SURPRISINGLY SHORT

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Abstract. This note is a compilation of several results in set theory which has surprisingly short proofs. From time to time (e.g., whenever I write down notes for my tutor lectures), more results will be added to this compilation.

Contents

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1. Diamond principles

1.1. **Jensen's diamond is (nearly) a cardinal arithmetic statement.** Recall that for a regular uncountable cardinal θ and a stationary subset $S \subseteq \theta$, $\diamondsuit_S$ is said to hold iff there exists a collection $\{S_\delta \mid \delta \in S\}$ such that for any $Z \subseteq \theta$, the set $\{\delta \in S \mid S_\delta = Z \cap \delta\}$ is stationary.

Evidently, if λ is a cardinal and $\{S_\delta \mid \delta \in \lambda^+\}$ is a collection witnessing $\diamondsuit_{\lambda^+}$, then $[\lambda^+]^{\leq\lambda} = \{Z \subseteq \lambda^+ \mid |Z| \leq \lambda\} \subseteq \{S_\delta \mid \delta \in \lambda^+\}$. In particular, $\diamondsuit_{\lambda^+}$ implies $2^\lambda = \lambda^+$. The result of this section deals with the inverse implication and has been established by Shelah [8]. Soon afterwards, Péter Komjáth found a more friendly proof [3], and sometime later, we found a considerably shorter proof which avoids the first ingredient of the two ingredients of Shelah's proof. The next proof is extracted from [5].

Theorem (Shelah). *Suppose λ is a cardinal satisfying $2^\lambda = \lambda^+$.*
Then $\diamondsuit_S$ holds for any stationary $S \subseteq \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}$.

Proof. For all $\delta < \lambda^+$, let $\{A^i_\delta \mid i < \mathrm{cf}(\lambda)\} \subseteq [\delta \times \delta]^{<\lambda}$ be an increasing chain converging to $\delta \times \delta$. By $2^\lambda = \lambda^+$, let $\{X_\beta \mid \beta < \lambda^+\}$ be an enumeration of $[\lambda \times \lambda \times \lambda^+]^{\leq\lambda}$. For all $(i,\tau) \in \lambda \times \lambda$ and $X \subseteq \lambda \times \lambda \times \lambda^+$, let $\pi_{i,\tau}(X) := \{\gamma < \lambda^+ \mid (i,\tau,\gamma) \in X\}$. For a set $B \subseteq \lambda^+ \times \lambda^+$ and $(i,\tau) \in \lambda \times \lambda$, denote:

$$(B)_{i,\tau} := \bigcup\{\pi_{i,\tau}(X_\beta) \mid (\alpha,\beta) \in B \text{ for some } \alpha\}.$$

Now, suppose $S \subseteq \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)\}$ is a given stationary set.

Claim. *There exists $(i,\tau) \in \lambda \times \lambda$, and for all $\delta \in S$, there exists $B_\delta \subseteq A^i_\delta$ such that $\langle (B_\delta)_{i,\tau} \mid \delta \in S\rangle$ is a $\diamondsuit_S$ sequence.*

Proof. Suppose not, we build by recursion on $\tau < \lambda$, three sequences:

(I) $\langle \{Z^i_\tau \mid i < \lambda\} \mid \tau < \lambda\rangle$;
(II) $\langle \{C^i_\tau \mid i < \lambda\} \mid \tau < \lambda\rangle$;
(III) $\langle \{A^i_\delta(\tau) \mid i < \lambda, \delta \in S \cap C^i_\tau\} \mid \tau < \lambda\rangle$.

Base case, $\tau = 0$. By the hypothesis, for all $i < \lambda$, $\langle (A^i_\delta)_{i,0} \mid \delta \in S\rangle$ is not a $\diamondsuit_S$ sequence, so pick a set $Z^i_0 \subseteq \lambda^+$ and a club $C^i_0 \subseteq \lambda^+$ witnessing that. Namely, fixing $i < \lambda$ and $\delta \in S \cap C^i_0$, we have:

$$Z^i_0 \cap \delta \neq (A^i_\delta)_{i,0} = \bigcup\{\pi_{i,0}(X_\beta) \mid (\alpha,\beta) \in A^i_\delta \text{ for some } \alpha\}.$$

In particular, either there exists some $(\alpha,\beta) \in A^i_\delta$ such that $Z^i_0 \cap \alpha \neq \pi_{i,0}(X_\beta)$, or $\sup\{\alpha < \delta \mid (\alpha,\beta) \in A^i_\delta\} < \alpha$. In the latter case, put $A^i_\delta(0) := A^i_\delta$. In the former case, pick $(\alpha,\beta) \in A^i_\delta$ with $Z^i_0 \cap \alpha \neq \pi_{i,0}(X_\beta)$ and let $A^i_\delta(0) := A^i_\delta \setminus \{(\alpha,\beta)\}$.

