Coloring vs. Chromatic

MFO workshop in Set Theory,
Oberwolfach, 13-Feb-2017

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A pointer

This talk centers around a joint paper with Chris Lambie-Hanson:


Grab it here:

HTTP://WWW.ASSAFRINOT.COM/PAPER/28
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The chromatic number of $G$, denoted $\text{Chr}(G)$, is the least cardinal $\kappa$ for which there exists a coloring $c : V(G) \to \kappa$ such that:

$$c(x) \neq c(y) \text{ for all } \{x, y\} \in E(G).$$
Graphs

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The coloring number of \( G \), denoted \( \text{Col}(G) \), is the least cardinal \( \kappa \) for which there exists a well-ordering \( \triangleleft \) of \( V(G) \) such that for all \( x \in V(G) \):

\[ |\{ y \triangleleft x \mid \{y, x\} \in E(G)\}| < \kappa. \]
Graphs

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$$|\{y \triangleleft x \mid \{y, x\} \in E(G)\}| < \kappa.$$

Note that $\text{Chr}(G) \leq \text{Col}(G)$, as one can define the coloring $c : V(G) \to \kappa$ by recursion over $(V(G), \triangleleft)$.
Compactness

Compactness for the chromatic number:

**Theorem (de Bruijn-Erdős, 1951)**

If $G$ is a graph, $k$ is a some positive integer, and all finite subgraphs of $G$ have chromatic number $\leq k$, then $\text{Chr}(G) \leq k$. 
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If $G$ is a graph, $\kappa < \chi$, $\chi$ strongly-compact, and all ($< \chi$)-sized subgraphs of $G$ have chromatic number $\leq \kappa$, then $\text{Chr}(G) \leq \kappa$.***
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Modulo large cardinals, may consistently replace the strongly-compact $\chi$ with the first cardinal fixed-point.*
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Singular compactness for the coloring number:

**Theorem (Shelah, 1975)**

If $G = (\theta, E)$ is a graph with $\theta$ a singular cardinal, $\aleph_0 \leq \kappa < \theta$, and all $(< \theta)$-sized subgraphs of $G$ have coloring number $\leq \kappa$, then $\text{Col}(G) \leq \kappa$.

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There are a few more consistent compactness phenomena (due to Foreman-Laver, Magidor, Shelah, Unger), but let us move on to incompactness.
Incompactness for the chromatic number

Say that a graph $G$ is almost countably chromatic ($\omega c$) if $\text{Chr}(G') \leq \aleph_0$ for every subgraph $G'$ of $G$ with $|V(G')| < |V(G)|$.

Small gaps

- (Erdős-Hajnal, 1968) CH $\implies \exists \omega c$ with $\text{Chr}(G) = \aleph_1, |V(G)| = \aleph_2$;
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- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \implies \exists \omega c$ with $\text{Chr}(G) = \aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
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- (Galvin, 1973) $2^{\aleph_0} = 2^{\aleph_1} < 2^{\aleph_2} \iff \exists \omega c$ with Chr($G$) = $\aleph_2$ and $|V(G)| = (2^{\aleph_1})^+$;
- (Todorcevic, 1983) $\theta$ regular uncountable and there is a nonreflecting stationary subset of $E_\omega^\theta \iff \exists \omega c$ with Chr($G$) > $\aleph_0$, $|V(G)| = \theta$;
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Say that a graph $G$ is almost countably chromatic (aωc) if $\text{Chr}(G') \leq \aleph_0$ for every subgraph $G'$ of $G$ with $|V(G')| < |V(G)|$.

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- (Komjáth, 1988) “$\exists \omega c$ $\text{Chr}(G) = \aleph_1$, $|V(G)| = \aleph_\omega$” cons. w/ $c = \aleph_{\omega_1+1}$;
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* (2015) $\text{CH}_\lambda + \square_\lambda \implies$ for all $\mu \leq \lambda \exists \text{a}\omega\text{c Chr}(G) = \mu, |V(G)| = \lambda^+$;
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Small gaps

(Shelah, 1975) Suppose $\kappa < \theta$ are infinite regular cardinals and there exists a nonreflecting stationary subset of $E_{\kappa}^\theta$.
Then there is a graph $G = (\theta, E)$ such that $\text{Col}(G \upharpoonright \gamma) \leq \kappa$ for all $\gamma < \theta$, but $\text{Col}(G) = \kappa^+$. 
Incompactness for the coloring number

