The search for diamonds

Saharon Shelah. *Middle Diamond.* ***Archive for Mathematical Logic***, vol. 44 (2005), pp. 527–560.
Saharon Shelah. *Diamonds.* ***Proceedings of the American Mathematical Society***, vol. 138 no. 6 (2010), pp. 2151–2161.
Martin Zeman. *Diamond, GCH and Weak Square.* ***Proceedings of the American Mathematical Society***, vol. 138 no. 5 (2010), pp. 1853–1859.

The Continuum Hypothesis (CH) implies, and is in fact equivalent to, the existence of an enumeration $\{A_\alpha : \alpha < \omega_1\}$ of *all* bounded subsets of ω_1. Given a set $A \subseteq \omega_1$, and an ordinal $\alpha < \omega_1$, we say that the above enumeration *predicts* A at α, if $A \cap \alpha = A_\alpha$. Jensen's diamond principle, $\diamondsuit$, is the strengthening of CH asserting the existence of an enumeration $\{A_\alpha : \alpha < \omega_1\}$ of all bounded subsets of ω_1 such that every subset $A \subseteq \omega_1$ gets predicted at some limit $\alpha < \omega_1$.

Now, consider the following generalization. Given a stationary set $S \subseteq \kappa$, $\diamondsuit_S$ asserts the existence of an enumeration $\{A_\alpha : \alpha \in S\}$ such that for every $A \subseteq \kappa$, the set $\{\alpha \in S : A_\alpha = A \cap \alpha\}$ is stationary. This concept has been introduced by R. B. Jensen in the late 60's of the last century; Jensen proved that $\diamondsuit_S$ holds in Gödel's constructible universe for every stationary $S \subseteq \kappa$ and every regular uncountable cardinal κ, and introduced the very first $\diamondsuit$-based construction of a complicated combinatorial object — a Souslin tree.

Since then, these principles have drawn a considerable attention, and have been used to solve problems, not just in logic, but also in real analysis, group theory, and topology.

Motivated by the utility of diamonds, the community began to study the validity of these principles, as well as weaker variants. Already in the 1970's it was known that $\diamondsuit_{\omega_1}$ is equivalent to several seemingly-weaker statements (Devlin, Kunen), that GCH $+\neg\diamondsuit_{\omega_1}$ is consistent (Jensen), that GCH $+\diamondsuit_{\omega_1} + \neg\diamondsuit_S$ for some stationary $S \subseteq \omega_1$ is consistent (Shelah), that $\diamondsuit_\kappa$ holds for every measurable cardinal κ (Kunen), that $\diamondsuit_{2^\omega}$ holds if 2^ω is real-valued measurable (Ketonen), and that GCH implies $\diamondsuit_{\kappa^+}$ for every uncountable cardinal κ (Jensen, Gregory, Shelah). In the 1980's, Woodin established the consistency of $\neg\diamondsuit_\kappa$ for a Mahlo cardinal κ, while Shelah, dealing with successor cardinals, established the consistency of GCH $+\neg\diamondsuit_{T_\kappa}$ for a regular uncountable κ and $T_\kappa := \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \mathrm{cf}(\kappa)\}$, as well as, GCH $+\neg\diamondsuit_S$ for a singular cardinal κ and a non-reflecting stationary subset $S \subseteq T_\kappa$. In the 1990's, Hauser, improving an earlier result of Woodin, showed that $\diamondsuit_{\mathrm{Reg}(\kappa)}$ may consistently fail for indescribable cardinals κ, where $\mathrm{Reg}(\kappa) := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \alpha\}$, and Shelah proved that $2^\kappa = \kappa^+$ entails $\diamondsuit_{\kappa^+}$ for every cardinal $\kappa \geq \beth_\omega$.[1]

The three papers under review continues this line of research. We commence with describing the main results of these papers, and subsequently, we shall be discussing the leading arguments of the involved proofs.

The main result of *Diamonds* is as follows: For a given subset $S \subseteq \kappa^+$ and an uncountable cardinal κ, if $S \setminus T_\kappa$ is stationary, then $2^\kappa = \kappa^+$ implies $\diamondsuit_S$. In particular, $2^\kappa = \kappa^+$ implies $\diamondsuit_{\kappa^+}$ for every uncountable cardinal κ.[2]

As $\diamondsuit_{\kappa^+}$ yields an enumeration witnessing that $2^\kappa = \kappa^+$, Shelah's theorem provides a complete understanding of the relation between the value of 2^κ, and the validity of diamond over stationary subsets of $\kappa^+ \setminus T_\kappa$, thus, concluding a 40 year old search for such an understanding(!).

