alpha,k} \neq \emptyset$. Therefore, $x \in \mathrm{cl}(I_{\alpha,k}) = F_{\alpha,k}$, as sought. □

Corollary 4.19. *For any $B \subseteq X$ of size λ^+:*

1. *There exists $D \subseteq B$ with $|D| = \lambda$ such that $B \subseteq \mathrm{cl}(D)$;*
2. *If B is τ-closed, then there is $(\beta, k) \in \lambda^+ \times \omega$ such that $(\lambda^+ \setminus \beta) \times (\omega \setminus k) \subseteq B$.*

Proof. Given B as above, fix the least $k < \omega$ such that $|B \cap (\lambda^+ \times \{k\})| = \lambda^+$. By Clause (2) of Definition 2.4, we can find an $\alpha \in E^{\lambda^+}_\lambda$ such that $\mathrm{dom}((A^{j,n}_{\alpha+i} \times \{k\}) \cap B)$ is cofinal in α for all $i < \lambda$ and $j \leq n < \omega$. Let $D := B \cap (\alpha \times \{k\})$. It follows that, for every $x \in [\alpha, \alpha + \lambda) \times \{k\}$ and every $N \in \mathcal{N}_x$, $\mathrm{dom}(N \cap D)$ is cofinal in α. So, $[\alpha, \alpha+\lambda) \times \{k\} \subseteq \mathrm{cl}(D)$ and hence $(\lambda^+ \setminus (\alpha+\lambda)) \times (\omega \setminus k) \subseteq \mathrm{cl}([\alpha, \alpha+\lambda) \times \{k\}) \subseteq \mathrm{cl}(D)$.

1. As $|B \setminus \mathrm{cl}(D)| \leq \lambda$, we see that $D \cup (B \setminus \mathrm{cl}(D))$ is a dense subset of B of cardinality λ.
2. If B is τ-closed, then for $\beta := \alpha + \lambda$, $(\lambda^+ \setminus \beta) \times (\omega \setminus k) \subseteq \mathrm{cl}(D) \subseteq B$. □

Corollary 4.20. $\mathrm{hd}(\mathbb{X}) \leq \lambda$ *and there are no two disjoint closed subspaces of $\mathbb{X}$ of cardinality λ^+.*

Proof. By Corollary 4.19(2). □

Lemma 4.21. *The space $\mathbb{X}$ is Hausdorff and collectionwise normal.*

Proof. As $\mathbb{X}$ is T_1, it suffices to verify that it is collectionwise normal. Fix a nonzero cardinal θ and a discrete sequence $\vec{K} = \langle K_i \mid i < \theta\rangle$ of closed sets. It follows from Clause (2) of Definition 2.4 that $\theta \leq \lambda$, so, using Corollary 4.20 and by possibly re-indexing, we may find a large enough $\xi \in E^{\lambda^+}_\lambda$ such that $K_i \subseteq \xi \times \omega$ for all nonzero $i < \theta$. Recall that $O := \xi \times \omega$ is an open set.

Claim 4.21.1. *There exists a sequence $\langle N_x \mid x \in X\rangle \in \prod_{x\in X} \mathcal{N}_x$ such that:*

(a) for all $x \in O$, $N_x \subseteq O$;
(b) for all $x \in O$ and $x' \in X \setminus \{x\}$, $N^-_x \cap N^-_{x'} = \emptyset$;
(c) for all $x \in X$ and $i < \theta$, if $N_x \cap K_i \neq \emptyset$ then $x \in K_i$.

Proof. Recalling Lemma 4.17, we should only worry about requirements (b) and (c). Let $x = (\beta, n)$ in X. There are three cases to consider:

▶ If $\beta < \lambda$, then set $N_x := \{x\}$. Evidently, requirement (c) is satisfied.

▶ If $\beta \geq \xi + \lambda$, then $\alpha_\beta > \xi$, so we let

$$\epsilon := \max(\{\xi, \sup(\mathrm{dom}[(\bigcup_{j\leq n}(A^{j,n}_\beta \times \{j\})) \cap K_0])\} \cap \alpha_\beta),$$

and then set $N_x := N^{\epsilon+1}_x$. Evidently, $N_x \cap O = \emptyset$. In particular, for all $i < \theta$, if $N_x \cap K_i \neq \emptyset$, then $i = 0$ and $\sup(\mathrm{dom}[(\bigcup_{j\leq n}(A^{j,n}_\beta \times \{j\})) \cap K_0]) = \alpha_\beta$, which means that $x \in K_0$, since the latter is closed. So, requirement (c) is satisfied.