Now, assume that the three sequences are defined up to some τ. Let $D := \bigcap\{C^i_\epsilon \mid \epsilon < \tau, i < \lambda\}$. For all $\delta \in D \cap S$, and $i < \lambda$, put $B^i_\delta := \bigcap\{A^i_\delta(\epsilon) \mid \epsilon < \tau\}$. By the hypothesis, for all $i < \lambda$, $\langle (B^i_\delta)_{i,\tau} \mid \delta \in S \cap D\rangle$ is not a $\diamondsuit_S$ sequence, so pick a set $Z^i_\tau \subseteq \lambda^+$ and a club $C^i_\tau \subseteq D$ witnessing that. As before, for all $i < \lambda$ and $\delta \in S \cap C^i_\tau$, put $A^i_\delta(\tau) := B^i_\delta$ in the case that $\sup\{\alpha < \delta \mid (\alpha, \beta) \in B^i_\delta\} < \delta$, or else, let $A^i_\delta(\tau) := B^i_\delta \setminus \{(\alpha, \beta)\}$ for some $(\alpha, \beta) \in B^i_\delta$ satisfying $Z^i_\tau \cap \alpha \neq \pi_{i,\tau}(X_\beta)$.

This completes the construction. To meet a contradiction, put $Z := \{(i, \tau, \gamma) \mid i < \lambda, \tau < \lambda, \gamma \in Z^i_\tau\}$ and define a function $f : \lambda^+ \to \lambda^+$ by letting:

$$f(\alpha) := \min\{\beta < \lambda^+ \mid Z \cap (\lambda \times \lambda \times \alpha) = X_\beta\}, \quad (\alpha < \lambda^+).$$

Since $\{\delta < \lambda^+ \mid f[\delta] \subseteq \delta\}$ is a club, pick $\delta \in S \cap \bigcap_{\tau<\lambda} \bigcap_{i<\lambda} C^i_\tau$ with $f[\delta] \subseteq \delta$. Since $f \restriction \delta \subseteq \delta \times \delta = \bigcup_{i<\mathrm{cf}(\lambda)} A^i_\delta$, let us define $g : \delta \to \mathrm{cf}(\lambda)$ as follows:

$$g(\alpha) = \min\{i < \mathrm{cf}(\lambda) \mid (\alpha, f(\alpha)) \in A^i_\delta\}, \quad (\alpha < \delta).$$

As $\delta = g^{-1}[\mathrm{cf}(\lambda)]$ and $\mathrm{cf}(\delta) \neq \mathrm{cf}(\lambda)$, there must exist some $i^* < \mathrm{cf}(\lambda)$ such that $H := g^{-1}[i^*]$ is cofinal in δ. By $A^{i^*}_\delta \supseteq \bigcup_{i<i^*} A^i_\delta$, this means that $f \restriction H \subseteq A^{i^*}_\delta$. Recall that by definition of f, if $\alpha \in H$, and $\beta = f(\alpha)$, then $Z^{i^*}_\tau \cap \alpha = \pi_{i^*,\tau}(X_\beta)$ for all $\tau < \lambda$. It now follows from $f \restriction H \subseteq A^{i^*}_\delta$ and the definition of the construction that $f \restriction H \subseteq A^{i^*}_\delta(\tau)$ for all $\tau < \lambda$. In particular, $\sup\{\alpha < \delta \mid (\alpha, \beta) \in A^{i^*}_\delta(\tau)\} \geq \sup(H) = \delta$ for all $\tau < \lambda$, and hence $\langle A^{i^*}_\delta(\tau) \mid \tau < \lambda\rangle$ must be a strictly decreasing sequence of subsets of $A^{i^*}_\delta$, contradicting the fact that $|A^{i^*}_\delta| < \lambda$. □

□

It is worth mentioning that for stationary subsets of $\{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$, there are consistency results concerning the failure of diamond. For instance, Jensen proved that CH is consistent with $\neg\diamondsuit_{\omega_1}$, and Shelah established the consistency of GCH with $\neg\diamondsuit_S$ for $S = \{\delta < \aleph_2 \mid \mathrm{cf}(\delta) = \aleph_1\}$. As for subsets of $\{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$, where λ is *singular* — the situation here is subtle, and we refer the interested reader to the following survey presentation:

`http://www.tau.ac.il/~rinot/rinot_best18.pdf`

1.2. **Magidor's notion of silly diamond.** In order to prove Corollary 1.3 below, Magidor have introduced the notion of *silly diamond*, where instead of guessing subsets of λ^+, we are only required to guess subsets of λ.

Theorem 1.1 (Magidor)**.** *Suppose $M \subseteq V$ is model of ZFC, λ is a cardinal in V, $\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)^V$ and $2^\lambda = \lambda^+$.*

Then for every $S \subseteq \lambda^+$ in M which is stationary in V, there exists a sequence in M which is a silly diamond sequence over S for V, that is, there exists a sequence $\vec{A} = \langle A_\delta \mid \delta \in S\rangle$ such that:

1. *$\vec{A} \in M$;*
2. *$V \models \forall X \subseteq \lambda\,(\{\delta \in S \mid A_\delta = X\}$ is stationary).*

Proof. Work in M. Fix a an enumeration $\{Y_\alpha \mid \alpha < \lambda^+\}$ of $\mathcal{P}(\lambda \times \lambda)$. For all $\delta < \lambda^+$, let $\{Y^i_\delta \mid i < \lambda\}$ be some enumeration of $\{Y_\alpha \mid \alpha < \delta\}$. For all $\delta \in S$, put $A^i_\delta = \{\alpha < \lambda \mid (i,\alpha) \in Y^i_\delta\}$.