Large gaps
No large gaps, really [LHR16]

For any infinite cardinal \( \kappa \), if \( G = (\theta, E) \) is a graph, and \( \text{Col}(G \upharpoonright \gamma) \leq \kappa \) for all \( \gamma < \theta \), then \( \text{Col}(G) \leq \kappa^{++} \).
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Gap 1

(Shelah, 1975) Suppose $\kappa < \theta$ are infinite regular cardinals and there exists a nonreflecting stationary subset of $E^\theta_\kappa$. Then there is a graph $G = (\theta, E)$ such that $\text{Col}(G \upharpoonright \gamma) \leq \kappa$ for all $\gamma < \theta$, but $\text{Col}(G) = \kappa^+$. 

Gap 2, anyone?

Is there a consistent example of an infinite cardinal $\kappa$ and a graph $G = (\theta, E)$ with $\text{Col}(G \upharpoonright \gamma) \leq \kappa$ for all $\gamma < \theta$, but $\text{Col}(G) = \kappa^{++}$?
Main questions

1. What’s the relationship between compactness for the chromatic number and compactness for the coloring number?
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In February 2013, Shelah uploaded a preprint to the arXiv (Sh:1018) that claimed to address Question 1, but Kojman and Komjath found some gaps (pun intended), and the status of this paper is unclear to me.
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2. All known examples of very large gaps requires the existence of a non-reflecting stationary set. Is this a coincidence?

Very recently, Fuchino announced that for all $\kappa = \kappa^{<\kappa}$, $(a) \implies (b)$:

(a) $\exists G$ with $\text{Col}(G) > \kappa$, yet $\text{Col}(G') \leq \kappa$ for $G'$ with $|V(G')| < |V(G)|$;

(b) $\exists G$ with $\text{Chr}(G) > \kappa$, yet $\text{Chr}(G') \leq \kappa$ for $G'$ with $|V(G')| < |V(G)|$. 
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We shall focus on the possible inverse implication $(a) \iff (b)$. 

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Specifically, we shall give strong counterexamples to $(a) \iff (b)$. 
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Specifically, we shall give strong counterexamples to $(a) \iff (b)$. In particular, answering Question 2 in the affirmative.
Yet another square principle

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Trying to reduce a \(\square_\lambda\)-based construction into a construction from a more relaxed hypothesis, one usually considers one of the following weakening:

1. There exists a non-reflecting stationary subset of \(\lambda^+\);
2. \(\square^*_\lambda\);
3. \(\square(\lambda^+)\).
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Trying to reduce a \( \square_\lambda \)-based construction into a construction from a more relaxed hypothesis, one usually considers one of the following weakening:

1. There exists a non-reflecting stationary subset of \( E_{\geq \chi}^{\lambda^+} \);
2. \( \square_{\lambda, \chi} \);
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1. There exists a non-reflecting stationary subset of \(E_{\geq \chi}^{\lambda^+}\);
2. \(\square_{\lambda, \chi}\);
3. \(\square(\lambda^+, \chi)\).

Let us introduce yet another variation...
$C$-sequences

A $C$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$.
\textbf{C-sequences}

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**C-sequences**

A coherent $C$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\bar{\alpha} \in \text{acc}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
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C-sequences

An $\subseteq \chi$-coherent $C$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\bar{\alpha} \in \text{acc}(C_\alpha)$ and $\text{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
**C-sequences**

A $\Box(\theta, \preceq_\chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\bar{\alpha} \in \text{acc}(C_\alpha)$ and $\text{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- For every club $D$ in $\theta$, there exists some $\alpha \in \text{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

\[\begin{array}{c}
\theta \\
\downarrow \\
\alpha \\
\downarrow \\
C_\alpha \\
\downarrow \\
\bar{\alpha} \\
\downarrow \\
C_{\bar{\alpha}} \\
\downarrow \\
\text{otp}(C_\alpha) \geq \chi
\end{array}\]
**C-sequences**

