Antichains in partially ordered sets of singular cofinality

Singular Cardinal Combinatorics
and Inner Model Theory

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Definitions

Suppose $\langle P, \leq \rangle$ is a poset.

Notation. For $A \subseteq P$, let:

- $\overline{A} := \{x \in P \mid \exists y \in A(y \leq x)\}$;
- $\underline{A} := \{x \in P \mid \exists y \in A(x \leq y)\}$.

Definition. $\mathrm{cf}(P, \leq) := \min\{|D| : D \subseteq P, P \subseteq \underline{D}\}$;
For $A \subseteq P$, $\mathrm{cf}_P(A) := \min\{|D| : D \subseteq P, A \subseteq \underline{D}\}$.

Definition. $A \subseteq P$ is said to be an *antichain* iff $x \not\leq y$ and $y \not\leq x$ for any two distinct $x, y \in A$.

Motivation

Theorem (Hausdorff, 1908). If $\langle P, \leq \rangle$ is a linearly ordered set, then cf$(P, \leq)$ is a regular cardinal.

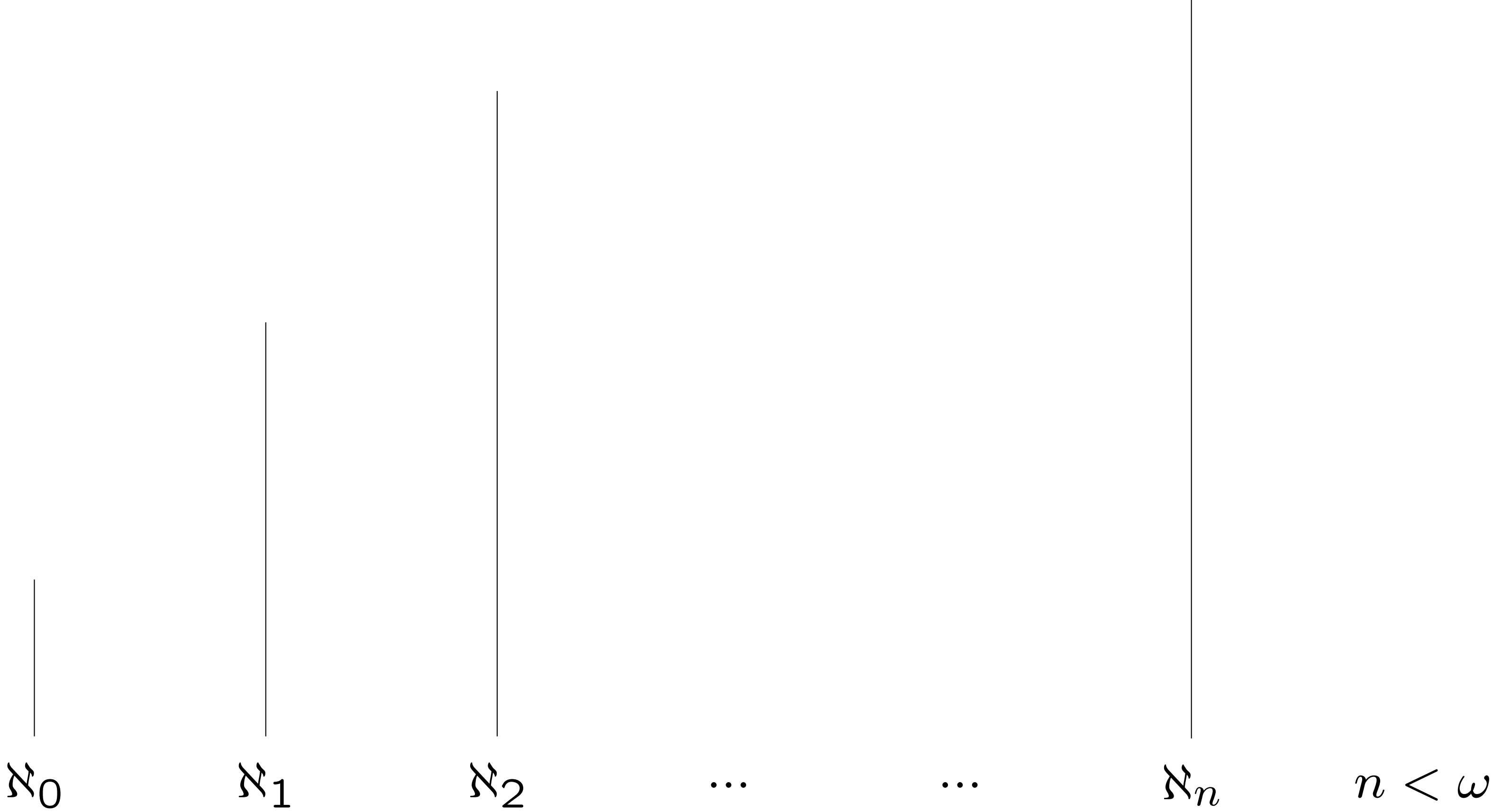
Theorem (Erdös-Tarski, 1943). If $\langle P, \leq \rangle$ is a poset with no infinite antichain, then P is the finite union of updirected posets.