▶ If $\lambda \leq \beta < \xi + \lambda$, then $A^{j,n}_\beta \in \mathcal{B}_\xi$ for all $j \leq n$, so that $\phi_x := \max\{f_\xi(A^{j,n}_\beta) \mid j \leq n\}$ is $< \alpha_\beta \leq \xi$. As $\vec{K}$ is discrete, let us pick an open neighborhood U of x for which $I := \{i < \theta \mid U \cap K_i \neq \emptyset\}$ contains at most one element, and then find a large enough $\varepsilon \in \alpha_\beta \setminus \phi_x$ such that $N^\varepsilon_x \subseteq U$. Let

$$\epsilon := \begin{cases} \varepsilon, & \text{if } I = \emptyset;\\ \max(\{\varepsilon, \sup(\mathrm{dom}[(\bigcup_{j\leq n}(A^{j,n}_\beta \times \{j\})) \cap K_i])\} \cap \alpha_\beta), & \text{if } I = \{i\},\end{cases}$$

and then set $N_x := N^{\epsilon+1}_x$. Evidently, $N_x \subseteq U$. So, for all $i < \theta$, if $N_x \cap K_i \neq \emptyset$, then $I = \{i\}$ and $\sup(\mathrm{dom}[(\bigcup_{j\leq n}(A^{j,n}_\beta \times \{j\})) \cap K_i]) = \alpha_\beta$, which means that $x \in K_i$. So, requirement (c) is satisfied.

We are left with verifying that $\langle N_x \mid x \in X\rangle$ satisfies requirement (b). Fix arbitrary $x \in O$ and $x' \in X \setminus \{x\}$. Say, $x = (\beta, n)$ and $x' = (\beta', n')$. Note that if $\beta < \lambda$, then N^-_x is empty, and if $\beta' \geq \xi + \lambda$, then $\min(\mathrm{dom}(N_{x'})) > \xi > \beta = \sup(\mathrm{dom}(N_x))$. Therefore, if $N^-_x \cap N^-_{x'} \neq \emptyset$, then $\lambda \leq \beta, \beta' < \xi + \lambda$. Thus, assume that $\lambda \leq \beta, \beta' < \xi + \lambda$, and fix $\epsilon \geq \phi_x$ and $\epsilon' \geq \phi_{x'}$ such that

- $N^-_x = \bigcup_{j\leq n}((A^{j,n}_\beta \setminus (\epsilon + 1)) \times \{j\})$, and
- $N^-_{x'} = \bigcup_{j\leq n'}((A^{j,n'}_{\beta'} \setminus (\epsilon' + 1)) \times \{j\})$.

So, if $N^-_x \cap N^-_{x'} \neq \emptyset$, then there exists $j \leq \min\{n, n'\}$ such that $(A^{j,n}_\beta \setminus \phi_x) \cap (A^{j,n'}_{\beta'} \setminus \phi_{x'}) \neq \emptyset$. In particular, $(A^{j,n}_\beta \setminus f_\xi(A^{j,n}_\beta)) \cap (A^{j,n'}_{\beta'} \setminus f_\xi(A^{j,n'}_{\beta'})) \neq \emptyset$, contradicting the fact that $(\beta, n) \neq (\beta', n')$. □

It now follows from Lemma 4.3 that there exists a sequence $\langle U_i \mid i < \theta\rangle$ of pairwise open sets such that $K_i \subseteq U_i$ for all $i < \theta$. □

Lemma 4.22. *The space $\mathbb{X}$ is Dowker.*

Proof. Denote $D_n := \lambda^+ \times (\omega \setminus n)$. Notice that $\langle D_n \mid n < \omega\rangle$ is a $\subseteq$-decreasing sequence of λ^+-sized τ-closed sets such that $\bigcap_{n<\omega} D_n = \emptyset$. By Corollary 4.20, there are no two disjoint closed sets of cardinality λ^+. So, by Corollary 4.21 and Lemma 3.1, the space (X,τ) is Dowker. $\square$

5. A Dowker space with a normal square

In [Sze10], Szeptycki proved that, assuming $\diamondsuit^*(S)$ for a stationary $S \subseteq E^{\omega_2}_{\omega_1}$, there exists a ladder-system over a subset of S whose corresponding ladder-system space is a Dowker space having a normal square. As seen in Section 3, the hypothesis may be reduced to $\clubsuit_{\mathrm{AD}}(\{E^{\omega_2}_{\omega_1}\},1,2)$ and still give a ladder-system whose corresponding space is Dowker,[3] since $E^{\omega_2}_{\omega_1}$ is a non-reflecting stationary subset of ω_2. But what about the normal square?