Claim 1.2. *There exists some $i < \lambda$ such that $\vec{A} = \langle A^i_\delta \mid \delta \in S\rangle$ works.*

Proof. Suppose not. Work in V. Then, for all $i < \lambda$, we may pick a set $X^i \subseteq \lambda$ and a club $C^i \subseteq \lambda^+$ such that $A^i_\delta \neq X^i$ for all $\delta \in S \cap C^i$. Put $Y := \bigcup_{i<\lambda}\{i\} \times X^i$, and find $\alpha < \lambda^+$ such that $Y = Y_\alpha$. Fix $\delta \in S \cap \bigcap_{i<\lambda} C^i$ with $\delta > \alpha$, and fix $j < \lambda$ such that $Y = Y^j_\delta$. Then in particular, $A^j_\delta = \{\alpha < \lambda \mid (j,\alpha) \in Y\} = X^j$, a contradiction to $\delta \in S \cap C^j$. □

□

Corollary 1.3. *Suppose $M \subseteq V$ is model of ZFC, λ is a cardinal in V, $\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)^V$ and $2^\lambda = \lambda^+$.*

Then, in M there exists a partition of $\{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ into λ^+ many V-stationary sets.

Proof. Put $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$. Notice that by $\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)^V$, we have that M and V agrees on the cardinal structure up to λ^+, that $S \in M$, and that S is stationary in V.

Now, let $\langle A_\delta \mid \delta \in S\rangle$ be a silly diamond sequence for S given by the preceding theorem. For all $X \subseteq \lambda$, denote $S_X := \{\delta \in S \mid A_\delta = X\}$. Then S_X is stationary in V, and the partition $S = \biguplus\{S_X \mid X \subseteq \lambda\}$ lies in M. □

Andrés Caicedo pointed out that the preceding corollary is actually a well-known consequence of a theorem by Erdös, Hajnal and Milner. Yet, it appears that Magidor was unaware of it.

We also thank Andrés Caicedo for communicating to us a result by Paul Larson, showing that the hypothesis "$2^\lambda = \lambda^+$" in the preceding corollary cannot be dropped.

2. SCH holds above a strongly-compact cardinal

Fix a regular cardinal λ. Recall that for a cardinal κ, $\mathcal{P}_\kappa(\lambda)$ denotes the family of subsets of λ of cardinality less than κ. An ultrafilter U over $\mathcal{P}_\kappa(\lambda)$ is *fine* iff for every $\alpha < \lambda$, the 'final segment' $\hat{\alpha} := \{X \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in X\}$ is a member of U.

Definition 2.1. A cardinal κ is λ*-strongly compact* iff there exists a κ-complete fine ultrafilter over $\mathcal{P}_\kappa(\lambda)$.

A cardinal κ is *strongly compact* iff it is λ-compact for all cardinals $\lambda \geq \kappa$.

This section will be dedicated to proving the following theorem from [9]:

Theorem (Solovay)**.** *The Singular Cardinal Hypothesis (SCH) holds above the first strongly compact cardinal (if exists).*

By a celebrated result of Silver from [7], to show that SCH holds above a cardinal κ, it suffices to prove that $\lambda^{\aleph_0} = \lambda$ for all regular $\lambda \geq \kappa$. The latter will be established in this section. The simplified proof is due to ???

Lemma 2.2. *Suppose κ is a λ-strongly compact cardinal, then every collection of less than κ many stationary subsets of $\{\beta < \lambda \mid \mathrm{cf}(\beta) < \kappa\}$ mutually reflects at some $\delta < \lambda$ of cofinality $< \kappa$.*

Proof. Let U is an ultrafilter witnessing that κ is λ-strongly compact. Consider ${}^{\mathcal{P}_\kappa(\lambda)}V/U$, an ultrapower of the universe. Since U is κ-complete, the ultrapower is well-founded, and we may identify it with its transitive collapse, M. For each x in V, let C_x denote the constant function $C_x : \mathcal{P}_\kappa(\lambda) \to \{x\}$. Now, defining $j : V \to M$ by letting $j(x) := [C_x]_U$ for all $x \in V$ yields an elementary embedding from the universe to a transitive proper class.

Consider the identity function, $\mathrm{id} : \mathcal{P}_\kappa(\lambda) \to \mathcal{P}_\kappa(\lambda)$. Since $\{X \in \mathcal{P}_\kappa(\lambda) \mid |\,\mathrm{id}(X)| < C_\kappa(X)\} = \mathcal{P}_\kappa(\lambda) \in U$, we get that $|[\mathrm{id}]_U| < j(\kappa)$. Since U is fine, for all $\alpha < \lambda$, we have $\{X \in \mathcal{P}_\kappa(\lambda) \mid C_\alpha(X) \in \mathrm{id}(X)\} \in U$, and hence $j(\alpha) \in [\mathrm{id}]_U$. Thus $j``\lambda \subseteq [\mathrm{id}]_U$ and $|j``\lambda| < j(\kappa)$. Put $\delta := \sup(j``\lambda)$. So $\mathrm{cf}(\delta) < j(\kappa)$. Also, since $\sup(X) < \lambda$ for all $X \in \mathcal{P}_\kappa(\lambda)$, we get from $j``\lambda \subseteq [\mathrm{id}]_U$ that $\delta < j(\lambda)$.