Theorem (Pouzet, 1979). If $\langle P, \leq \rangle$ is an updirected poset with no infinite antichain, then it contains a cofinal subset which is isomorphic to a product of finitely many regular cardinals.

Corollary (Pouzet, 1979). If $\langle P, \leq \rangle$ is a poset and cf$(P, \leq)$ is a singular cardinal, then P has an infinite antichain.

Canonical example :

$\bigoplus_{n<\omega} \aleph_n$



The Milner-Sauer conjecture

Conjecture (Milner-Sauer, 1981). If $\langle P, \leq \rangle$ is a poset and $\mathrm{cf}(P, \leq) = \lambda > \mathrm{cf}(\lambda) = \kappa$, then P must contain an antichain of size κ.

(Appears implicitly already in Pouzet, 1979)

Consistency results

The conjecture is consistent and was known to follow from GCH-type assumptions, e.g., for $\lambda > \mathrm{cf}(\lambda) = \kappa$:

Theorem (Milner-Prikry '81). If $\mu^{<\kappa} < \lambda$ for all $\mu < \lambda$, then any poset of cofinality λ contains an antichain of size κ.

Theorem (Milner-Pouzet '82). If $\lambda^{<\kappa} = \lambda$, then any poset of cofinality λ contains an antichain of size κ.

Theorem (Hajnal-Sauer '86). If λ is a strong limit, then any poset of cofinality λ contains λ^κ antichains of size κ.

More recent consistency results

Theorem (Milner-Pouzet, 1997).
If $\lambda^{<\kappa} = \lambda$, then any poset of cofinality λ contains λ^κ antichains of size κ.

Theorem (Magidor, 2002, *unpublished*).
If cf$([\lambda]^{<\kappa}, \subseteq) = \lambda$, then any poset of cofinality λ contains an antichain of size κ.

and independently, with a completely different proof:

Theorem (Gitik-R., 2005).
If cf$([\lambda]^{<\kappa}, \subseteq) = \lambda$, then any poset of cofinality λ contains λ^κ antichains of size κ.

$\neg\,\mathrm{cf}([\lambda]^{<\mathrm{cf}(\lambda)}, \subseteq) = \lambda$ does not suffice

Theorem (Gitik, 2006). Assuming GCH and the existence of a cardinal θ being $\theta^{+\omega_1+1}$-strong, there exists a cardinals-preserving forcing notion $\mathbb{P}$ such that:
$V^{\mathbb{P}} \models \exists\lambda.\, \mathrm{cf}([\lambda]^{<\mathrm{cf}(\lambda)}, \subseteq) > \lambda > \mathrm{cf}(\lambda)$ + "any poset of cofinality λ contains an antichain of size $\mathrm{cf}(\lambda)$".

Some ZFC observations

✓ Any tree (or even pseudotree) of cofinality $\lambda > \text{cf}(\lambda)$ contains an antichain of size $\text{cf}(\lambda)$.