A $\square(\theta, \subseteq \chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\check{\alpha} \in \text{acc}(C_\alpha)$ and $\text{otp}(C_\alpha) \geq \chi$, then $C_{\check{\alpha}} = C_\alpha \cap \check{\alpha}$;
- For every club $D$ in $\theta$, there exists some $\alpha \in \text{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

**Theorem (Shelah, 1991)**

If $\theta$ is regular cardinal and $E_\geq\aleph_2^\theta$ admits a nonreflecting stationary set, then there is a $\theta$-cc poset whose square is not $\theta$-cc.
**C-sequences**

A $\square(\theta, \subseteq \chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\bar{\alpha} \in \text{acc}(C_\alpha)$ and $\text{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- For every club $D$ in $\theta$, there exists some $\alpha \in \text{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

**Theorem (Shelah, 1991)**

If $\theta$ is regular cardinal and $E_{\geq \aleph_2}^\theta$ admits a nonreflecting stationary set, then there is a $\theta$-cc poset whose square is not $\theta$-cc.

**Theorem (2014)**

If $\theta > \chi$ are infinite regular cardinals, $\theta \geq \aleph_2$, and $\square(\theta, \subseteq \chi)$ holds, then there is a $\theta$-cc poset whose square is not $\theta$-cc.
C-sequences

A □(θ, ⊆_χ)-sequence is a sequence  \( \langle C_\alpha \mid \alpha < \theta \rangle \) such that:

- For every limit \( \alpha < \theta \), \( C_\alpha \) is a club in \( \alpha \);
- If \( \bar{\alpha} \in \text{acc}(C_\alpha) \) and \( \text{otp}(C_\alpha) \geq \chi \), then \( C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha} \);
- For every club \( D \) in \( \theta \), there exists some \( \alpha \in \text{acc}(D) \) with \( D \cap \alpha \neq C_\alpha \).

Theorem (Gregory, 1976)

If GCH holds and \( E^{\aleph_2}_{\aleph_0} \) admits a nonreflecting stationary set, then there is an \( \aleph_2 \)-Souslin tree.
C-sequences

A $\square(\theta, \kappa)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\bar{\alpha} \in \text{acc}(C_\alpha)$ and $\text{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- For every club $D$ in $\theta$, there exists some $\alpha \in \text{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Gregory, 1976)

If GCH holds and $E^{\aleph_2}_{\aleph_0}$ admits a nonreflecting stationary set, then there is an $\aleph_2$-Souslin tree.

Theorem (2016)

If GCH holds and so does $\square(\aleph_2, \kappa)$, then there is an $\aleph_2$-Souslin tree.
C-sequences

A $\square(\theta, \subseteq \chi)$-sequence is a sequence $\langle C_\alpha \mid \alpha < \theta \rangle$ such that:

- For every limit $\alpha < \theta$, $C_\alpha$ is a club in $\alpha$;
- If $\bar{\alpha} \in \text{acc}(C_\alpha)$ and $\text{otp}(C_\alpha) \geq \chi$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- For every club $D$ in $\theta$, there exists some $\alpha \in \text{acc}(D)$ with $D \cap \alpha \neq C_\alpha$.

Theorem (Magidor, 1982)

Modulo the existence of a weakly compact cardinal, it is consistent that $\aleph_2$ cannot be partitioned into two fat sets.
**C-sequences**

A □(θ, ⊆χ)-sequence is a sequence \( \langle C_\alpha \mid \alpha < \theta \rangle \) such that:

- For every limit \( \alpha < \theta \), \( C_\alpha \) is a club in \( \alpha \);
- If \( \bar{\alpha} \in \text{acc}(C_\alpha) \) and \( \text{otp}(C_\alpha) \geq \chi \), then \( C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha} \);
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**Theorem (Magidor, 1982)**

Modulo the existence of a weakly compact cardinal, it is consistent that \( \aleph_2 \) cannot be partitioned into two fat sets.

**Theorem (2017)**

If \( \theta > \chi \) are infinite regular cardinals and □(θ, ⊆χ) holds, then any fat subset of \( \theta \) may be partitioned into \( \theta \) many fat sets.
The $C$-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

**Definition**

Given a $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{ \{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta) \}.$$

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The $C$-sequence graph

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**Theorem ([LHR16])**

*If $\vec{C}$ is coherent, then $G(\vec{C})$ is $\omega_c$.***
The \( C \)-sequence graph

Let \( \theta > \chi \) denote infinite regular cardinals.

