Coloring vs. Chromatic

MFO workshop in Set Theory,
Oberwolfach, 13-Feb-2017

Assaf Rinot
Bar-Ilan University

A pointer

This talk centers around a joint paper with Chris Lambie-Hanson:

[LHR16] Reflection on the coloring and chromatic numbers, *submitted December 2016.*

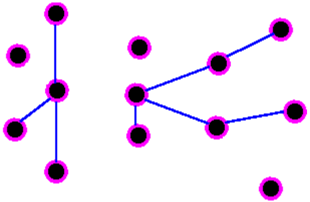
Grab it here:

HTTP://WWW.ASSAFRINOT.COM/PAPER/28

Graphs

Definition

A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
Elements of V are called the vertices of G;
Elements of E are called the edges of G.



Graphs

Definition

A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
Elements of V are called the vertices of G;
Elements of E are called the edges of G.

For an arbitrary graph G, the set of vertices of G is often denoted by $V(G)$, and the set of edges by $E(G)$.

Graphs

Definition

A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
Elements of V are called the vertices of G;
Elements of E are called the edges of G.

For an arbitrary graph G, the set of vertices of G is often denoted by $V(G)$, and the set of edges by $E(G)$.

Definition

The chromatic number of G, denoted $\mathrm{Chr}(G)$, is the least cardinal κ for which there exists a coloring $c : V(G) \to \kappa$ such that:

$$c(x) \neq c(y) \text{ for all } \{x, y\} \in E(G).$$

Graphs

Definition

A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
Elements of V are called the vertices of G;
Elements of E are called the edges of G.

For an arbitrary graph G, the set of vertices of G is often denoted by $V(G)$, and the set of edges by $E(G)$.

Definition

The coloring number of G, denoted $\mathrm{Col}(G)$, is the least cardinal κ for which there exists a well-ordering $\lhd$ of $V(G)$ such that for all $x \in V(G)$:

$$|\{y \lhd x \mid \{y, x\} \in E(G)\}| < \kappa.$$

Graphs

Definition

A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
Elements of V are called the vertices of G;
Elements of E are called the edges of G.

For an arbitrary graph G, the set of vertices of G is often denoted by $V(G)$, and the set of edges by $E(G)$.

Definition

The coloring number of G, denoted $\mathrm{Col}(G)$, is the least cardinal κ for which there exists a well-ordering $\lhd$ of $V(G)$ such that for all $x \in V(G)$:

$$|\{y \lhd x \mid \{y, x\} \in E(G)\}| < \kappa.$$

Note that $\mathrm{Chr}(G) \leq \mathrm{Col}(G)$, as one can define the coloring $c : V(G) \to \kappa$ by recursion over $(V(G), \lhd)$.

Compactness

Compactness for the chromatic number:

Theorem (de Bruijn-Erdős, 1951)

If G is a graph, k is a some positive integer, and all finite subgraphs of G have chromatic number $\leq k$, then $\mathrm{Chr}(G) \leq k$.

Compactness

Compactness for the chromatic number:

Theorem (de Bruijn-Erdős, 1951)

If G is a graph, k is a some positive integer, and all finite subgraphs of G have chromatic number $\leq k$, then $\mathrm{Chr}(G) \leq k$.
If G is a graph, $\kappa < \chi$, χ strongly-compact, and all $(< \chi)$-sized subgraphs of G have chromatic number $\leq \kappa$, then $\mathrm{Chr}(G) \leq \kappa$.

Compactness

Compactness for the chromatic number:

Theorem (de Bruijn-Erdős, 1951)

If G is a graph, k is a some positive integer, and all finite subgraphs of G have chromatic number $\leq k$, then $\mathrm{Chr}(G) \leq k$.
If G is a graph, $\kappa < \chi$, χ strongly-compact, and all $(< \chi)$-sized subgraphs of G have chromatic number $\leq \kappa$, then $\mathrm{Chr}(G) \leq \kappa$.

