$\aleph_3$-trees

P.O.I. Workshop in pure and descriptive set theory
Università di Torino, Italy
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Partial bibliography

This talk centers around the following works:

[BR1] A. Brodsky and A. Rinot, A microscopic approach to Souslin-tree constructions, *in preparation*.

[BR2] A. Brodsky and A. Rinot, Reduced powers of Souslin trees, *submitted July 2015*.

[RS] A. Rinot and R. Schindler, Square with built-in diamond-plus, *in preparation*.

The second paper is available at http://www.assafrinot.com

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By convention, all trees in this lecture are Hausdorff trees:

$$(x_{\downarrow} = y_{\downarrow}) \Rightarrow (x = y).$$

Particular trees of interest

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- A κ-Kurepa tree is a κ-tree is having at least κ^+ many cofinal branches;
- A κ-Souslin tree is a κ-Aronszajn tree having no antichains of size κ;
- A λ^+-tree is special if it is the union of λ many antichains.

I've got the power

The I-power of a tree

Given a tree $(T, \lhd)$ and a set I, let

$$T^I := \{f : I \to T \mid ht \circ f \text{ is constant}\},$$

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Lemma (Kurepa, 1952)

For every κ-tree $(T, \lhd)$, T^2 is not κ-Souslin.

Reduced powers

The reduced I-power of a tree

Given a tree $(T, \triangleleft)$, an infinite set I, and a uniform ultrafilter $\mathcal{U}$ over I, let

$$T^I/\mathcal{U} := \{[f]_{\mathcal{U}} \mid f \in T^I\},$$

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Lemma (essentially Kurepa, 1952)

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Question

But is the reduced I-power of a κ-Souslin tree at least κ-Aronszajn?

Reduced powers of Souslin tree

The good news (Devlin, 1983)

Consistently, there exists an $\aleph_2$-Souslin tree whose reduced $\aleph_0$-power is $\aleph_2$-Aronszajn.

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Theorem (Cummings, 1997)

If $\boxdiamond_\lambda$ holds for an uncountable cardinal $\lambda^{<\lambda} = \lambda$, then there exists a λ-complete λ^+-Souslin tree whose reduced $\aleph_0$-power is not λ^+-Aronszajn.

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Various powers

In Luminy 2010, I told Cummings that I can tweak his construction so that for every infinite cardinal $\theta < \lambda$, the reduced θ-power is not λ^+-Aronszajn.

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Theorem ([BR2])

There consistently exist an $\aleph_6$-Souslin tree $(T, \lhd)$ and a sequence of uniform ultrafilters $\langle \mathcal{U}_n \mid n < 6 \rangle$ such that for all $n < 6$, $T^{\aleph_n}/\mathcal{U}_n$ is $\aleph_6$-Aronszajn iff n is not a prime number.

Kurepa's lemma revisited

Almost Souslin trees

Recall

For every $\aleph_1$-tree $(T, \lhd)$, T^2 is not $\aleph_1$-Souslin.

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Definition (Devlin-Shelah, 1977)

An $\aleph_1$-tree is Almost Souslin if for every of its antichains A, we have that $\{ht(x) \mid x \in A\}$ is nonstationary.

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Note

An almost Souslin $\aleph_1$-tree cannot contain a special tree.

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Consistently, $\exists$ $\aleph_1$-Souslin tree whose square is almost Souslin.

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A test

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Better seen at the level of $\aleph_3$

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Assume $V = L$.
Then there exist trees T_0, T_1, T_2, T_3, and uniform ultrafilters $\mathcal{U}_0$ over $\aleph_0$, $\mathcal{U}_1$ over $\aleph_1$, such that:

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	T	$T^{\aleph_0}/\mathcal{U}_0$	$T^{\aleph_1}/\mathcal{U}_1$
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T_2	$\aleph_3$-S.	$\aleph_3$-Aronszajn + almost S.	$\aleph_3$-Kurepa + $\neg$ almost S.

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T_2	$\aleph_3$-*S.*	$\aleph_3$-*Aronszajn* + *almost S.*	$\aleph_3$-*Kurepa* + $\neg$ *almost S.*
T_3	$\aleph_3$-*S.*	$\neg\aleph_3$-*Aronszajn*	$\aleph_3$-*Aronszajn* + *almost S.*

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Let us now describe the concepts and tools that are used in proving the above.

Preventing objects from appearing in the reduced power

Definition

Suppose that $(T, \lhd)$ is a tree, and I is a nonempty set.
For every $g \in T^I$, the derived tree along g is the collection:

$$\{f \in T^I \mid \forall i \in I(f(i) \text{ is } \lhd\text{-compatible with } g(i))\}.$$

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Definition

A κ-Souslin tree $(T, \lhd)$ is said to be χ-free if for every nonzero $\tau < \chi$, and every injective $g \in T^\tau$, the derived tree along g is again κ-Souslin.