**Definition**

Given a \( C \)-sequence \( \vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle \), define a graph \( G(\vec{C}) := (\theta, E) \) by:

\[
E := \{ \{ \alpha, \beta \} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta) \}.
\]

**Theorem ([LHR16])**

If \( \vec{C} \) is \( \preceq_\chi \)-coherent, then \( G(\vec{C}) \) is a\( \chi \)c.
The $C$-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

**Definition**

Given a $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$ 

**Theorem ([LHR16])**

If $\vec{C}$ is $\preceq_\chi$-coherent, then $G(\vec{C})$ is a $\chi c$.

**Theorem ([LHR16])**

If $\vec{C}$ is a generic $\preceq_\chi$-coherent $C$-sequence, then $\text{Chr}(G(\vec{C})) = \theta$. 
The C-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

**Definition**

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}$.

**Theorem ([LHR16])**

If $\vec{C}$ is $\subseteq_\chi$-coherent, then $G(\vec{C})$ is a $\chi c$.

**Theorem ([LHR16])**

Assume GCH + $\square(\theta, \subseteq_\chi)$, and $\theta = \lambda^+$.

$\blacktriangleright \lambda$ is singular $\implies \exists \subseteq_\chi$-coherent C-sequence $\vec{C}$ with $\text{Chr}(G(\vec{C})) = \lambda^+$. 
The $C$-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

**Definition**

Given a $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

**Theorem ([LHR16])**

If $\vec{C}$ is $\sqsubseteq_\chi$-coherent, then $G(\vec{C})$ is a $\chi c$.

**Theorem ([LHR16])**

Assume GCH $+ \Box(\theta, \subseteq_\chi)$, and $\theta = \lambda^+$.

- $\lambda$ is singular $\implies \exists \subseteq_\chi$-coherent $C$-sequence $\vec{C}$ with $\text{Chr}(G(\vec{C})) = \lambda^+$;
- $\lambda$ is regular $\implies \exists \subseteq_\chi$-coherent $C$-sequence $\vec{C}$ with $\text{Chr}(G(\vec{C})) \geq \lambda$.  
The $C$-sequence graph

Let $\theta > \chi$ denote infinite regular cardinals.

**Definition**

Given a $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$, define a graph $G(\vec{C}) := (\theta, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\theta]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$

**Theorem ([LHR16])**

If $\vec{C}$ is $\leq_\chi$-coherent, then $G(\vec{C})$ is a $\chi c$.

**Theorem ([LHR16])**

Assume $GCH + \Box(\theta, \leq_\chi)$, and $\theta = \lambda^+$. 

- $\lambda$ is singular $\implies$ $\exists \leq_\chi$-coherent $C$-sequence $\vec{C}$ with $\text{Chr}(G(\vec{C})) = \lambda^+$;
- $\lambda$ is regular $\implies$ $\exists \leq_\chi$-coherent $C$-sequence $\vec{C}$ with $\text{Chr}(G(\vec{C})) \geq \lambda$;
- $\exists S \subseteq \theta \text{ Refl}(S) \implies \exists \leq_\chi$-coherent $C$-sequence $\vec{C}$ with $\text{Chr}(G(\vec{C})) = \lambda^+$. 

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Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\subseteq^\chi$-coherent $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:
Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:

**Theorem ([LHR16])**

Assuming large cardinal axioms, the following are consistent:

1. $\chi$ is a supercompact, and $\mathcal{E}(\chi, \theta)$ holds for all regular $\theta > \chi$;

Recall (de Bruijn-Erdős, 1951)

If $\chi$ is strongly-compact, then $\mathcal{E}(\kappa, \theta)$ fails whenever $\theta > \chi > \kappa$. 
Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\subseteq^\chi$-coherent $C$-sequence $\vec{C} = \langle C_\alpha | \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:

**Theorem ([LHR16])**

Assuming large cardinal axioms, the following are consistent:

1. $\chi$ is a supercompact, and $\mathcal{E}(\chi, \theta)$ holds for all regular $\theta > \chi$;
2. FRP($\aleph_2$) together with $\mathcal{E}(\aleph_0, \aleph_2)$;