Compactness for the coloring number:

Theorem (Magidor-Shelah, 1994)

If G is a graph, $\kappa < \chi$, χ is strongly-compact, and all $(< \chi)$-sized subgraphs of G have coloring number $\leq \kappa$, then $\mathrm{Col}(G) \leq \kappa$.

Compactness

Compactness for the chromatic number:

Theorem (de Bruijn-Erdős, 1951)

If G is a graph, k is a some positive integer, and all finite subgraphs of G have chromatic number $\leq k$, then $\mathrm{Chr}(G) \leq k$.
If G is a graph, $\kappa < \chi$, χ strongly-compact, and all $(< \chi)$-sized subgraphs of G have chromatic number $\leq \kappa$, then $\mathrm{Chr}(G) \leq \kappa$.

Compactness for the coloring number:

Theorem (Magidor-Shelah, 1994)

If G is a graph, $\kappa < \chi$, χ is strongly-compact, and all $(< \chi)$-sized subgraphs of G have coloring number $\leq \kappa$, then $\mathrm{Col}(G) \leq \kappa$.
Modulo large cardinals, may consistently replace the strongly-compact χ with the first cardinal fixed-point.

Compactness

Singular compactness for the coloring number:

Theorem (Shelah, 1975)

If $G = (\theta, E)$ is a graph with θ a singular cardinal, $\aleph_0 \leq \kappa < \theta$, and all $(< \theta)$-sized subgraphs of G have coloring number $\leq \kappa$, then $\mathrm{Col}(G) \leq \kappa$.

Compactness for the coloring number:

Theorem (Magidor-Shelah, 1994)

If G is a graph, $\kappa < \chi$, χ is strongly-compact, and all $(< \chi)$-sized subgraphs of G have coloring number $\leq \kappa$, then $\mathrm{Col}(G) \leq \kappa$.
Modulo large cardinals, may consistently replace the strongly-compact χ with the first cardinal fixed-point.

Compactness

Singular compactness for the coloring number:

Theorem (Shelah, 1975)

If $G = (\theta, E)$ is a graph with θ a singular cardinal, $\aleph_0 \leq \kappa < \theta$, and all $(< \theta)$-sized subgraphs of G have coloring number $\leq \kappa$, then $\mathrm{Col}(G) \leq \kappa$.

There are a few more consistent compactness phenomena (due to Foreman-Laver, Magidor, Shelah, Unger), but let us move on to incompactness.

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies \exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies \exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies \exists$a$\omega$c with $\mathrm{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
- (Todorcevic, 1983) θ regular uncountable and there is a nonreflecting stationary subset of $E^\theta_\omega \implies \exists$a$\omega$c with $\mathrm{Chr}(G) > \aleph_0, |V(G)| = \theta$;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
- (Todorcevic, 1983) θ regular uncountable and there is a nonreflecting stationary subset of $E^\theta_\omega \implies$ $\exists$aωc with $\mathrm{Chr}(G) > \aleph_0, |V(G)| = \theta$;
- (Baumgartner, 1984) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ GCH;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies \exists$a$\omega$c with $\mathrm{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
- (Todorcevic, 1983) θ regular uncountable and there is a nonreflecting stationary subset of $E^\theta_\omega \implies \exists$a$\omega$c with $\mathrm{Chr}(G) > \aleph_0, |V(G)| = \theta$;
- (Baumgartner, 1984) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ GCH;
- (Komjáth, 1988) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ $\mathfrak{c} = \aleph_3$;