But there are further questions. For instance, the preceding theorem leaves the following questions untouched:

[1] Let us stress that the above list is far from being complete.

[2] Recall that by Shelah's works from the 80's, diamond may indeed fail over stationary subsets of T_κ.

1. The mathematical question: For a singular cardinal κ satisfying $2^\kappa = \kappa^+$, how large is the class of subsets of T_κ on which diamond may fail? In particular, is GCH $+\neg\diamondsuit_{T_\kappa}$ consistent?
2. The metamathematical question: Can one find a non-trivial prediction principle for κ^+ which is provable even assuming $2^\kappa > \kappa^+$?[3]

One of the standard ways of attacking the first question is to study sufficient conditions that impose a link between elements of $\check{T}_\kappa := \kappa^+ \setminus T_\kappa$ (on which, diamond holds), to elements of T_κ (on which, we would like to establish diamond). Of course, reflection of stationary sets yields a link of this kind, and indeed, in the second and third papers under review, instances of reflection are shown to entail a partial answer to the first question.

In *Diamonds*, Shelah proves that for every singular cardinal κ of uncountable cofinality, $2^\kappa = \kappa^+$ implies $\diamondsuit_{T_\kappa}$ provided that κ^+ weakly reflects at $\mathrm{cf}(\kappa)$. In *Diamond, GCH and Weak Square*, Zeman, improving a theorem from an early paper of Shelah (*Diamonds, uniformization*, JSL XLIX 1022), proves that for every singular cardinal κ, if $2^\kappa = \kappa^+$ and $\square^*_\kappa$ holds, then $\{S \subseteq T_\kappa : \diamondsuit_S \text{ fails}\}$ omits any set that reflects stationarily often.[4]

Next, let us deal with the second, metamathematical, question.

One of the earliest attempts to provide an answer to this type of question may be found in a paper by Devlin and Shelah (*A weak form of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, ***Israel Journal of Mathematics*** XXIX 239). To describe their result, let us first point out that $\diamondsuit_\kappa$ is equivalent to the existence of an enumeration of functions $\{g_\alpha : \alpha < \kappa\}$ such that for every function $f : \kappa \to 2$, the set $\{\alpha < \kappa : g_\alpha = f \restriction \alpha\}$ is stationary. Thus, in their paper, Devlin and Shelah introduce the following prediction principle:

(Φ) For every coloring $F : \bigcup_{\alpha<\omega_1} {}^\alpha 2 \to 2$, there exists a coloring $g : \omega_1 \to 2$, such that for every function $f : \omega_1 \to 2$, the set $\{\alpha < \omega_1 : g(\alpha) = F(f \restriction \alpha)\}$ is stationary.

They named this principle by *weak diamond*, and proved that it follows from $2^{\aleph_0} < 2^{\aleph_1}$. So, not only that weak diamond follows outright from CH, but, more importantly, it is a reminiscent of diamond which is consistent with $\neg$ CH.

Skipping a vast body of work concerning generalizations (as well as applications) of the weak diamond, we now arrive to Shelah's paper, entitled *Middle Diamond* — a paper in which a striking answer to the metamathematical question is given.

Let us say that κ has the (θ, χ)-*middle diamond*, if there exists an enumeration $\{g_\alpha : \alpha < \kappa\}$ such that for every function $f : \kappa \to \chi$, the set of all $\alpha < \kappa$ satisfying:

- $\mathrm{cf}(\alpha) = \theta$, and
- there exists a club subset $c_\alpha \subseteq \alpha$, such that $g_\alpha = f \restriction c_\alpha$,

is stationary.

Nota bene that the above clubs c_α could be of a rather small size, and this is why it is not a-priori impossible for κ^+ to enjoy middle diamond while satisfying $2^\kappa > \kappa^+$.

Indeed, in the first paper under review, Shelah introduces middle diamond and establishes that many instances of it are consequences of ZFC. To exemplify: Suppose that μ is a limit of strong limit cardinals, and $\chi < \mu$; then for every regular cardinal $\kappa > \mu$ which is not strongly inaccessible, the set $\{\theta < \mu : \kappa \text{ has the } (\theta, \chi)\text{-middle diamond}\}$ contains a non-empty final segment of regular cardinals.

---

[3] Of course, the Ostaszewski principle, $\clubsuit_{\kappa^+}$, is a non-trivial guessing principle which is consistent with $2^\kappa > \kappa^+$. The point is that the Ostaszewski principle is not provable in ZFC.

[4] Recall that $\square^*_\kappa$ asserts the existence of a *coherent* local clubs sequence $\langle c_\alpha : \alpha < \kappa^+\rangle$. That is, each c_α is a club in α of type $\leq \kappa$, and $\{c_\alpha \cap \beta : \alpha < \kappa^+\}$ has size $< \kappa^+$ for all $\beta < \kappa^+$.

