Nets of spaces having singular density

Workshop on Set Theory
and its Applications

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Assaf Rinot
Tel-Aviv University

http://www.tau.ac.il/~rinot

Definitions

Suppose $\langle X, O \rangle$ is a topological space.

Weight: $w(X) :=$ min$\{|\mathcal{B}| : \mathcal{B} \subseteq O$ is a base $\} + \aleph_0$.

Density: $d(X) :=$ min$\{|D| : D \subseteq X$ is dense $\} + \aleph_0$.

Hereditary compactness degree:
$hC(X) :=$ min$\{\mu :$ any open cover of any $Y \subseteq X$ contains a subcover of size $< \mu\} + \aleph_0$.

The effect of singular cardinals on usual mathematics is unusual

Theorem (Hajnal-Juhász, 1969).
Suppose λ is a singular strong-limit cardinal.
Any Hausdorff space of cardinality λ contains a discrete subspace of size λ.

In particular, if X is Hausdorff and $d(X) = \lambda$ is SSL, then $hC(X) > \lambda$.

History

Theorem (Juhász-Shelah, 2002).
For any relevant ordinal α, the following is consistent:

$$\exists X \text{ regular}, \; hC(X) = \aleph_1, \; d(X) = \aleph_\alpha = \mathrm{cf}(\aleph_\alpha).$$

Theorem (Moore, 2004).
The following is a consequence of ZFC:

$$\exists X \text{ regular}, \; hC(X) = \aleph_1, \; d(X) = \aleph_1.$$

Question (Juhász). Is the following consistent:

$$\exists X \text{ regular}, \; hC(X) = \aleph_1, \; d(X) = \aleph_{\aleph_1} \text{ ??}$$

Theorem (Gitik-Rinot, 2005).

If $\exists X$ arbitrary, $hC(X) = \aleph_1$, $d(X) = w(X) = \aleph_{\aleph_1}$,

then there exists an inner model with a measurable cardinal.

Theorem (Juhász-Shelah, 2007).
For any ordinal α, the following is consistent:

$$\exists X \text{ regular}, hC(X) = \aleph_1, d(X) = \aleph_\alpha.$$

Theorem (Rinot, 2007).
In all currently known models of ZFC:

$$\neg\exists X \text{ arbitrary}, hC(X) = \aleph_1, d(X) = w(X) = \aleph_{\aleph_1}.$$

Question (Juhász). Does the same follow from ZFC?

Prevalent Singular Cardinals Hypothesis

Definition. A singular cardinal λ is a *prevalent singular cardinal* iff there exists a family $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ with $|\mathcal{A}| = \lambda$ and $\sup\{|A| : A \in \mathcal{A}\} < \lambda$ such that any $B \subseteq \lambda$ with $|B| < \mathrm{cf}(\lambda)$ is contained in some $A \in \mathcal{A}$.

PSH states that any singular cardinal is a prevalent singular cardinal.

Prevalent Singular Cardinals Hypothesis

Fact 1. PSH is a consequence of GCH, PFA, and more.

Fact 2. PSH holds in all known models of set theory. It is unknown whether $\neg$PSH is consistent.

Probabilistic view: The following event occurs a.e.: There exists a cardinal κ such that any singular $\lambda > \kappa$ is a prevalent singular cardinal.
Reasoning: the complementary event implies the existence of an inner model with a *class* of measurables.

Restricted net weight

Definition. For a space $\langle X, O\rangle$ and a cardinal θ, let:

$$nw_\theta(X) := \min\left\{|\mathcal{N}| : \mathcal{N} \subseteq \mathcal{P}(X), O \subseteq \{\bigcup\mathcal{U} \mid \mathcal{U} \in [\mathcal{N}]^{<\theta}\}\right\}$$

Main Theorem. (PSH) If $\langle X, O\rangle$ is an arbitrary space (no separation axioms assumed) and $d(X) = \lambda$ is a singular cardinal, then $nw_{\mathrm{cf}(\lambda)}(X) > \lambda$.

Proof of the main theorem

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Corollary

Theorem. In all currently known models of ZFC, if λ is singular and $\langle X, O \rangle$ is an arbitrary topological space with $d(X) = w(X) = \lambda$, then $hC(X) > \mathrm{cf}(\lambda)$.

Thank you!