Incompactness for the chromatic number

Say that a graph G is <u>almost countably chromatic</u> (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies$ $\exists$aωc with $\mathrm{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
- (Todorcevic, 1983) θ regular uncountable and there is a nonreflecting stationary subset of $E^\theta_\omega \implies$ $\exists$aωc with $\mathrm{Chr}(G) > \aleph_0, |V(G)| = \theta$;
- (Baumgartner, 1984) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ GCH;
- (Komjáth, 1988) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ $\mathfrak{c} = \aleph_3$;
- (Komjáth, 1988) "$\exists$aωc $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_{\omega_1}$" cons. w/ $\mathfrak{c} = \aleph_{\omega_1+1}$;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies \exists$aωc with $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies \exists$a$\omega$c with $\mathrm{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
- (Todorcevic, 1983) θ regular uncountable and there is a nonreflecting stationary subset of $E^\theta_\omega \implies \exists$a$\omega$c with $\mathrm{Chr}(G) > \aleph_0, |V(G)| = \theta$;
- (Baumgartner, 1984) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ GCH;
- (Komjáth, 1988) "$\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \aleph_2$" cons. w/ $\mathfrak{c} = \aleph_3$;
- (Komjáth, 1988) "$\exists$aωc $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_{\omega_1}$" cons. w/ $\mathfrak{c} = \aleph_{\omega_1+1}$;
- (Shelah, 1990) "$\exists$aωc $\mathrm{Chr}(G) = \aleph_1, |V(G)| = \aleph_{\omega_1}$" cons. w/ GCH.

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Large gaps

- (Todorcevic, 1986) MA $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}$;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Large gaps

- (Todorcevic, 1986) MA $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}$;
- (L. Soukup, 1990) "$\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}^+$" is consistent with arbitrarily large $\mathfrak{c}$.

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Large gaps

- (Todorcevic, 1986) MA $\implies \exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}$;
- (L. Soukup, 1990) "$\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}^+$" is consistent with arbitrarily large $\mathfrak{c}$.

Very large gaps

- (Shelah, 1990) $V = L \implies \exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \theta$ provided that θ is regular not weakly-compact;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Large gaps

- (Todorcevic, 1986) MA $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}$;
- (L. Soukup, 1990) "$\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}^+$" is consistent with arbitrarily large $\mathfrak{c}$.

Very large gaps

- (Shelah, 1990) $V = L \implies \exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \theta$ provided that θ is regular not weakly-compact;
- (2015) $\mathrm{CH}_\lambda + \square_\lambda \implies$ for all $\mu \leq \lambda$ $\exists$aωc $\mathrm{Chr}(G) = \mu, |V(G)| = \lambda^+$;

Incompactness for the chromatic number

Say that a graph G is almost countably chromatic (aωc) if $\mathrm{Chr}(G') \leq \aleph_0$ for every subgraph G' of G with $|V(G')| < |V(G)|$.

Large gaps

- (Todorcevic, 1986) MA $\implies$ $\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}$;
- (L. Soukup, 1990) "$\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \mathfrak{c}^+$" is consistent with arbitrarily large $\mathfrak{c}$.

Very large gaps

- (Shelah, 1990) $V = L \implies$ $\exists$aωc with $\mathrm{Chr}(G) = |V(G)| = \theta$ provided that θ is regular not weakly-compact;
- (2015) $\mathrm{CH}_\lambda + \square_\lambda \implies$ for all $\mu \leq \lambda$ $\exists$aωc $\mathrm{Chr}(G) = \mu, |V(G)| = \lambda^+$;
- (2015) $\mathrm{CH}_\lambda + \square_\lambda + \lambda$ singular $\implies$ $\exists$aωc $\mathrm{Chr}(G) = |V(G)| = \lambda^+$.

Incompactness for the coloring number

Small gaps

(Shelah, 1975) Suppose $\kappa < \theta$ are infinite regular cardinals and there exists a nonreflecting stationary subset of E^θ_κ.
Then there is a graph $G = (\theta, E)$ such that $\mathrm{Col}(G \restriction \gamma) \leq \kappa$ for all $\gamma < \theta$, but $\mathrm{Col}(G) = \kappa^+$.