In particular, this shows that for every successor cardinal $\kappa > \beth_{\omega_1}$, there exists a sequence of local clubs $\langle c_\alpha : \alpha < \kappa \rangle$, and an enumeration $\{g_\alpha : \alpha < \kappa\}$, such that for every function $f : \kappa \to 2$, the set $\{\alpha < \kappa : g_\alpha = f \restriction c_\alpha\}$ is stationary.

We now turn to discuss some of the proofs from the papers under review. For this, it is helpful to establish a jargon. We do so by revisiting the following well-known claims:

- Kunen's theorem: $\diamondsuit_{\kappa^+}$ is equivalent to the existence of a matrix enumeration $\{A^i_\alpha : (\alpha, i) \in \kappa^+ \times \kappa\}$ such that $\{\alpha < \kappa^+ : \exists i < \kappa(A^i_\alpha = A \cap \alpha)\}$ is stationary for every $A \subseteq \kappa^+$.
- Shelah's *club guessing* theorem: If κ is an uncountable cardinal, then there exists a collection $\{C_\alpha : \alpha < \kappa^+\}$ with $\sup(C_\alpha) = \alpha$ for all $\alpha < \kappa^+$, such that $\{\alpha < \kappa^+ : C_\alpha \subseteq E \cap \alpha\}$ is stationary for every club subset $E \subseteq \kappa^+$. (So, unlike diamond, where all subsets of κ^+ gets predicted, here, we settle for *guessing* just *closed and unbounded* subsets of κ^+.)

The idea of Kunen's proof is to use some coding of the sets A^i_α, together with projection maps π_i, in such a way that allows to argue that for some $i < \kappa$, $\{\pi_i(A^i_\alpha) : \alpha < \kappa^+\}$ successfully predicts all subsets of κ^+. Indeed, if the latter fails, then we may collect κ many independent counterexamples — one to each i — and then unify them into a single encoded set whose projections are these counterexamples. By the hypothesis, this encoded set gets predicted stationarily often, hence by Fodor's lemma it gets predicted on some fixed $i < \kappa$, and in particular, its i_{th} projection gets predicted, contradicting the choice of the i_{th} counterexample.

The idea of the club guessing proof is to start with an arbitrary potential guessing sequence $\{C_\alpha : \alpha < \kappa\}$, together with an operation $\mathrm{g}\ell_D$, and to recursively construct a decreasing sequence of clubs $\langle D_i : i < \kappa\rangle$, in such a way that allows to argue that for some $i < \kappa$, $\{\mathrm{g}\ell_{D_i}(C_\alpha) : \alpha < \kappa^+\}$ successfully guesses all club subsets of κ^+. As in Kunen's proof, towards a contradiction, we collect κ many counterexamples. However, here, we do so recursively, where at stage $i + 1$ of the recursion, we first improve our potential guessing sequence to make it correctly guess the i_{th} counterexample, and then we pick a new counterexample, witnessing that the just constructed sequence still fails to guess all clubs. The contradiction we meet here is that a potential guessing sequence cannot be improved "forever".

Now, the proof of the main theorem from *Diamonds* may be viewed as a combination of the above two. The outline of the proof is as follows. Since $S \cap \breve{T}_\kappa$ is statinoary, we may pick a matrix indexed by $S \times \kappa$ that has a very rough, yet non-trivial, predicting feature, commonly dubbed as *hitting*. By $2^\kappa = \kappa^+$, we can fix for each $i < \kappa$, a certain projection map $\mathrm{cd}_i : \kappa^+ \to \kappa^+$. Then, by an argument a-la Kunen, we find some $i < \kappa$, such that the projection cd_i of the i_{th} column alone enjoys the very same hitting feature of the whole matrix. Next, we run a club guessing type of argument, using this projected column as our initial potential predicting sequence, which we keep improving recursively via counterexamples. Finally, a second projection argument is invoked, establishing that the recursive improvement process cannot be carried endlessly.

It is worth mentioning that the above argument is not really limited to stationary subsets of $\breve{T}_\kappa$. In fact, this proof applies to any stationary set (or just normal filters, concentrating on a set) that admits a hitting matrix. Recalling that Džamonja and Shelah showed that T_κ admits a hitting matrix whenever κ^+ weakly reflects at $\mathrm{cf}(\kappa)$, it should now be clear how to prove the other result from the *Diamonds* paper.

