CRM-Fields-PIMS Prize Lecture

Stevo Todorcevic on Walks on Ordinals and Their Characteristics

THE 2012 CRM-FIELDS-PIMS PRIZE LECTURE BEGAN with a brief overview of Stevo Todocevic's work, given by his former student Justin Tatch Moore. Moore spoke about Todorcevic's 1998 ICM lecture and how it has served as an inspiration for some of his own work, and invited anyone with background in analysis to read Todorcevic's 1999 paper on compact subsets of the first Baire class, where set-theoretic forcing was applied to this field in an unconventional way (in this paper, theorems, rather than consistency results, concerning compact sets of Baire class-one functions are obtained by analyzing the corresponding objects in forcing extensions of the universe).

Todorcevic's lecture at the Fields Institute (his third lecture related to the 2012 CRM-Fields-PIMS prize) focused on his method of minimal walks on ordinals. This method was discovered in the early 1980's when Todorcevic learned from Galvin about the problem of finding a Ramsey basis for structures on the first uncountable ordinal ω_1. This is of course a part of a very general problem, but a simple form in this particular context asks whether there (consistently) exists a natural number $n = n(\omega_1)$ such that for any symmetric function $f:[\omega_1]^2 \to K$ with some finite range K there exists a set $C \subseteq K$ of size at most $n(\omega_1)$ together with an uncountable set X, such that $f(x,y) \in C$ for all distinct x, y in X. Galvin and Shelah proved in 1973 that $n(\omega_1)$ — if it exists — must be bigger than or equal to 4. While analyzing this problem in the early 1980s, Todorcevic showed that several well-known basis problems in mathematics could be solved if such a number exists. For example, he showed that $n(\omega_1) = 4$ would entail that the class of uncountable regular topological spaces has a 3-element basis, and that the class of uncountable linear orderings has a 5-element basis. However, not long afterwards, in a surprising turn of events in 1984, Todorcevic provided a negative answer to the fundamental Ramsey problem for ω_1, exhibiting a function $c:[\omega_1]^2 \to \omega_1$ with the remarkable property that c is *surjective* over all uncountable squares. In the construction of such a map, he used as motivation the "walks below ε_0" method which originated in proof theory, and which is one-dimensional in nature, and came up with a two-dimensional variation that goes all the way up to ω_1 (and, in fact, could be implemented on any given ordinal). The method of "minimal walks on ordinals" was born.

After discussing the above, Todorcevic gave a technical account on the method of minimal walks, and the associated characteristic functions, highlighting two of its main features: metric triangle inequalities, and canonicity. The former turned out to be a key in many constructions (for instance, in constructions of Banach spaces with no infinite unconditional basic subsequence), whereas the latter centers around comparing derivatives of minimal walks (e.g., Lipschitz trees) with objects that were previously constructed in an ad-hoc fashion. A very interesting instance of canonicity is the fact that minimal walks may be utilized to derive (in the presence of a forcing axiom) a selective ultrafilter on the set of natural numbers, which is Σ_1-definable in the structure $(H(\omega_2), \in)$ of sets that have hereditary cardinality no more than the first uncountable ordinal.

In conclusion of his lecture, Todorcevic also mentioned related works of his former students. On the classification side, Moore proved that the class of uncountable regular spaces does not admit a finite basis, and that, in the presence of a standard forcing axiom, the class of uncountable linear orderings has a 5-element basis, and the class of Aronszajn orderings has a 2-element basis. On the canonicity side, Martinez-Ranero proved that the same forcing axiom implies that the class of Aronszajn orderings is well-quasi-ordered by the relation of isomorphic embeddability.

The CRM-Fields-PIMS prize is awarded annually. Its main criterion for selection is outstanding contribution to the advancement of research. A video of the talk can be found at www.fields.utoronto.ca/video-archives

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