In this short section, we point out that the $\diamondsuit^*$ hypothesis may be reduced to an assumption in the language of the Brodsky-Rinot proxy principle (see Definition 5.2 below). For the rest of this section, let λ denote an infinite regular cardinal.

Proposition 5.1. $\mathrm{P}^-_\lambda(\lambda^+,2,{}_{\lambda^+}\sqsubseteq,\lambda,\{E^{\lambda^+}_\lambda\})$ *entails the existence of a ladder-system over a subset of* $E^{\lambda^+}_\lambda$ *whose corresponding ladder-system space* (λ^+,τ) *is a Dowker space having a normal square.*

The point is that the hypothesis of the preceding already follows from $\diamondsuit(E^{\lambda^+}_\lambda)$, but it is also consistent with its failure (see Clauses (1) and (11) of [BR21, Theorem 6.1]).

Definition 5.2 (Brodsky-Rinot, [BR21])**.** For a family $\mathcal{S} \subseteq \mathcal{P}(\kappa)$, and a cardinal $\theta < \kappa$, $\mathrm{P}^-_\xi(\kappa,2,{}_\kappa\sqsubseteq,\theta,\mathcal{S})$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

- for every $\alpha \in \operatorname{acc}(\kappa)$, C_α is a club in α of order-type $\le \xi$;
- for every $S \in \mathcal{S}$ and every sequence $\langle B_i \mid i < \theta\rangle$ of cofinal subsets of κ, there exist stationarily many $\alpha \in S$ such that, for all $i < \theta$,
$$\sup\{\delta \in B_i \cap \alpha \mid \min(C_\alpha \setminus (\delta+1)) \in B_i\} = \alpha.$$

Note that for every $\mathcal{S} \subseteq \mathcal{P}(E^{\lambda^+}_\lambda)$, any $\mathrm{P}^-_\lambda(\lambda^+,2,{}_{\lambda^+}\sqsubseteq,\theta,\mathcal{S})$-sequence witnesses the validity of $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,\theta)$.

Fact 5.3 (Brodsky-Rinot, [BR21, Theorem 4.15])**.** *For a family* $\mathcal{S} \subseteq \mathcal{P}(\kappa)$*, and a cardinal* $\theta < \kappa$*,* $\mathrm{P}^-_\xi(\kappa,2,{}_\kappa\sqsubseteq,\theta,\mathcal{S})$ *entails the existence of a sequence* $\langle C_\alpha \mid \alpha < \kappa\rangle$ *such that:*

(1) for every $\alpha \in \operatorname{acc}(\kappa)$*,* C_α *is a club in* α *of order-type* $\le \xi$*;*
(2) for every $S \in \mathcal{S}$ *and every sequence* $\langle \mathcal{B}_i \mid i < \theta\rangle$ *with* $\mathcal{B}_i \subseteq [\kappa]^{<\omega}$ *and* $\operatorname{mup}(\mathcal{B}_i) = \kappa$ *for all* $i < \theta$*, there exist stationarily many* $\alpha \in S$ *such that, for all* $i < \theta$*,*
$$\operatorname{mup}\{x \in \mathcal{B}_i \mid x \subseteq C_\alpha\} = \alpha.$$

For an ordinal α, let us say that a subset x of the product $\alpha \times \alpha$ is *dominating* iff for every $(\beta,\gamma) \in (\alpha,\alpha)$, there exists $(\beta',\gamma') \in x$ with $\beta \le \beta'$ and $\gamma \le \gamma'$. Now, we are ready to prove the main lemma.