While weak reflection is used to *pump up* diamond from $\breve{T}_\kappa$ to T_κ, in Zeman's paper, we go in the inverse direction. The outline is as follows. If $S \subseteq T_\kappa$ reflects stationarily often, then it necessarily reflects within $\breve{T}_\kappa$. So, by the coherence of the square sequence, it is possible to *pull* the arguments that work successfully for $\breve{T}_\kappa$, *down* to the set S.

Zeman's paper commences with the introduction of a new principle, denoted $\bigcirc_\kappa(S)$, which has been isolated from a simplified presentation, due to P. Komjáth, of the second paper under review. Then it is proved that $\bigcirc_\kappa(S)$ implies $\diamondsuit_S$ for every stationary $S \subseteq \kappa$ and every regular uncountable cardinal κ (including the case that κ is inaccessible!). And, finally, it is proved that the above-mentioned hypotheses entails $\bigcirc_{\kappa^+}(S)$.

We now turn to describe the structure of the first paper under review.

In Section 1 of *Middle Diamond*, a sufficient condition for the validity of middle diamond over cardinals of the form $\mathrm{cf}(2^\mu)$, is given. Of course, if $2^\mu = \mu^+$, then we have natural projection maps that enable us to run a Kunen-type of argument, but how do we diagonalize otherwise?

Shelah's solution to this problem is in the form of a newly introduced principle, denoted $\mathrm{Sep}(\mu,\chi)$, which commutes nicely with certain projection maps, $\mathrm{cd}_i : 2^\mu \to 2^\mu$. The *main claim* of this section is: If $\lambda = 2^\mu$, $\kappa = \mathrm{cf}(\lambda)$ and some stationary $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \theta\}$ carries a reasonably-coherent local clubs sequence, $\overline{c} = \langle c_\alpha : \alpha \in S\rangle$, then $\mathrm{Sep}(\mu,\chi)$ together with an arithmetic hypothesis involving λ and θ, yields the following relative of weak diamond:

($\Phi_{\overline{c},\lambda,\chi}$) For every coloring $F : \bigcup_{\alpha\in S} {}^{(c_\alpha)}\lambda \to \chi$, there exists a coloring $g : S \to \chi$, such that for every $f : \kappa \to \lambda$, the set $\{\alpha \in S : g(\alpha) = F(f \restriction c_\alpha)\}$ is stationary.[5]

The proof of the latter may be viewed as a fine, yet natural, abstraction of the Kunen argument, where each ingredient of the original argument admits a subtle counterpart that allows to overcome the unrelaxed working conditions.

Apparently, at this point of the paper, the value of the main claim is not completely understood; for instance, it is unclear whether its hypothesis is commonly satisfiable, and whether its assertion entails that κ has the (θ,χ)-middle diamond. Fortunately, the rest of the paper is devoted to clarifying these aspects.

At the second part of Section 1, sufficient conditions for $\mathrm{Sep}(\mu,\chi)$ to hold, and for $\Phi_{\overline{c},\lambda,\chi}$ to imply middle diamond, are provided. In addition, Shelah addresses the above-mentioned arithmetic hypothesis "involving λ and θ" by recalling results from his seminal paper *The generalized continuum hypothesis revisited*, ***Israel Journal of Mathematics*** CXVI 285.

In Section 2, the existence of a stationary subset of $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \theta\}$ that carries a "reasonably-coherent" local clubs sequence is studied. For a cardinal κ which is the successor of a regular cardinal, the existence of such a stationary set has already been established in Shelah's paper *Reflecting stationary sets and successors of singular cardinals*, ***Archive for Mathematical Logic***, XXXI 25. As for cardinals which are not of the above form, the existence of such a stationary set is obtained here as a corollary to a generalization of the Engelking-Karłowicz theorem, whose proof occupies a large portion of this section.

By the end of the paper, it is understood that the hypothesis of the main claim is indeed commonly satisfiable, and that (θ,χ)-middle diamond is derivable from $\Phi_{\overline{c},\lambda,\chi'}$ for a large enough χ'. Thus, the only gap from our original goal is that the main claim applies to cardinals of the form $\mathrm{cf}(2^\mu)$, while we wanted to yield middle diamond over cardinals of additional patterns. For this, the paper is concluded with two theorems that address the question of lifting up the middle diamond from cardinals of the form $\mathrm{cf}(2^\mu)$, to arbitrary regular cardinals.

The moral of this paper is that while GCH, $\square_\kappa$ or $\diamondsuit_\kappa$ may fail everywhere, non-trivial approximations of these principles are almost always available.

Assaf Rinot
School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. review01@rinot.com.

[5] So $\Phi \equiv \Phi_{\overline{c},\lambda,\chi}$, for $\overline{c} = \langle \alpha : \alpha \in \omega_1\rangle$, and $\lambda = \chi = 2$.