Incompactness for the coloring number

Large gaps

Incompactness for the coloring number

No large gaps, really [LHR16]

For any infinite cardinal κ, if $G = (\theta, E)$ is a graph, and $\mathsf{Col}(G \restriction \gamma) \leq \kappa$ for all $\gamma < \theta$, then $\mathsf{Col}(G) \leq \kappa^{++}$.

Incompactness for the coloring number

No large gaps, really [LHR16]

For any infinite cardinal κ, if $G = (\theta, E)$ is a graph, and $\mathrm{Col}(G \restriction \gamma) \leq \kappa$ for all $\gamma < \theta$, then $\mathrm{Col}(G) \leq \kappa^{++}$.

Gap 1

(Shelah, 1975) Suppose $\kappa < \theta$ are infinite regular cardinals and there exists a nonreflecting stationary subset of E^θ_κ.
Then there is a graph $G = (\theta, E)$ such that $\mathrm{Col}(G \restriction \gamma) \leq \kappa$ for all $\gamma < \theta$, but $\mathrm{Col}(G) = \kappa^+$.

Gap 2, anyone?

Is there a consistent example of an infinite cardinal κ and a graph $G = (\theta, E)$ with $\mathrm{Col}(G \restriction \gamma) \leq \kappa$ for all $\gamma < \theta$, but $\mathrm{Col}(G) = \kappa^{++}$?

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?
2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?
2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

In February 2013, Shelah uploaded a preprint to the arXiv (Sh:1018) that claimed to address Question 1, but Kojman and Komjath found some gaps (pun intended), and the status of this paper is unclear to me.

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?
2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

Very recently, Fuchino announced that for all $\kappa = \kappa^{<\kappa}$, $(a) \implies (b)$:

(a) $\exists G$ with $\mathrm{Col}(G) > \kappa$, yet $\mathrm{Col}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$;

(b) $\exists G$ with $\mathrm{Chr}(G) > \kappa$, yet $\mathrm{Chr}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$.

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?
2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

Very recently, Fuchino announced that for all $\kappa = \kappa^{<\kappa}$, $(a) \implies (b)$:

(a) $\exists G$ with $\mathrm{Col}(G) > \kappa$, yet $\mathrm{Col}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$;

(b) $\exists G$ with $\mathrm{Chr}(G) > \kappa$, yet $\mathrm{Chr}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$.

We shall focus on the possible inverse implication $(a) \impliedby (b)$.

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?
2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

Very recently, Fuchino announced that for all $\kappa = \kappa^{<\kappa}$, $(a) \implies (b)$:

(a) $\exists G$ with $\mathrm{Col}(G) > \kappa$, yet $\mathrm{Col}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$;

(b) $\exists G$ with $\mathrm{Chr}(G) > \kappa$, yet $\mathrm{Chr}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$.

Specifically, we shall give strong counterexamples to $(a) \impliedby (b)$.

Main questions

1. What's the relationship between compactness for the chromatic number and compactness for the coloring number?
2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

Very recently, Fuchino announced that for all $\kappa = \kappa^{<\kappa}$, $(a) \implies (b)$:

(a) $\exists G$ with $\mathrm{Col}(G) > \kappa$, yet $\mathrm{Col}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$;

(b) $\exists G$ with $\mathrm{Chr}(G) > \kappa$, yet $\mathrm{Chr}(G') \leq \kappa$ for G' with $|V(G')| < |V(G)|$.

Specifically, we shall give strong counterexamples to $(a) \impliedby (b)$.
In particular, answering Question 2 in the affirmative.

Yet another square principle

$\square_\lambda$ is a very powerful principle and comes handy in many combinatorial constructions. However it is too strong in the sense that it decides too many statements (such as TP, RP, AP, incompactness for both coloring and chromatic numbers).

Yet another square principle

$\square_\lambda$ is a very powerful principle and comes handy in many combinatorial constructions. However it is too strong in the sense that it decides too many statements (such as TP, RP, AP, incompactness for both coloring and chromatic numbers).