Fix $\mu < \kappa$ and suppose $\mathcal{S} = \{S_\alpha \mid \alpha < \mu\}$ is a collection of stationary subsets of $\{\beta < \lambda \mid \mathrm{cf}(\beta) < \kappa\}$. Since U is κ-complete, we get that $j \restriction \kappa$ is the identity function. In particular, $j(\mu) = \mu$ and $j(\mathcal{S}) = \{j(S_\alpha) \mid \alpha < \mu\}$.

Let $\alpha < \mu$ be arbitrary. We have $j``S_\alpha \subseteq j(S_\alpha)$ and actually $j``S_\alpha \subseteq j(S_\alpha) \cap \delta$. Suppose now C is a club in δ. Since j is elementary and $j \restriction \kappa$ is the identity function, we get that $j(\beta) = \sup(j``\beta)$ for every limit ordinal $\beta$ with $\mathrm{cf}(\beta) < \kappa$. So, $j``\lambda$ is a κ-club in δ, and hence $C \cap j``\lambda$ is a $\kappa$-club. Let $D \subseteq \lambda$ in $V$ be such that $j``D = C \cap j``\lambda$. Then D is a κ-club. Since S_α is stationary, and $\beta \in S_\alpha \Rightarrow \mathrm{cf}(\beta) < \kappa$, S_α must meet the κ-club D. So $S_\gamma \cap D \neq \emptyset$, and hence $j(S_\alpha) \cap C \neq \emptyset$.

Thus, we have shown that:

$$M \models \exists \delta < j(\lambda)\,(\mathrm{cf}(\delta) < j(\kappa), \forall S \in j(\mathcal{S})(S \cap \delta \text{ is stationary}))\,.$$

Consequently:

$$V \models \exists \delta < \lambda\,(\mathrm{cf}(\delta) < \kappa, \forall S \in \mathcal{S}(S \cap \delta \text{ is stationary}))\,. \quad \square$$

Corollary 2.3. *Suppose $\lambda \geq \kappa$ are regular cardinals.*
If κ is strongly compact, then $\lambda^{\aleph_0} = \lambda$.

Proof. Clearly, it suffices to prove that $\lambda^{<\kappa} = \lambda$. Let $\langle S_\alpha \mid \alpha < \lambda\rangle$ be a partition of $\{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \omega\}$ into λ many mutually-disjoint stationary sets. For each $\delta < \lambda$, consider the set $A_\delta = \{\alpha < \lambda \mid S_\alpha \cap \delta \text{ is stationary }\}$. Clearly, if $\delta < \lambda$ and $\mathrm{cf}(\delta) < \kappa$, then there exists a club subset of δ of cardinality $< \kappa$. Consequently, $|A_\delta| < \kappa$ for such δ (because the stationary subsets are mutually-disjoint), and also $|\mathcal{P}(A_\delta)| < \kappa$, because κ is strongly inaccessible.

Finally, by the previous lemma, for all $a \in [\lambda]^{<\kappa}$, there exists some $\delta < \lambda$ such that $a \subseteq A_\delta$. Thus, $[\lambda]^{<\kappa} \subseteq \bigcup\{\mathcal{P}(A_\delta) \mid \delta < \lambda, \mathrm{cf}(\delta) < \kappa\}$, and so $\lambda^{<\kappa} = \lambda$. $\square$

3. Partition relations

Recall the arrow notation: for ordinals $\kappa \leq \lambda$, $(\lambda) \to (\kappa)^2_2$ asserts that for any function $f : [\lambda]^2 \to 2$, there exists some $H \subseteq \lambda$ of order-type κ such that $|f``[H]^2| = 1$.[1]

A well-known theorem of Frank Ramsey states that $(\omega) \to (\omega)^2_2$, and its proof can be found in any relevant textbook. Consequently, $(\omega_1) \to (\omega)^2_2$.

In the next subsection it is shown that $(\omega_1) \to (\omega_1)^2_2$ does not hold, and actually a strong negation of it, due to Todorcevic, holds.

So, how about $(\omega_2) \to (\omega_1)^2_2$? In subsection 3.2 it is proved that the latter happens to be equivalent to the Continuum Hypothesis.

3.1. **A strong negation of** $(\omega_1) \to (\omega_1)^2_2$**.** This subsection will be dedicated to proving the following somewhat surprising theorem.

Theorem (Todorčević, [10])**.** *There exists a coloring $h : [\omega_1]^2 \to \omega_1$ such that every uncountable $Z \subseteq \omega_1$ satisfies $h``[Z]^2 = \omega_1$.*

The simplified presentation given here is due to Dan Velleman [11].