Preventing objects from appearing in the reduced power

Remark

Why injective?

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Why injective? Because of Kurepa's lemma.

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A filter $\mathcal{F}$ over a cardinal θ is said to be selective if it is uniform, and for every function f with $\mathrm{dom}(f) \in \mathcal{F}$, one of the following holds:

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Remarks

- for every infinite cardinal θ, $\mathcal{F}^{bd}_\theta := \{Z \subseteq \theta \mid \sup(\theta \setminus Z) < \theta\}$ is a selective filter;

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- (essentially Rudin, 1956) If θ is regular, and $2^\theta = \theta^+$, then there exists a selective ultrafilter over θ;
- (Kunen, 1976) after adding $\aleph_2$ random reals to a model of GCH, there are no selective ultrafilters over ω.

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Lemma 1

If $(T, \lhd)$ is a θ^+-free κ-Souslin tree, then for every selective ultrafilter $\mathcal{U}$ over θ, the reduced θ-power $T^\theta/\mathcal{U}$ is κ-Aronszajn.

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Lemma 2

If $(T, \lhd)$ is a θ^+-free λ^+-Souslin tree, then for every selective ultrafilter $\mathcal{U}$ over θ, the reduced θ-power $T^\theta/\mathcal{U}$ is almost Souslin.

The tree T_0

Theorem (essentially Jensen, 1960's)

If $\diamondsuit(E^{\aleph_3}_{\aleph_2})$ *holds and* $\aleph_3^{\aleph_2} = \aleph_3$*, then there exists an* $\aleph_2$*-complete* $\aleph_2$*-free* $\aleph_3$*-Souslin tree.*

The tree T_0

Theorem (essentially Jensen, 1960's)

If $\diamondsuit(E^{\aleph_3}_{\aleph_2})$ holds and $\aleph_3^{\aleph_2} = \aleph_3$, then there exists an $\aleph_2$-complete $\aleph_2$-free $\aleph_3$-Souslin tree.

Corollary ([BR2])

If $\diamondsuit(E^{\aleph_3}_{\aleph_2}) + GCH$ holds, then there exist an $\aleph_3$-Souslin tree T_0, and selective ultrafilters $\mathcal{U}_0 \subseteq \mathcal{P}(\aleph_0)$, $\mathcal{U}_1 \subseteq \mathcal{P}(\aleph_1)$ such that $T_0^{\aleph_0}/\mathcal{U}_0$ and $T_0^{\aleph_1}/\mathcal{U}_1$ are $\aleph_3$-Aronszajn and almost Souslin.

Injecting objects to the reduced power

Definition (essentially Laver, 1980's)

Suppose that $X \subseteq {}^{<\kappa}\kappa$ is a downward-closed family such that $(X, \subset)$ is a κ-tree, and $\mathcal{F}$ is a filter over some index set I.

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Proposition

If $(T, \lhd)$ admits an $(\mathcal{F}^{bd}_\theta, X)$-ascent path, then the reduced θ-power tree (by any uniform ultrafilter over θ) contains a copy of the tree $(X, \subset)$.

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Notation

For an infinite cardinal θ, let $\mathcal{F}^{fin}_\theta := \{Z \subseteq \theta \mid |\theta \setminus Z| < \aleph_0\}$.

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An $(\mathcal{F}, X)$-ascent path through a κ-tree $(T, \lhd)$ is a sequence $\vec{f} = \langle f_x \mid x \in X \rangle$ such that:

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Proposition

If $(T, \lhd)$ admits an $(\mathcal{F}^{fin}_{\theta}, X)$-ascent path, then for every infinite $\mu \leq \theta$, the reduced μ-power tree (by any uniform ultrafilter over μ) contains a copy of the tree $(X, \subset)$.

The tree T_1

Theorem ([BR2])

Suppose that $\square_{\aleph_2} + \diamondsuit^*(\aleph_3)$ *holds and* $\aleph_3^{\aleph_2} = \aleph_3$.
Then there are:

- *an* $\aleph_3$*-Souslin tree* $T_1 \subseteq {}^{<\aleph_3}2$;
- *an* $\aleph_3$*-Kurepa tree* $K \subseteq {}^{<\aleph_3}2$;
- *a special* $\aleph_3$*-tree* $S \subseteq {}^{<\aleph_3}(\aleph_2 \setminus 2)$,

such that $(T_1, \subset)$ *admits an* $(\mathcal{F}^{fin}_{\aleph_2}, X)$*-ascent path, for* $X := K \uplus S$.

In particular, the reduced $\aleph_0$-power and $\aleph_1$-power (by any uniform ultrafilters) are $\aleph_3$-Kurepa and not almost Souslin.

Intertwining the two strategies

1+1=?