Trying to reduce a $\square_\lambda$-based construction into a construction from a more relaxed hypothesis, one usually considers one of the following weakening:

1. There exists a non-reflecting stationary subset of λ^+;
2. $\square^*_\lambda$;
3. $\square(\lambda^+)$.

Yet another square principle

$\square_\lambda$ is a very powerful principle and comes handy in many combinatorial constructions. However it is too strong in the sense that it decides too many statements (such as TP, RP, AP, incompactness for both coloring and chromatic numbers).

Trying to reduce a $\square_\lambda$-based construction into a construction from a more relaxed hypothesis, one usually considers one of the following weakening:

1. There exists a non-reflecting stationary subset of $E^{\lambda^+}_{\geq\chi}$;
2. $\square_{\lambda,\chi}$;
3. $\square(\lambda^+,\chi)$.

Yet another square principle

$\square_\lambda$ is a very powerful principle and comes handy in many combinatorial constructions. However it is too strong in the sense that it decides too many statements (such as TP, RP, AP, incompactness for both coloring and chromatic numbers).

Trying to reduce a $\square_\lambda$-based construction into a construction from a more relaxed hypothesis, one usually considers one of the following weakening:

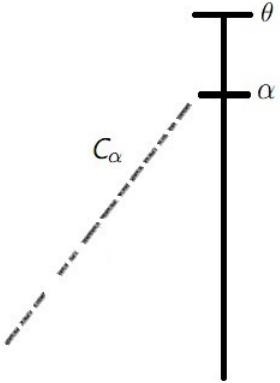
1. There exists a non-reflecting stationary subset of $E^{\lambda^+}_{\geq\chi}$;
2. $\square_{\lambda,\chi}$;
3. $\square(\lambda^+,\chi)$.

Let us introduce yet another variation...

C-sequences

A C-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

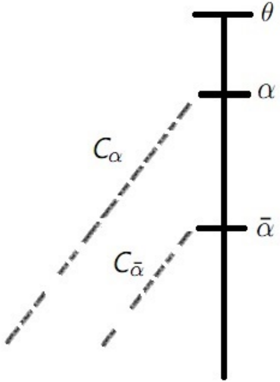
- For every limit $\alpha < \theta$, C_α is a club in α.



C-sequences

A C-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

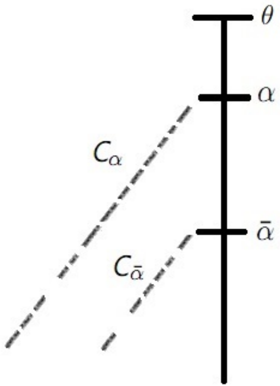
- For every limit $\alpha < \theta$, C_α is a club in α.



C-sequences

A coherent C-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta\rangle$ such that:

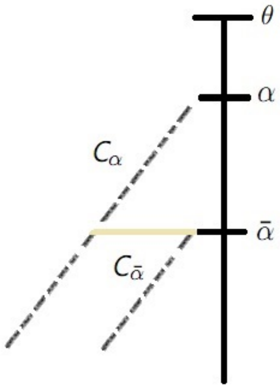
- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.



C-sequences

A coherent C-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

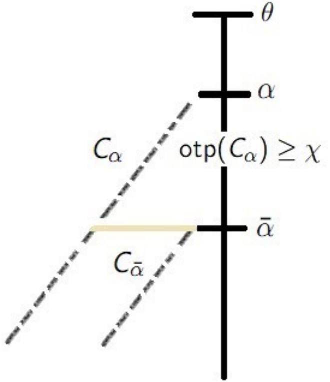
- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.



C-sequences

An $\sqsubseteq_\chi$-coherent C-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta\rangle$ such that:

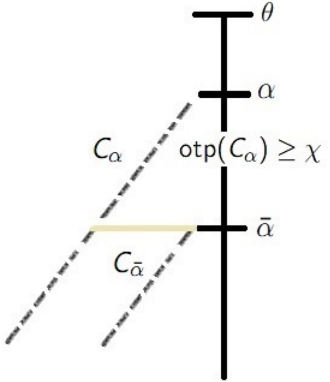
- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;



C-sequences

A $\square(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.