Definition 3.1. For every $B \subseteq \omega_1$, denote:

$$C^B := \{\delta < \omega_1 \mid B \subseteq \delta\} \cup \{\delta < \omega_1 \mid |B| = \aleph_1, \delta = \sup(B \cap \delta)\}.$$

Lemma 3.2. *For all $B \subseteq \omega_1$, C^B is a club.*

Proof. Fix $B \subseteq \omega_1$. To see that C^B is closed, suppose $\langle \alpha_n \mid n < \omega\rangle$ is an increasing sequence of elements of C^B, and let $\alpha := \sup_{n<\omega} \alpha_n$. If $B \subseteq \alpha$, then we are done. Otherwise, $B \not\subseteq \alpha_n$ for all $n < \omega$, then $|B| = \aleph_1$, and $\alpha_n = \sup(B \cap \alpha_n)$ for all $n < \omega$. To see that $\alpha = \sup(B \cap \alpha)$, we need to show that for all $\beta < \alpha$ there exists some $\gamma \in B \cap \alpha$ above β. Now, simply notice that if $\beta < \alpha$, then there exists some $n < \omega$ with $\beta < \alpha_n < \alpha$ and by $\alpha_n = \sup(B \cap \alpha_n)$, we may find some $\gamma \in B \cap \alpha_n$ above β.

To see that C^B is unbounded, pick $\beta < \omega_1$. If B is countable, then we may find some $\delta < \omega_1$ such that $B \subseteq \delta$, and then $\beta + \delta \in C^B$ as well. If B is uncountable, then just pick an increasing sequence of elements $\langle \alpha_n \mid n < \omega\rangle$ from B with $\alpha_0 > \beta$. Thus $\alpha := \sup_{n<\omega} \alpha_n$ satisfies $\alpha \in C^B$. □

For the sake of this proof, fix ω_1 many distinct functions $\{r_\alpha \mid \alpha < \omega_1\} \subseteq {}^{\omega}2$. For every $Z \subseteq \omega_1$, $n < \omega$ and $g : n \to 2$, denote $B^Z_g := \{\alpha \in Z \mid g \subseteq r_\alpha\}$.

Corollary 3.3. *For any $Z \subseteq \omega_1$, the following set is a club:*

$$C_Z := \left\{\delta < \omega_1 \mid \forall g \in {}^{<\omega}2 \left((B^Z_g \subseteq \delta) \text{ or } (|B^Z_g| = \aleph_1 \text{ and } \delta = \sup(B^Z_g \cap \delta))\right)\right\}.$$

Proof. Since $C_Z = \bigcap_{g \in {}^{<\omega}2} C^{B^Z_g}$ and the countable intersection of clubs is a club. □

[1] For a function g and a set $A \subseteq \mathrm{dom}(g)$, $g``A$ denotes the set $\{g(a) \mid a \in A\}$.

Theorem 3.4. *There exists a function $f : [\omega_1]^2 \to \omega_1$ such that $f``[Z]^2$ contains a club for any unbounded $Z \subseteq \omega_1$.*

Proof. For each $\alpha < \omega_1$, by $|\alpha| \leq \aleph_0$, let us fix some injection $e_\alpha : \alpha \to \omega$.

For any $\alpha < \beta < \omega_1$, let:

$$\Delta(\alpha,\beta) := \min\{n < \omega \mid r_\alpha(n) \neq r_\beta(n)\},$$

$$\Gamma(\alpha,\beta) := \{\gamma < \beta \mid e_\beta(\gamma) \leq \Delta(\alpha,\beta)\} \setminus \alpha.$$

Thus $\Gamma(\alpha,\beta) \subseteq [\alpha,\beta)$. If $\Gamma(\alpha,\beta)$ is empty, let $f(\alpha,\beta) := 0$, otherwise, let $f(\alpha,\beta) := \min\Gamma(\alpha,\beta)$. We claim that $C_Z \subseteq f[Z]$ for any unbounded $Z \subseteq \omega_1$.

Indeed, fix an unbounded $Z \subseteq \omega_1$ and some $\delta \in C_Z$. Since Z is unbounded, let us pick an arbitrary $\beta \in Z$ satisfying $\beta > \delta$. We now aim at finding some $\alpha < \beta$ with $\alpha \in Z$ such that $f(\alpha,\beta) = \delta$. Put $n := e_\beta(\delta)$, $g := r_\beta \restriction n$. Then $g \subseteq r_\beta$ and $\beta \in B^Z_g$. In particular, $B^Z_g \not\subseteq \delta < \beta$, so by $\delta \in C_Z$, we have $|B^Z_g| = \aleph_1$.

For each $\gamma \in B^Z_g$, put $m_\gamma := \Delta(\gamma,\beta)$. Clearly, $m_\gamma \geq n$. Evidently, there exists some uncountable subset $B \subseteq B^Z_g$ such that m_γ equals to some fixed $m < \omega$ for all $\gamma \in B$.

By shrinking further, we may also assume that $r_\gamma \restriction m+1 = h$ for some fixed $h : m+1 \to \omega$ for all $\gamma \in B$.

Thus, for B^Z_h, we have $B \subseteq B^Z_h$, i.e., $|B^Z_h| = \aleph_1$, and by $\delta \in C_Z$, we get that $\sup(B^Z_h \cap \delta) = \delta$. Let $F = \{\gamma < \delta \mid e_\beta(\gamma) \leq m\}$. Since e_β is an injection, F is finite and $\sup(F) < \delta$, and so there exists some $\alpha \in B^Z_h \cap \delta$ with $F \subseteq \alpha$. We claim that α works.