So far, we have described a strategy for constructing κ-Souslin trees whose θ_0-power omits prescribed objects, and another strategy for constructing κ-Souslin trees whose θ_1-power contains a prescribed object. Could these strategies live side by side?

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The answer is clearly negative if $\theta_0 = \theta_1$. But even if $\theta_0 \neq \theta_1$, there are further obstructions.

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Suppose we would like to construct an $\aleph_3$-Souslin tree $(T, \lhd)$ whose reduced $\aleph_0$-power is $\aleph_3$-Aronszajn and almost Souslin, and whose reduced $\aleph_1$-power is $\aleph_3$-Kurepa and not almost Souslin.

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Given the preceding strategies, it would be best if we can construct an $\aleph_3$-Souslin tree which is $\aleph_1$-free, and admits an $(\mathcal{F}^{bd}_{\aleph_1}, X)$-ascent path for $X = K \uplus S$, where K is $\aleph_3$-Kurepa, and S is special.

The tree T_2

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The base case $\alpha = 0$ is trivial: let $T_0 := \{\emptyset\}$, and $f_\emptyset = \aleph_1 \times \{\emptyset\}$.
The successor case $\alpha + 1$: let $T_{\alpha+1} = \{t^\frown \langle \varepsilon \rangle \mid t \in T_\alpha, \varepsilon < \varphi(\alpha)\}$, and $f_x(i) = f_{x \restriction \alpha}(i)^\frown \langle x(\alpha) \rangle$ for all $x \in X_{\alpha+1}$ and $i < \aleph_1$.

The tree T_2 (cont.)

On limit stage α, we need to construct:

(1) T_α so that every $t \in T \restriction \alpha$ admits an extension in T_α;

(2) for each $x \in X_\alpha$, $f_x : \aleph_1 \to T_\alpha$ s.t. $\{i < \aleph_1 \mid f_{x\restriction\beta}(i) \lhd f_x(i)\}$ is co-countable for all $\beta < \alpha$.

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Clause (1) usually involves the process of constructing for every $t \in (T \restriction \alpha)$, some *canonical branch* $\mathbf{b}_t^\alpha$ which goes through t and is cofinal in $T \restriction \alpha$.

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Altogether T_α would consist of limits of canonical branches $\mathbf{b}_t^\alpha$ for $t \in T \restriction \alpha$, and limits of branches $\{f_{x\restriction\beta}(i) \mid \beta < \alpha\}$ for $x \in X_\alpha$, and "$i < \aleph_1$".

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Altogether T_α would consist of limits of canonical branches $\mathbf{b}_t^\alpha$ for $t \in T \restriction \alpha$, and limits of branches $\{f_{x\restriction\beta}(i) \mid \beta \in U_{\alpha,i}\}$ for $x \in X_\alpha$, unbounded $U_{\alpha,i} \subseteq \alpha$, and "$i < \aleph_1$".

The tree T_2 (cont.)

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For instance, consider $g : 2 \to T_\alpha$, where:

- $g(0) = t \in T_\beta$ for some $\beta < \alpha$, and
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To be able to seal antichains in the derived tree along this g, the construction of the canonical branch $\mathbf{b}_t^\alpha$ would have to know about the future limit $\{f_{x \restriction \beta}(i) \mid \beta < \alpha\}$.

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To be able to seal antichains in the derived tree along this g, the construction of the canonical branch $\mathbf{b}_t^\alpha$ would have to know about the future limit $\{f_{x \restriction \beta}(i) \mid \beta < \alpha\}$. But is this at all possible?

The tree T_2 (cont.)

Question

Can the construction of a canonical branch $\mathbf{b}_t^\alpha$ for $t \in T \restriction \alpha$ anticipate the future limit $\{f_{x\restriction\beta}(i) \mid \beta < \alpha\}$ for some $x \in X_\alpha$?

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Answer

Yes, provided that all involved constructions are done in some strictly canonical way AND (of course) that t is aware of x.

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Can $t \in T \restriction \alpha$ be aware of $x \in X_\alpha$?

Answer

No, unless the tree $(X, \subset)$ happens to be the outcome of a construction, where any $x \in X_\alpha$ is the limit of a canonical branch $\mathbf{b}_y^\alpha$ for some $y \in X \restriction \alpha$, using the very same ladder system $\vec{C}$.

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We call such trees $\vec{C}$-respecting.

