Nets of spaces having singular density

Workshop on Set Theory
and its Applications

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Assaf Rinot
Tel-Aviv University

http://www.tau.ac.il/~rinot
Definitions

Suppose \( \langle X, O \rangle \) is a topological space.

**Weight:** \( w(X) := \min\{|\mathcal{B}| : \mathcal{B} \subseteq O \text{ is a base} \} + \aleph_0. \)

**Density:** \( d(X) := \min\{|D| : D \subseteq X \text{ is dense} \} + \aleph_0. \)

**Hereditary compactness degree:**
\[ hC(X) := \min\{\mu : \text{any open cover of any } Y \subseteq X \text{ contains a subcover of size } < \mu \} + \aleph_0. \]
The effect of singular cardinals on usual mathematics is unusual

**Theorem** (Hajnal-Juhász, 1969). Suppose $\lambda$ is a singular strong-limit cardinal. Any Hausdorff space of cardinality $\lambda$ contains a discrete subspace of size $\lambda$.

In particular, if $X$ is Hausdorff and $d(X) = \lambda$ is SSL, then $hC(X) > \lambda$. 
**History**

**Theorem** (Juhász-Shelah, 2002).
For any relevant ordinal $\alpha$, the following is consistent:

$$\exists X \text{ regular}, \; hC(X) = \aleph_1, \; d(X) = \aleph_{\alpha} = \text{cf}(\aleph_{\alpha}).$$

**Theorem** (Moore, 2004).
The following is a consequence of ZFC:

$$\exists X \text{ regular}, \; hC(X) = \aleph_1, \; d(X) = \aleph_1.$$

**Question** (Juhász). Is the following consistent:

$$\exists X \text{ regular}, \; hC(X) = \aleph_1, \; d(X) = \aleph_{\aleph_1}.$$
Theorem (Gitik-Rinot, 2005).

If $\exists X$ arbitrary, $hC(X) = \aleph_1$, $d(X) = w(X) = \aleph_{\aleph_1}$,
then there exists an inner model with a measurable cardinal.

Theorem (Juhász-Shelah, 2007).
For any ordinal $\alpha$, the following is consistent:

$\exists X$ regular, $hC(X) = \aleph_1$, $d(X) = \aleph_\alpha$.

Theorem (Rinot, 2007).
In all currently known models of ZFC:

$\neg \exists X$ arbitrary, $hC(X) = \aleph_1$, $d(X) = w(X) = \aleph_{\aleph_1}$.

Question (Juhász). Does the same follow from ZFC?
Prevalent Singular Cardinals Hypothesis

**Definition.** A singular cardinal $\lambda$ is a *prevalent singular cardinal* iff there exists a family $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ with $|\mathcal{A}| = \lambda$ and $\operatorname{sup}\{|A| : A \in \mathcal{A}\} < \lambda$ such that any $B \subseteq \lambda$ with $|B| < \operatorname{cf}(\lambda)$ is contained in some $A \in \mathcal{A}$.

**PSH** states that any singular cardinal is a prevalent singular cardinal.
Prevalent Singular Cardinals Hypothesis

**Fact 1.** PSH is a consequence of GCH, PFA, and more.

**Fact 2.** PSH holds in all known models of set theory. It is unknown whether $\neg$PSH is consistent.

**Probabilistic view:** The following event occurs a.e.: There exists a cardinal $\kappa$ such that any singular $\lambda > \kappa$ is a prevalent singular cardinal.

**Reasoning:** the complementary event implies the existence of an inner model with a class of measurables.
Restricted net weight

**Definition.** For a space \( \langle X, O \rangle \) and a cardinal \( \theta \), let:

\[
nw_\theta(X) := \min \left\{ |\mathcal{N}| : \mathcal{N} \subseteq \mathcal{P}(X), O \subseteq \{ \bigcup U \mid U \in [\mathcal{N}]^{<\theta} \} \right\}
\]

**Main Theorem.** (PSH) If \( \langle X, O \rangle \) is an arbitrary space (no separation axioms assumed) and \( d(X) = \lambda \) is a singular cardinal, then \( nw_{\text{cf}(\lambda)}(X) > \lambda \).
Proof of the main theorem
Corollary

**Theorem.** In all currently known models of ZFC, if $\lambda$ is singular and $\langle X, O \rangle$ is an arbitrary topological space with $d(X) = w(X) = \lambda$, then $hC(X) > \text{cf}(\lambda)$. 
Thank you!