Indeed, by $\alpha \in B^Z_h$, we have $\Delta(\alpha,\beta) \geq m \geq n = e_\beta(\delta)$, i.e., $\delta \in \Gamma(\alpha,\beta)$. We need to show that δ is minimal in that sense. Pick $\gamma \in \Gamma(\alpha,\beta)$. Then $e_\beta(\gamma) \leq m$. If, in addition, $\gamma < \delta$, then $\gamma \in F$, but then $\gamma < \alpha$, contradicting the fact that $\Gamma(\alpha,\beta) \cap \alpha = \emptyset$. □

Lemma 3.5. *There exists a function $\psi : \omega_1 \to \omega_1$ such that $\psi[C] = \omega_1$ for any club $C \subseteq \omega_1$.*

Proof. Let $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be a partition of ω_1 into mutually disjoint stationary sets. Now, for $\alpha < \omega_1$, let $\psi(\alpha) = \beta$ where β is the unique ordinal such that $\alpha \in S_\beta$.

Finally, notice that if C is a club, then $C \cap S_\beta \neq \emptyset$ for all $\beta < \omega_1$. □

Corollary 3.6. *There exists a function $h : [\omega_1]^2 \to \omega_1$ such that $h``[Z]^2 = \omega_1$ for any unbounded $Z \subseteq \omega_1$.*

Proof. Put $h = \psi \circ f$, for some f as in Theorem 3.4 and ψ as in Lemma 3.5. □

3.2. The Continuum Hypothesis and $(\omega_2) \to (\omega_1)^2_2$.

Theorem 3.7 (Erdös-Rado [2], Sierpinski [6]). *The following are equivalent:*

(1) *CH, i.e.,* $2^{\aleph_0} = \aleph_1$*;*
(2) $(\aleph_2) \to (\aleph_1)^2_2$.

Proof. (1)⇒(2) The simplified proof given here is due to Simpson. Suppose $f : [\omega_2]^2 \to 2$ is given. For every countable $X \subseteq \omega_2$ and $y \in \omega_2 \setminus X$, define $f^X_y : X \to 2$ by letting $f^X_y(x) := f(x, y)$ for all $x \in X$. Notice that by CH, the following set is of cardinality $\leq \aleph_1$:

$$\phi^X := \{f^X_y \mid y \in \omega_2 \setminus \sup(X)\}.$$

We now define a continuous and increasing chain $\{M_\alpha \mid \alpha < \omega_1\} \subseteq [\omega_2]^{\aleph_1}$ by induction on $\alpha < \omega_1$.

Let M_0 be an arbitrary subset of ω_2 of cardinality $\aleph_1$. Given M_α, by $|[M_\alpha]^{\leq\aleph_0}| = \aleph_1$ and the above remark, pick some $M_{\alpha+1} \in [\omega_2]^{\aleph_1}$ extending M_α such that $\phi^X = \{f^X_y \mid y \in M_{\alpha+1} \setminus \sup(X)\}$ for every countable $X \subseteq M_\alpha$.

Finally, let $M = \bigcup_{\alpha<\omega_1} M_\alpha$. Since $|M| = \aleph_1$, let us pick some $y^* \in \omega_2 \setminus \sup(M)$. We shall now define an increasing sequence of ordinals $\langle y_\alpha \mid \alpha < \omega_1\rangle \in \prod_{\alpha<\omega_1} M_{\alpha+1}$.

Let y_0 be an arbitrary element of M_1. Suppose $X_\beta := \{y_\alpha \mid \alpha < \beta\}$ is defined for some $\beta < \omega_1$. By the construction of $M_{\beta+1}$, there exists some $y_\beta \in M_{\beta+1} \setminus \sup(M_\beta)$ such that $f^{X_\beta}_{y_\beta} = f^{X_\beta}_{y^*}$, so let y_β be such element. This completes the description of the construction.

Let $i < 2$, put $H_i := \{y_\alpha \mid \alpha < \omega_1, f(y_\alpha, y^*) = i\}$. Notice that if $\alpha < \beta$ and $y_\alpha, y_\beta \in H_i$, then $f(y_\alpha, y_\beta) = f(y_\alpha, y^*) = i$. Consequently, $f``[H_i]^2 = \{i\}$.

The proof is complete noticing that $\aleph_1 \in \{|H_0|, |H_1|\}$.

$(\neg 1) \Rightarrow (\neg 2)$ Pick $X \subseteq \mathbb{R}$ of cardinality $\aleph_2$. Let $\leq$ denote the natural ordering of the real line and let $\trianglelefteq$ be some well-ordering of X.

Define $f : [X]^2 \to 2$ by letting $f(x, y) = 1$ iff $\{(x, y), (y, x)\} \cap \leq \cap \trianglelefteq \neq \emptyset$, that is, iff $\leq$ and $\trianglelefteq$ agrees on the relation of x and y. Now assume towards a contradiction that $Y \subseteq X$ is a set of cardinality $\aleph_1$, such that $f``[Y]^2 = \{1\}$.[2]

By thinning out, we may assume that Y has no maximal element, so for all $y \in Y$, define the *successor of* y to be the minimal element above it:

$$\hat{y} := \min_{\trianglelefteq}\{z \in Y \mid z \neq y, y \trianglelefteq z\}.$$

But $y \trianglelefteq \hat{y}$ implies that $y \leq \hat{y}$, and so $\{(y, \hat{y}) \mid y \in Y\}$ is an uncountable family of mutually-disjoint open intervals of the real line, but this is impossible, because each such interval contains a (distinct) rational point, and there are only countably many rational numbers. □

[2]The other case is handled in the same fashion.