Respecting trees

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Putting all technologies together

Corollary ([BR2])

Assume $\boxtimes^+_{\aleph_2} + CH$ *(e.g.,* $V = L$*).*
Then there exists an $\aleph_3$*-Souslin tree* T_2*, a selective ultrafilter* $\mathcal{U}_0 \subseteq \mathcal{P}(\aleph_0)$*, and a uniform ultrafilter* $\mathcal{U}_1 \subseteq \mathcal{P}(\aleph_1)$*, such that*

- $T_2^{\aleph_0}/\mathcal{U}_0$ *is* $\aleph_3$*-Aronszajn and almost Souslin, and*
- $T_2^{\aleph_1}/\mathcal{U}_1$ *is* $\aleph_3$*-Kurepa and not almost Souslin.*

The good-bad-good tree

Remember that T_3 denotes an $\aleph_3$-Souslin tree whose reduced $\aleph_0$-power is not $\aleph_3$-Aronszajn, and its reduced $\aleph_1$-power is $\aleph_3$-Aronszajn and Almost Souslin.

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- For the $\aleph_0$-power, it is natural to require that T_3 admits an $(\mathcal{F}^{bd}_{\aleph_0}, X)$-ascent path, where $(X, \subset)$ is isomorphic to $(\omega_3, \in)$.

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Unfortunately, the two requirements are contradictory.
For this, we refine the second requirement, and introduce a two-cardinals version of freeness.

Two-cardinals freeness

Recall

A κ-Souslin tree $(T, \lhd)$ is said to be χ-free if $\forall$ nonzero $\tau < \chi$, and every injective $g \in T^\tau$, the derived tree along g is again κ-Souslin.

Two-cardinals freeness

Recall

A κ-Souslin tree $(T, \lhd)$ is said to be χ-free if $\forall$ nonzero $\tau < \chi$, and every injective $g \in T^\tau$, for every κ-sized subset A of the derived tree along g, there exist $\vec{x}$ and $\vec{y}$ in A such that

$$\{i < \tau \mid \neg(\vec{x}(i) \lhd \vec{y}(i))\} = \emptyset.$$

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Definition

A κ-tree $(T, \lhd)$ is said to be (χ, η)-free if $\forall$ nonzero $\tau < \chi$, and every injective $g \in T^\tau$, for every κ-sized subset A of the derived tree along g, there exist $\vec{x}$ and $\vec{y}$ in A such that

$$|\{i < \tau \mid \neg(\vec{x}(i) \lhd \vec{y}(i))\}| < \eta.$$

Two-cardinals freeness

Note

1. A κ-Souslin tree is χ-free iff it is $(\chi, 1)$-free;
2. If $\chi_0 \geq \chi_1$ and $\eta_0 \leq \eta_1$, then (χ_0, η_0)-free implies (χ_1, η_1)-free.

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Two-cardinals freeness

Assume GCH.

Lemma 1 (refined)

If $(T, \lhd)$ is a (θ^+, θ)-free κ-Souslin tree, then for every selective ultrafilter $\mathcal{U}$ over θ, the reduced θ-power $T^\theta/\mathcal{U}$ is κ-Aronszajn.

Lemma 2 (refined)

If $(T, \lhd)$ is a (θ^+, θ)-free λ^+-Souslin tree, then for every selective ultrafilter $\mathcal{U}$ over θ, the reduced θ-power $T^\theta/\mathcal{U}$ is almost Souslin.

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A κ-tree $(T, \lhd)$ is said to be (χ, η)-free if $\forall$ nonzero $\tau < \chi$, and every injective $g \in T^\tau$, for every κ-sized subset A of the derived tree along g, there exist $\vec{x}$ and $\vec{y}$ in A such that

$$|\{i < \tau \mid \neg(\vec{x}(i) \lhd \vec{y}(i))\}| < \eta.$$

Putting everything together

Theorem ([BR2])

Assume $\boxed{\diamondsuit}_{\aleph_2} + GCH$.
Then there exists an $\aleph_3$*-Souslin tree* T_3*, which is* $(\aleph_2, \aleph_1)$*-free and admits an* $(\mathcal{F}^{bd}_{\aleph_0}, X)$*-ascent path, where* $(X, \subset) \cong (\omega_3, \in)$*.*

Putting everything together

Theorem ([BR2])

Assume $\boxtimes\!\!\!\!\diamondsuit_{\aleph_2}$ + GCH.
Then there exists an $\aleph_3$-Souslin tree T_3, which is $(\aleph_2, \aleph_1)$-free and admits an $(\mathcal{F}^{bd}_{\aleph_0}, X)$-ascent path, where $(X, \subset) \cong (\omega_3, \in)$.
In particular, by taking a uniform ultrafilter $\mathcal{U}_0 \subseteq \mathcal{P}(\aleph_0)$, and a selective ultrafilter $\mathcal{U}_1 \subseteq \mathcal{P}(\aleph_1)$, we get that:

- *$T_3^{\aleph_0}/\mathcal{U}_0$ is not $\aleph_3$-Aronszajn;*
- *$T_3^{\aleph_1}/\mathcal{U}_1$ is $\aleph_3$-Aronszajn and almost Souslin.*

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Unfortunately, I am out of time now, and cannot elaborate further. So let me just thank you for your attention!