**Theorem (Fuchino, Sakai, L. Soukup, Usuba, 2012)**

FRP($\aleph_2$) holds iff any $\aleph_2$-sized graph of uncountable coloring number contains an $\aleph_1$-sized subgraph of uncountable coloring number.
Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\sqsubseteq^\chi$-coherent $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:

**Theorem ([LHR16])**

Assuming large cardinal axioms, the following are consistent:

1. $\chi$ is a supercompact, and $\mathcal{E}(\chi, \theta)$ holds for all regular $\theta > \chi$;
2. $\text{FRP}(\aleph_2)$ together with $\mathcal{E}(\aleph_0, \aleph_2)$;
3. $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega+1})$;

**Theorem ([LHR16])**

$(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ implies that if $\aleph_0 \leq \kappa < \theta = \text{cf}(\theta) \leq \aleph_{\omega+1}$, and $G = (\theta, E)$ is a graph such that $\text{Col}(G \upharpoonright \gamma) \leq \kappa$ for all $\gamma < \theta$, then $\text{Col}(G) \leq \kappa^+$. 
Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\subseteq^\chi$-coherent $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:

**Theorem ([LHR16])**

Assuming large cardinal axioms, the following are consistent:

1. $\chi$ is a supercompact, and $\mathcal{E}(\chi, \theta)$ holds for all regular $\theta > \chi$;
2. FRP($\aleph_2$) together with $\mathcal{E}(\aleph_0, \aleph_2)$;
3. $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega+1})$;
4. RC, and $\mathcal{E}(\aleph_2, \theta)$ holds for all regular $\theta > \aleph_2$;

**Theorem (Todorcevic, 1983)**

RC holds iff any tree whose comparability graph is uncountably chromatic has an $\aleph_1$-sized subtree whose comparability graph is uncountably chromatic.
Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\subseteq \chi$-coherent $C$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:

### Theorem ([LHR16])

**Assuming large cardinal axioms, the following are consistent:**

1. $\chi$ is a supercompact, and $\mathcal{E}(\chi, \theta)$ holds for all regular $\theta > \chi$;
2. $\text{FRP}(\aleph_2)$ together with $\mathcal{E}(\aleph_0, \aleph_2)$;
3. $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega+1})$;
4. $\text{RC}$, and $\mathcal{E}(\aleph_2, \theta)$ holds for all regular $\theta > \aleph_2$;
5. $\Delta_{\aleph_2, \aleph_{2+1}}$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega^2+1})$.

### Theorem (Magidor-Shelah, 1994)

$\Delta_{\chi, \theta}$ implies that for every infinite $\kappa < \chi$, every $\theta$-sized graph of coloring number $> \kappa$, has a ($< \theta$)-sized subgraphs of coloring number $> \kappa$. 

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Corollaries

Let $\mathcal{E}(\chi, \theta)$ stand for the assertion that there exists an $\sqsubseteq_\chi$-coherent $\mathcal{C}$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \theta \rangle$ with $\text{Chr}(G(\vec{C})) = \theta$. Then:

**Theorem ([LHR16])**

Assuming large cardinal axioms, the following are consistent:

1. $\chi$ is a supercompact, and $\mathcal{E}(\chi, \theta)$ holds for all regular $\theta > \chi$;
2. $\text{FRP}(\aleph_2)$ together with $\mathcal{E}(\aleph_0, \aleph_2)$;
3. $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega+1})$;
4. $\text{RC}$, and $\mathcal{E}(\aleph_2, \theta)$ holds for all regular $\theta > \aleph_2$;
5. $\Delta_{\aleph_0^{\aleph_2}, \aleph_{\omega^2+1}}$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega^2+1})$;
6. $\Delta_{\theta, \mu}$ holds for all regular $\mu \geq \theta$, $\theta$ is inaccessible, and $\mathcal{E}(\aleph_0, \theta)$ holds;
7. $\text{Refl}(E_{\aleph_2}^{\aleph_0})$ together with $\mathcal{E}(\aleph_0, \aleph_2)$;
8. $\text{Refl}(\aleph_{\omega+1})$ together with $\mathcal{E}(\aleph_0, \aleph_{\omega+1})$;
9. $\text{Refl}(\theta)$ with $\mathcal{E}(\aleph_0, \theta)$, where $\theta$ is the least inaccessible cardinal.