Lemma 5.4. *Suppose that* $\mathrm{P}^-_\lambda(\lambda^+,2,{}_{\lambda^+}\sqsubseteq,\lambda,\{E^{\lambda^+}_\lambda\})$ *holds. Then there exist a partition* $\langle S_n \mid n < \omega\rangle$ *of* λ^+ *into stationary sets and a sequence* $\langle s_\alpha \mid \alpha \in E^{\lambda^+}_\lambda\rangle$ *such that:*

---

[3] Recall that by Remark 2.2 and Lemma 2.10, for $S \subseteq E^{\omega_2}_{\omega_1}$, $\diamondsuit^*(S) \implies \diamondsuit(E^{\omega_2}_{\omega_1}) \implies \clubsuit(E^{\omega_2}_{\omega_1}) \implies \clubsuit_{\mathrm{AD}}(E^{\omega_2}_{\omega_1},\omega_1,{<}\omega) \implies \clubsuit_{\mathrm{AD}}(E^{\omega_2}_{\omega_1},1,2)$.

(1) For each $\alpha \in E^{\lambda^+}_\lambda$, s_α is either empty or a cofinal subset of α of order-type λ;
(2) For all $n < \omega$ and $\alpha \in S_{n+1}$, $s_\alpha \subseteq \bigcup_{i\le n} S_i$;
(3) For every $k < \omega$, every λ-sized subfamily $\mathcal{F} \subseteq [\bigcup_{i\le k} S_i]^{\lambda^+}$, and every $n \in \omega \setminus (k+1)$, the following set is stationary:

$$\{\alpha \in S_n \mid \forall F \in \mathcal{F}\ [\sup(s_\alpha \cap F) = \alpha]\};$$

(4) For every two dominating subsets B_0, B_1 of $\lambda^+ \times \lambda^+$, there exists $m < \omega$ such that for every $n \in \omega \setminus m$ the following set is stationary:

$$\{\alpha \in S_n \mid \forall i < 2\ [(s_\alpha \times s_\alpha) \cap B_i \text{ dominates } (\alpha, \alpha)]\}.$$

Proof. Let $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+\rangle$ be a $\mathrm{P}^-_\lambda(\lambda^+, 2, {}_{\lambda^+}{\sqsubseteq}, \lambda, \{E^{\lambda^+}_\lambda\})$-sequence. Let $\mathcal{I}$ denote the collection of all $T \subseteq E^{\lambda^+}_\lambda$ such that $\vec{C}$ is not a $\mathrm{P}^-_\lambda(\lambda^+, 2, {}_{\lambda^+}{\sqsubseteq}, \lambda, \{T\})$-sequence. Evidently, $\mathcal{I}$ is a λ^+-complete ideal over $E^{\lambda^+}_\lambda$. So, by Ulam's theorem, $\mathcal{I}$ is not weakly λ^+-saturated. This means that there exists a sequence $\langle S_\iota \mid \iota < \lambda^+\rangle$ of pairwise disjoint subsets of $E^{\lambda^+}_\lambda$, such that, for each $\iota < \lambda^+$, $\vec{C}$ is a $\mathrm{P}^-_\lambda(\lambda^+, 2, {}_{\lambda^+}{\sqsubseteq}, \lambda, \{S_\iota\})$-sequence. In particular, we may fix a family $\mathcal{S}$ consisting of $\aleph_0$-many pairwise disjoint stationary subsets of $E^{\lambda^+}_\lambda$ such that $\mathrm{P}^-_\lambda(\lambda^+, 2, {}_{\lambda^+}{\sqsubseteq}, \lambda, \mathcal{S})$ holds and yet $S_0 := \lambda^+ \setminus \bigcup \mathcal{S}$ is stationary. Fix an injective enumeration $\langle S_{n+1} \mid n < \omega\rangle$ of $\mathcal{S}$. For every $\alpha \in E^{\lambda^+}_\lambda$, let $n(\alpha)$ be such that $\alpha \in S_{n(\alpha)}$. For each $n < \omega$, let $W_n := \bigcup_{i\le n} S_i$. Now, let $\vec{D} = \langle D_\alpha \mid \alpha < \lambda^+\rangle$ be a $\mathrm{P}^-_\lambda(\lambda^+, 2, {}_{\lambda^+}{\sqsubseteq}, \lambda, \{S_{n+1} \mid n < \omega\})$-sequence as in Fact 5.3. For every $\alpha \in E^{\lambda^+}_\lambda$, let

$$s_\alpha := \begin{cases} W_{n(\alpha)-1} \cap D_\alpha, & \text{if } n(\alpha) > 0 \ \&\ \sup(W_{n(\alpha)-1} \cap D_\alpha) = \alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$

We claim that the sequence $\langle s_\alpha \mid \alpha \in E^{\lambda^+}_\lambda\rangle$ is as sought. Notice that Clauses (1) and (2) hold by our very construction, as D_α has order-type λ for every $\alpha \in E^{\lambda^+}_\lambda$.