✓ If $\langle P, \leq \rangle$ is of cofinality λ, and $\sup\left\{|\underline{\{x\}}| : x \in P\right\} < \lambda$, then P contains an antichain of size λ.

✓ If there exists a counter-example of cofinality λ, then there exists a poset $\langle P, \leq \rangle$ such that:
$\Rightarrow$ $\text{cf}_P(\overline{\{x\}}) = |P| = \lambda$ for any $x \in P$;
$\Rightarrow$ P does not contain a chain of size λ;
$\Rightarrow$ P does not contain an antichain of size $\text{cf}(\lambda)$.

Prevalent Singular Cardinals

Definition. A singular cardinal λ is a *prevalent singular cardinal* iff $\exists \mu \in [\mathrm{cf}(\lambda), \lambda)$ with $\mathrm{cov}(\lambda, \mu, \mathrm{cf}(\lambda), 2) = \lambda$, i.e., if there exists a family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ such that $|\mathcal{F}| = \lambda$, $\sup\{|A| : A \in \mathcal{F}\} < \lambda$, and $[\lambda]^{<\mathrm{cf}(\lambda)} \subseteq \bigcup_{A \in \mathcal{F}} \mathcal{P}(A)$.

In Gitik's model, indeed $\mathrm{cf}([\lambda]^{<\mathrm{cf}(\lambda)}, \subseteq) > \lambda$, however, λ remained a prevalent singular cardinal.

Conventions

From now on, λ denotes a singular cardinal, $\kappa := \mathrm{cf}(\lambda)$, and $\langle P, \leq \rangle$ denotes a poset of cofinality λ.

Main Result

Theorem. If λ is a prevalent singular cardinal then any poset of cofinality λ contains λ^κ antichains of size κ.

Corollary. The negation of the Milner-Sauer conjecture - if consistent - requires knowledge in cardinal arithmetic that is not yet available.

The notion of a stable subposet

Definition. We say that a subset $P' \in [P]^\lambda$ is **stable** iff $\mathrm{cf}_P(P' \setminus \overline{X}) = \lambda$ for all $X \in [P']^{<\kappa}$.

Lemma 1. If P has a stable subposet, then P contains an antichain of size κ.

Proof. Let $P' \subseteq P$ be stable. We build an antichain $\{x_\alpha \mid \alpha < \kappa\}$ by induction on $\alpha < \kappa$.

Suppose $X := \{x_\beta \mid \beta < \alpha\} \subseteq P'$ has been defined. By $X \in [P']^{<\kappa}$, $\mathrm{cf}_P(P' \setminus \overline{X}) = \lambda$. Since $\mathrm{cf}_P(\underline{X}) \leq |X| < \lambda$, we may find $x_\alpha \in P'$ such that $x_\alpha \notin (\overline{X} \cup \underline{X})$. $\square$

The family of bad subsets

Definition. For $P' \subseteq P$, let:

$$\wp(P') := \{X \in [P']^{<\kappa} \mid \mathrm{cf}_P(P' \setminus \overline{X}) < \lambda\}.$$

Lemma 2. The following are equivalent:
(a) P contains a stable subset.
(b) There are $P', Y \subseteq P$, $\mathrm{cf}_P(P') = |P'| = \lambda > \mathrm{cf}_P(Y)$, such that $Y \cap X \neq \emptyset$ for all $X \in \wp(P')$.

Lemma 3. For any $\mathcal{F} \subseteq \wp(P)$ of cardinality $\leq \lambda$, there exists $Y \in [P]^\kappa$ with $\underline{Y} \cap A \neq \emptyset$ for all $A \in \mathcal{F}$.
Proof. Y is constructed by diagonalization, very much like the construction of a Luzin set. $\square$

Towards a proof of the main theorem

Theorem 2. If λ is a prevalent singular cardinal then any poset of cofinality λ contains a stable subset.

Proof (sketch). By considering a cofinal subset, we may assume that $|P| = \lambda$. By hypothesis, we may find $\mu < \lambda$ and $\mathcal{F} \subseteq [P]^{<\mu}$ of cardinality λ such that each $X \in \wp(P)$ is contained in some $B \in \mathcal{F}$.