C-sequences

A $\square(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Shelah, 1991)

If θ is regular cardinal and $E^\theta_{\geq \aleph_2}$ admits a nonreflecting stationary set, then there is a θ-cc poset whose square is not θ-cc.

C-sequences

A $\Box(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Shelah, 1991)

If θ is regular cardinal and $E^\theta_{\geq\aleph_2}$ admits a nonreflecting stationary set, then there is a θ-cc poset whose square is not θ-cc.

Theorem (2014)

If $\theta > \chi$ are infinite regular cardinals, $\theta \geq \aleph_2$, and $\Box(\theta, \sqsubseteq_\chi)$ holds, then there is a θ-cc poset whose square is not θ-cc.

C-sequences

A $\square(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Gregory, 1976)

If GCH *holds and $E^{\aleph_2}_{\aleph_0}$ admits a nonreflecting stationary set, then there is an $\aleph_2$-Souslin tree.*

C-sequences

A $\square(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta\rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Gregory, 1976)

If GCH holds and $E^{\aleph_2}_{\aleph_0}$ admits a nonreflecting stationary set, then there is an $\aleph_2$-Souslin tree.

Theorem (2016)

If GCH holds and so does $\square(\aleph_2, \sqsubseteq_{\aleph_0})$, then there is an $\aleph_2$-Souslin tree.

C-sequences

A $\square(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Magidor, 1982)

Modulo the existence of a weakly compact cardinal,
it is consistent that $\aleph_2$ cannot be partitioned into two fat sets.

C-sequences

A $\square(\theta, \sqsubseteq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, C_α is a club in α;
- If $\bar\alpha \in \mathrm{acc}(C_\alpha)$ and $\mathrm{otp}(C_\alpha) \geq \chi$, then $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;
- For every club D in θ, there exists some $\alpha \in \mathrm{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Magidor, 1982)

*Modulo the existence of a weakly compact cardinal,
it is consistent that $\aleph_2$ cannot be partitioned into two fat sets.*

Theorem (2017)

*If $\theta > \chi$ are infinite regular cardinals and $\square(\theta, \sqsubseteq_\chi)$ holds,
then any fat subset of θ may be partitioned into θ many fat sets.*

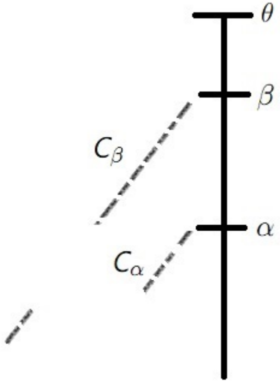
The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$



The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

Theorem ([LHR16])

If $\vec{C}$ is coherent, then $G(\vec{C})$ is aωc.

The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

Theorem ([LHR16])

If $\vec{C}$ is $\sqsubseteq_\chi$-coherent, then $G(\vec{C})$ is aχc.

The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

Theorem ([LHR16])

If $\vec{C}$ is $\sqsubseteq_\chi$-coherent, then $G(\vec{C})$ is aχc.

Theorem ([LHR16])

If $\vec{C}$ is a generic $\sqsubseteq_\chi$-coherent C-sequence, then $\mathrm{Chr}(G(\vec{C})) = \theta$.

The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

Theorem ([LHR16])

If $\vec{C}$ is $\sqsubseteq_\chi$-coherent, then $G(\vec{C})$ is aχc.

Theorem ([LHR16])

Assume GCH $+\square(\theta, \sqsubseteq_\chi)$, *and* $\theta = \lambda^+$.

- λ *is singular* $\implies \exists \sqsubseteq_\chi$*-coherent* C*-sequence* $\vec{C}$ *with* $\mathrm{Chr}(G(\vec{C})) = \lambda^+$.