Claim 5.4.1. *Let $k < \omega$ and $\mathcal{F} \subseteq [W_k]^{\lambda^+}$ be a family of size λ. For every integer $n > k$, $\{\alpha \in S_n \mid \forall F \in \mathcal{F}\ [\sup(s_\alpha \cap F) = \alpha]\}$ is stationary.*

Proof. Let $\{\mathcal{B}_i \mid i < \lambda\}$ be some enumeration of $\{[F]^1 \mid F \in \mathcal{F}\}$. Evidently, for every $i < \lambda$, $\mathcal{B}_i \subseteq [\lambda^+]^{<\omega}$ and $\mathrm{mup}(\mathcal{B}_i) = \lambda^+$. Now, by Fact 5.3, Clause (2), for every $n < \omega$, the set G_n of all $\alpha \in S_n$ such that, for all $i < \lambda$,

$$\mathrm{mup}\{x \in \mathcal{B}_i \mid x \subseteq D_\alpha\} = \alpha,$$

is stationary. In particular, for all $n > k$, $\alpha \in G_n$ and $F \in \mathcal{F}$:

$$\sup(s_\alpha \cap F) = \sup(W_{n(\alpha)-1} \cap D_\alpha \cap F) = \alpha. \qquad \square$$

Claim 5.4.2. *Let B_0, B_1 be two dominating subsets of $\lambda^+ \times \lambda^+$. Then there exists some $m < \omega$ such that for every $n \in \omega \setminus m$ the following set is stationary:*

$$\{\alpha \in S_n \mid \forall i < 2\ [(s_\alpha \times s_\alpha) \cap B_i \text{ dominates } (\alpha, \alpha)]\}.$$

Proof. For every $\epsilon < \lambda^+$ and $i < 2$, fix $(\xi^i_\epsilon, \zeta^i_\epsilon) \in B_i$ with $\min\{\xi^i_\epsilon, \zeta^i_\epsilon\} > \epsilon$. For every $n < \omega$, let

$$\mathcal{B}_n := \{\{\xi^0_\epsilon, \zeta^0_\epsilon, \xi^1_\epsilon, \zeta^1_\epsilon\} \mid \epsilon < \lambda^+, \max\{n(\xi^0_\epsilon), n(\zeta^0_\epsilon), n(\xi^1_\epsilon), n(\zeta^1_\epsilon)\} = n\}.$$

By the pigeonhole principle, we may find $m < \omega$ such that $|\mathcal{B}_m| = \lambda^+$. In particular, $\mathcal{B}_m \subseteq [\lambda^+]^{<\omega}$ and $\mathrm{mup}(\mathcal{B}_m) = \lambda^+$. Now, by Fact 5.3, Clause (2), for every $n < \omega$, the set G_n of all $\alpha \in S_n$ such

$$\mathrm{mup}\{x \in \mathcal{B}_m \mid x \subseteq D_\alpha\} = \alpha,$$

is stationary. In particular, for all $n > m$, $\alpha \in G_n$ and $i < 2$, $(s_\alpha)^2 \cap B_i = (D_\alpha \cap W_{n(\alpha)-1})^2 \cap B_i$ and it dominates (α, α). □

This completes the proof. □

At this point, the proof of Proposition 5.1 continues exactly as in [Sze10, Theorem 4], using the sequences $\langle S_n \mid n < \omega \rangle$ and $\langle s_\alpha \mid \alpha < \lambda^+ \rangle$ constructed in the preceding lemma.

Remark 5.5. Lemma 5.4 also implies that $\mathrm{P}^-_\omega(\omega_1, 2, {}_{\omega_1}{\sqsubseteq}, \omega, \{\omega_1\})$ is a weakening of $\clubsuit^*$ sufficient for the constructions of [Goo95a].

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