4. The consistency of the negation of Borel's conjecture

Recall that a subset $X \subseteq \mathbb{R}$ is said to be a *null set* iff for every positive $\varepsilon \in \mathbb{R}$ there exists a sequence of open intervals $\langle I_n \mid n < \omega\rangle$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and also $\sum_{n<\omega} |I_n| < \varepsilon$.[3]

A subset $X \subseteq \mathbb{R}$ is said to be a *strongly null set* iff for every sequence of positive real numbers, $\langle \varepsilon_n \mid n < \omega\rangle$, there exists a sequence of open intervals $\langle I_n \mid n < \omega\rangle$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and also $|I_n| < \varepsilon_n$ for all $n < \omega$.

Clearly, any strongly null set is a null set, and any countable subset of $\mathbb{R}$ is strongly null. Also, there are well-known examples of uncountable null sets, but what about uncountable strongly null sets?

Émile Borel conjectured that a set is strongly null iff it is countable. In [4], Richard Laver proved the consistency of this conjecture. In this section, we shall include a proof for the consistency of the negation of Borel's conjecture.

This result is considered as folklore, but its ingredients are due to Luzin.

Theorem. $\mathsf{CH} \Rightarrow$ *There exists an uncountable strongly null set.*

Proof. By $2^{\aleph_0} = \aleph_1$, let $\mathcal{F} = \{F_\alpha \mid \alpha < \omega_1\}$ be an enumeration of all closed nowhere dense subsets of $\mathbb{R}$.

Fix $\alpha < \omega_1$. Since α is countable, by Baire category theorem, $\bigcup_{\beta<\alpha} F_\beta \neq \mathbb{R}$, so let us pick some $x_\alpha \in \mathbb{R} \setminus \bigcup_{\beta<\alpha} F_\beta$.

Let $X := \{x_\alpha \mid \alpha < \omega_1\}$. Clearly, for all $\alpha < \omega_1$, $X \cap F_\alpha \subseteq \{x_\beta \mid \beta < \alpha\}$. In particular, since all singletons are in $\mathcal{F}$, the set X is uncountable.

Finally, suppose $\langle \varepsilon_n \mid n < \omega\rangle$ is a sequence of positive real numbers. Let $\{q_{2n} \mid n < \omega\}$ be some enumeration of $\mathbb{Q}$. For all $n < \omega$, pick an interval I_{2n} such that $q_{2n} \in I_{2n}$ and $|I_{2n}| < \varepsilon_{2n}$.

Consider the set $G = \bigcup_{n<\omega} I_{2n}$; this is an open and dense set, and hence there exists some $\alpha < \omega_1$ such that $\mathbb{R} \setminus G = F_\alpha$. By $|X \cap F_\alpha| \leq \aleph_0$, let $\{y_{2n+1} \mid n < \omega\}$ be some enumeration of $X \setminus G$. For all $n < \omega$, pick an interval I_{2n+1} such that $y_{2n+1} \in I_{2n+1}$ and $|I_{2n+1}| < \varepsilon_{2n+1}$.

Clearly, $X \subseteq \bigcup_{n<\omega} I_n$. □

[3]Here, $|I|$ denotes the diameter of the open interval I.

5. Square Sequences

Definition 5.1. We say that a stationary set S *carries a square sequence type-bounded by* η iff there exists a collection $\{C_\alpha \mid \alpha \in S\}$ such that for all limit $\alpha \in S$:

(1) C_α is a club subset of S, and $\mathrm{otp}(C_\alpha) \leq \eta$;
(2) if $\beta \in \mathrm{acc}(C_\alpha)$, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$.

Here, $\mathrm{acc}(C_\alpha) := \{\beta \in C_\alpha \mid \sup(C_\alpha \cap \beta) = \beta\}$. Jensen's square principle, $\square_\lambda$, is the assertion that λ^+ carries a partial square sequence. Note that $\square_\omega$ holds trivially.

In this section, we would like to present Shelah's results that, quite often, many stationary subsets of successor cardinals carries a partial square sequence. For this, we first need the following well-known lemma.

Theorem 5.2 (Engelking-Karłowicz)**.** *For cardinals $\kappa \leq \lambda \leq \mu \leq 2^\lambda$, the following are equivalent:*

(1) $\lambda^{<\kappa} = \lambda$;
(2) *there exists a collection of functions, $\langle f_i : \mu \to \lambda \mid i < \lambda\rangle$, such that for every $X \in [\mu]^{<\kappa}$ and every function $f : X \to \lambda$, there exists some $i < \lambda$ with $f \subseteq f_i$.*

Proof. (2)⇒(1) Suppose $\langle f_i : \mu \to \lambda \mid i < \lambda\rangle$ is a given collection. Then

$$|\{f_i \restriction \theta \mid i < \lambda, \theta < \kappa\}| \leq \lambda < \lambda^{<\kappa},$$

so there must exists some $f \in {}^{<\kappa}\lambda$ with $f \not\subseteq f_i$ for all $i < \lambda$.