We recursively construct a sequence $\{Y_\alpha \mid \alpha < \mu\} \subseteq [P]^\kappa$ using Lemma 3. Then argue that $Y := \bigcup_{\alpha<\kappa} \underline{Y_\alpha}$ is such that $\mathrm{cf}_P(Y) \leq \mu < \lambda$ and $Y \cap X \neq \emptyset$ for all $X \in \wp(P)$.

Recalling Lemma 2, our proof is complete. $\square$

Towards a proof of the main theorem (cont.)

Corollary. If λ is a prevalent singular cardinal then any poset of cofinality λ contains an antichain of size κ.

Thus, we have established the existence of a single antichain. Next, we shall improve the result, obtaining λ^κ many antichains.

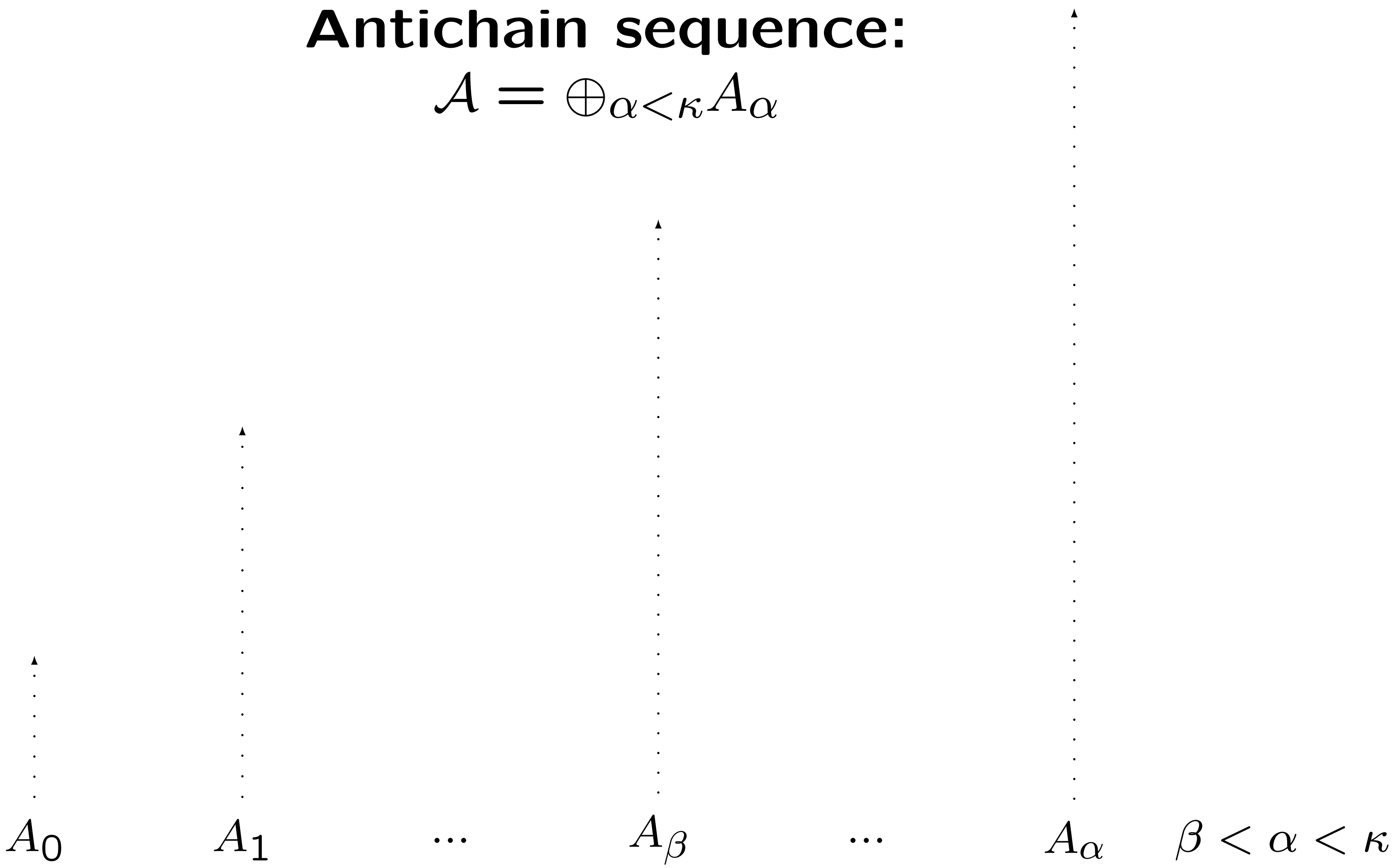
Antichain sequences

Definition (Hajnal-Sauer '86).
Assume $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is a family of sets.
$\mathcal{A}$ is said to be an **antichain sequence** iff:
(a) For all $\beta < \alpha < \kappa$, $|A_\beta| \leq |A_\alpha|$ and $A_\alpha \subseteq P$.
(b) Any $X \subseteq \bigcup_{\alpha<\kappa} A_\alpha$ such that $|X \cap A_\alpha| \leq 1$ for all $\alpha < \kappa$, is an antichain.

The **cofinality** of $\mathcal{A}$ is $\mathrm{cf}_P(\bigcup_{\alpha<\kappa} A_\alpha)$.

Antichain sequence:

$$\mathcal{A} = \oplus_{\alpha<\kappa} A_\alpha$$



Why everybody likes antichain sequences

Lemma 4. If P has an antichain sequence of cofinality λ, then P contains λ^κ antichains of size κ.

Proof. Suppose $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ is an antichain sequence. For all $\alpha < \kappa$, set $\lambda_\alpha = |A_\alpha|$. Finally, since $\langle \lambda_\alpha \mid \alpha < \kappa \rangle$ is non-decreasing, converging to λ:

$$|\{\mathrm{Im}(f) \mid f \in \prod_{\alpha<\kappa} A_\alpha\}| = \prod_{\alpha<\kappa} \lambda_\alpha = \lambda^\kappa. \quad \square$$

Surprise, Surprise

Not only that a stable subset exemplifies the existence of a single antichain, but actually:

Theorem 3. The following are equivalent:
(a) P contains a stable subset.
(b) P contains an antichain sequence of cofinality λ.

Main Result

Corollary. If λ is a prevalent singular cardinal then any poset of cofinality λ contains λ^κ antichains of size κ.

A related problem

Problem. Does there (consistently) exists a topological space of density and weight λ such that its hereditary compactness degree equals $\text{cf}(\lambda)$?

Observation. A counter-example of cofinality λ (to the Milner-Sauer conjecture) would induce such space.

Theorem. If λ is a prevalent singular cardinal, then the answer to the above problem is "No".

Where do the weight assumption come from?

The Milner-Sauer conjecture concerns posets of singular cofinality, so you would expect its topological analogue would concern spaces of singular density.

Theorem. If $\aleph_{\omega_1}$ is a prevalent singular cardinal, then any topological space of density <u>and weight</u> $\aleph_{\omega_1}$ is not hereditarily Lindelöf. (No separation axioms assumed)

Theorem (Juhász-Shelah, 2007).
There consistently exists a regular topological space of density $\aleph_{\omega_1}$, being hereditarily Lindelöf.

Open problem

For $\lambda > \mathrm{cf}(\lambda) = \kappa$,

Theorem (Gitik, 2005). Suppose $\lambda^{<\kappa} > \lambda$. If $A \subseteq \lambda$ codes the cardinals structure up to λ, and a stationary subset of $\mathcal{P}_\kappa(\lambda)$ of size λ, then $L[A] \models \lambda^{<\kappa} = \lambda$.

Problem. Suppose λ is not a prevalent cardinal. Does there exist a set $A \subseteq \lambda$ coding the cardinals structure up to λ, and $L[A] \models$ "λ **is** a prevalent singular cardinal"?

References

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[2] E.C. Milner, N. Sauer, *Remarks on the cofinality of a partially ordered set, and a generalization of König's lemma*. Discrete Math., Vol. 35 (1981), 165-171. http://dx.doi.org/10.1016/0012-365X(81)90205-3