The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

Theorem ([LHR16])

If $\vec{C}$ is $\sqsubseteq_\chi$-coherent, then $G(\vec{C})$ is aχc.

Theorem ([LHR16])

Assume GCH $+\square(\theta, \sqsubseteq_\chi)$, *and* $\theta = \lambda^+$.

- λ *is singular* $\implies \exists \sqsubseteq_\chi$-*coherent C-sequence $\vec{C}$ with* $\mathrm{Chr}(G(\vec{C})) = \lambda^+$;
- λ *is regular* $\implies \exists \sqsubseteq_\chi$-*coherent C-sequence $\vec{C}$ with* $\mathrm{Chr}(G(\vec{C})) \geq \lambda$.

The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

Definition

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

Theorem ([LHR16])

If $\vec{C}$ is $\sqsubseteq_\chi$-coherent, then $G(\vec{C})$ is aχc.

Theorem ([LHR16])

Assume GCH $+\square(\theta, \sqsubseteq_\chi)$, *and* $\theta = \lambda^+$.

- λ *is singular* $\implies \exists \sqsubseteq_\chi$*-coherent C-sequence $\vec{C}$ with* $\mathrm{Chr}(G(\vec{C})) = \lambda^+$;
- λ *is regular* $\implies \exists \sqsubseteq_\chi$*-coherent C-sequence $\vec{C}$ with* $\mathrm{Chr}(G(\vec{C})) \geq \lambda$;
- $\exists S \subseteq \theta$ $\mathrm{Refl}(S) \Rightarrow \exists \sqsubseteq_\chi$*-coherent C-sequence $\vec{C}$ with* $\mathrm{Chr}(G(\vec{C})) = \lambda^+$.

Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Theorem ([LHR16])

Assuming large cardinal axioms, the following are consistent:

1. χ *is a supercompact, and* $\mathcal{E}(\chi, \theta)$ *holds for all regular* $\theta > \chi$*;*

Recall (de Bruijn-Erdős, 1951)

If χ is strongly-compact, then $\mathcal{E}(\kappa, \theta)$ fails whenever $\theta > \chi > \kappa$.

Corollaries

Let $\mathcal{E}(\chi,\theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta\rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Theorem ([LHR16])

Assuming large cardinal axioms, the following are consistent:

1. χ *is a supercompact, and* $\mathcal{E}(\chi,\theta)$ *holds for all regular* $\theta > \chi$*;*
2. $\mathrm{FRP}(\aleph_2)$ *together with* $\mathcal{E}(\aleph_0,\aleph_2)$*;*

Theorem (Fuchino, Sakai, L. Soukup, Usuba, 2012)

$\mathrm{FRP}(\aleph_2)$ holds iff any $\aleph_2$-sized graph of uncountable coloring number contains an $\aleph_1$-sized subgraph of uncountable coloring number.

Corollaries

Let $\mathcal{E}(\chi,\theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta\rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Theorem ([LHR16])

Assuming large cardinal axioms, the following are consistent:

1. χ *is a supercompact, and* $\mathcal{E}(\chi,\theta)$ *holds for all regular* $\theta > \chi$;
2. $\mathrm{FRP}(\aleph_2)$ *together with* $\mathcal{E}(\aleph_0,\aleph_2)$;
3. $(\aleph_{\omega+1},\aleph_\omega) \twoheadrightarrow (\aleph_1,\aleph_0)$ *together with* $\mathcal{E}(\aleph_0,\aleph_{\omega+1})$;

Theorem ([LHR16])

$(\aleph_{\omega+1},\aleph_\omega) \twoheadrightarrow (\aleph_1,\aleph_0)$ implies that if $\aleph_0 \leq \kappa < \theta = \mathrm{cf}(\theta) \leq \aleph_{\omega+1}$, and $G = (\theta, E)$ is a graph such that $\mathrm{Col}(G \restriction \gamma) \leq \kappa$ for all $\gamma < \theta$, then $\mathrm{Col}(G) \leq \kappa^+$.

Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Theorem ([LHR16])

Assuming large cardinal axioms, the following are consistent:

1. χ *is a supercompact, and* $\mathcal{E}(\chi, \theta)$ *holds for all regular* $\theta > \chi$*;*
2. $\mathrm{FRP}(\aleph_2)$ *together with* $\mathcal{E}(\aleph_0, \aleph_2)$*;*
3. $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ *together with* $\mathcal{E}(\aleph_0, \aleph_{\omega+1})$*;*
4. RC, *and* $\mathcal{E}(\aleph_2, \theta)$ *holds for all regular* $\theta > \aleph_2$*;*

Theorem (Todorcevic, 1983)

RC holds iff any tree whose comparability graph is uncountably chromatic has an $\aleph_1$-sized subtree whose comparability graph is uncountably chromatic.

Corollaries

Let $\mathcal{E}(\chi,\theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta\rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Theorem ([LHR16])

Assuming large cardinal axioms, the following are consistent:

1. χ *is a supercompact, and* $\mathcal{E}(\chi,\theta)$ *holds for all regular* $\theta > \chi$;
2. $\mathsf{FRP}(\aleph_2)$ *together with* $\mathcal{E}(\aleph_0,\aleph_2)$;
3. $(\aleph_{\omega+1},\aleph_\omega) \twoheadrightarrow (\aleph_1,\aleph_0)$ *together with* $\mathcal{E}(\aleph_0,\aleph_{\omega+1})$;
4. RC, *and* $\mathcal{E}(\aleph_2,\theta)$ *holds for all regular* $\theta > \aleph_2$;
5. $\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$ *together with* $\mathcal{E}(\aleph_0,\aleph_{\omega^2+1})$;

Theorem (Magidor-Shelah, 1994)

$\Delta_{\chi,\theta}$ implies that for every infinite $\kappa < \chi$, every θ-sized graph of coloring number $> \kappa$, has a $(< \theta)$-sized subgraphs of coloring number $> \kappa$.

Corollaries

Let $\mathcal{E}(\chi,\theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta\rangle$ with $\mathrm{Chr}(G(\vec{C})) = \theta$. Then:

Theorem ([LHR16])

Assuming large cardinal axioms, the following are consistent:

1. χ *is a supercompact, and* $\mathcal{E}(\chi,\theta)$ *holds for all regular* $\theta > \chi$*;*
2. $\mathrm{FRP}(\aleph_2)$ *together with* $\mathcal{E}(\aleph_0,\aleph_2)$*;*
3. $(\aleph_{\omega+1},\aleph_\omega) \twoheadrightarrow (\aleph_1,\aleph_0)$ *together with* $\mathcal{E}(\aleph_0,\aleph_{\omega+1})$*;*
4. RC, *and* $\mathcal{E}(\aleph_2,\theta)$ *holds for all regular* $\theta > \aleph_2$*;*
5. $\Delta_{\aleph_{\omega^2},\aleph_{\omega^2+1}}$ *together with* $\mathcal{E}(\aleph_0,\aleph_{\omega^2+1})$*;*
6. $\Delta_{\theta,\mu}$ *holds for all regular* $\mu \geq \theta$*,* θ *is inaccessible, and* $\mathcal{E}(\aleph_0,\theta)$ *holds;*
7. $\mathrm{Refl}(E^{\aleph_2}_{\aleph_0})$ *together with* $\mathcal{E}(\aleph_0,\aleph_2)$*;*
8. $\mathrm{Refl}(\aleph_{\omega+1})$ *together with* $\mathcal{E}(\aleph_0,\aleph_{\omega+1})$*;*
9. $\mathrm{Refl}(\theta)$ *with* $\mathcal{E}(\aleph_0,\theta)$*, where* θ *is the least inaccessible cardinal.*