(1)⇒(2) The simplified proof here is due to Shelah. Put:

$$W := \{(a, \mathcal{A}, g) \mid a \in [\lambda]^{<\kappa}, \mathcal{A} \in [a]^{<\kappa}, g \in {}^{\mathcal{A}}\lambda\}.$$

Then $|W| = \lambda^{<\kappa} = \lambda$, and we may fix an enumeration

$$W = \{(a_i, \mathcal{A}_i, g_i) \mid i < \lambda\}.$$

By $\mu \leq 2^\lambda$, let $\langle B_\alpha \mid \alpha < \mu\rangle$ be a sequence of distinct subsets of λ. For all $i < \lambda$, we now define $f_i : \mu \to \lambda$, by letting for all $\alpha < \mu$:

$$f_i(\alpha) = \begin{cases} g(a_i \cap B_\alpha), & a_i \cap B_\alpha \in \mathcal{A}_i \\ 0, & \text{otherwise} \end{cases}.$$

Finally, suppose that a set $X \in [\mu]^{<\kappa}$ and a function $f : X \to \lambda$ are given. For all distinct $\alpha, \beta \in X$, pick $x(\alpha, \beta) \in B_\alpha \Delta B_\beta$. Put $a = \{x(\alpha, \beta) \mid \alpha, \beta \in X, \alpha \neq \beta\}$. Then, $|a| < \kappa$ and for all distinct $\alpha, \beta \in X$, we have $a \cap B_\alpha \neq a \cap B_\beta$. It follows that $|\mathcal{A}| = |a|$, where $\mathcal{A} := \{a \cap B_\alpha \mid \alpha \in X\}$. It also follows that we may well-define a function $g : \mathcal{A} \to \lambda$ by letting:

$$g(a \cap B_\alpha) := f(\alpha), \quad (\alpha \in X).$$

Pick $i < \lambda$ such that $(a, \mathcal{A}, g) = (a_i, \mathcal{A}_i, g_i)$. Then, $f \subseteq f_i$. □

Theorem 5.3 (Shelah, [**?**]). *Suppose $\kappa \leq \lambda$ are cardinals such that $\lambda^{<\kappa} = \lambda$.*

Denote $T := \{\alpha < \lambda^+ \mid \omega \leq \mathrm{cf}(\alpha) < \kappa\}$; then T is the union of λ many sets, each carrying a partial square sequence type-bounded by κ. That is, there exists sequences $\langle\langle C^i_\alpha \mid \alpha \in S_i\rangle \mid i < \lambda\rangle$ such that:

(1) $\bigcup_{i<\lambda} S_i = T$;
(2) *for all $i < \lambda$ and $\alpha \in S_i$:*
  (a) *C^i_α is a club subset of α, with $\mathrm{otp}(C^i_\alpha) < \kappa$;*
  (b) *$C^i_\beta = C^i_\alpha \cap \beta$ for all $\beta \in \mathrm{acc}(C^i_\alpha)$.*

Proof. By $\lambda^{<\kappa} = \lambda$, let us fix for each $\alpha < \lambda^+$, an enumeration $\{A^j_\alpha \mid j < \lambda\}$ of $[\alpha]^{<\kappa}$. By $\lambda^{<\kappa} = \lambda$, and the Engelking-Karłowicz theorem, we may pick a collection of functions $\langle f_i : \lambda^+ \to \lambda \mid i < \lambda\rangle$ such that for every $X \in [\lambda^+]^{<\kappa}$ and every function $f : X \to \lambda$, there exists some $i < \lambda$ with $f \subseteq f_i$.

For each $i < \lambda$ and $\alpha \in T$, denote $C^i_\alpha := A^{f_i(\alpha)}_\alpha$.

Now, for each $i < \lambda$, put:

$$S_i := \{\alpha \in T \mid C^i_\alpha \text{ satisfies } 2(\mathrm{a}) + 2(\mathrm{b})\}.$$

Finally, we fix some $\alpha \in T$, and show that $\alpha \in \bigcup_{i<\lambda} S_i$.

Let D_α be a club subset of α with $\mathrm{otp}(D_\alpha) < \kappa$. Consider the function $f : D_\alpha \cup \{\alpha\} \to \lambda$ defined by:

$$f(\beta) := \min\{j < \lambda \mid D_\alpha \cap \beta = A^j_\beta\}, \quad (\beta \in D_\alpha \cup \{\alpha\}).$$

Pick $i < \lambda$ such that $f \subseteq f_i$, then for all $\beta \in D_\alpha \cup \{\alpha\}$, we have:

$$D_\alpha \cap \beta = A^{f(\beta)}_\beta = A^{f_i(\beta)}_\beta = C^i_\beta,$$

and hence $\alpha \in S_i$. □

Corollary 5.4. *If $\lambda^{<\lambda} = \lambda$, then $\{\alpha < \lambda^+ \mid \omega \leq \mathrm{cf}(\alpha) < \lambda\}$ is the union of λ many sets, each carrying a partial square sequence type-bounded by λ.*

In future versions of this section, we shall be presenting Shelah's improvements to the preceding corollary.

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