Chain conditions, unbounded colorings and the $C$-sequence spectrum

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Bibliography

Most results are taken from the following joint papers with Chris Lambie-Hanson:

2. Knaster and friends II: The C-sequence number, *to be submitted*.
Conventions

- $\kappa$ and $\lambda$ denote infinite cardinals;
- $\text{Reg}(\kappa) := \{\theta < \kappa \mid \text{cf}(\theta) = \theta \geq \aleph_0\}$;
- $E^\kappa_{\geq \chi} := \{\alpha < \kappa \mid \text{cf}(\alpha) \geq \chi\}$ and $E^\kappa_{> \chi} := \{\alpha < \kappa \mid \text{cf}(\alpha) > \chi\}$;
- $[A]^\chi := \{a \subseteq A \mid |a| = \chi\}$ and $[A]^{< \chi} := \{a \subseteq A \mid |a| < \chi\}$;
- For $a$, $b$, nonempty sets of ordinals, $a < b$ means that $\sup(a) < \min(b)$. 
Chain conditions

Let $\mathbb{P} := \langle P, \leq \rangle$ denote a poset.

**Definition**
For a subset $X \subseteq P$, we write $\bigwedge X := \{ z \in P \mid \forall x \in X(z \leq x) \}$. We say that $x, y \in P$ are **compatible** iff $\bigwedge\{x, y\} \neq \emptyset$.

**Definition**

- $\mathbb{P}$ satisfies the **$\kappa$-cc** iff $\forall A \in [P]^{\kappa} \exists X \in [A]^2 \ \bigwedge X \neq \emptyset$;
- $\mathbb{P}$ is **$\kappa$-Knaster** iff $\forall A \in [P]^{\kappa} \exists B \in [A]^{\kappa} \ \forall X \in [B]^2 \ \bigwedge X \neq \emptyset$;
- $\mathbb{P}$ has **precaliber $\kappa$** iff $\forall A \in [P]^{\kappa} \exists B \in [A]^{\kappa} \ \forall X \in [B]^{<\omega} \ \bigwedge X \neq \emptyset$;
- $\mathbb{P}$ is **$\kappa$-stationarily layered** iff $\{ Q \in [P]^{<\kappa} \mid \langle Q, \leq \rangle \text{ is a regular suborder of } \mathbb{P} \}$ is stationary in $[P]^{<\kappa}$.
The product order (aka, coordinatewise order)

Given posets $\langle P_1, \leq_1 \rangle, \langle P_2, \leq_2 \rangle$, consider their product $\langle P_1 \times P_2, \sqsubseteq \rangle$, where $(x, y) \sqsubseteq (x', y')$ iff $x \leq_1 x'$ and $y \leq_2 y'$. (Longer products are defined analogously.)

Question
Suppose that $\langle P_1, \leq_1 \rangle, \langle P_2, \leq_2 \rangle$ satisfy the $\kappa$-cc. Must their product satisfy the $\kappa$-cc?

Sufficient condition
If one of the posets is moreover $\kappa$-Knaster, then “yes”.

Definition
Let $C_\kappa$ denote the assertion that the product of any two $\kappa$-cc posets is again $\kappa$-cc.

Note: It suffices to consider squares $C_\kappa$ iff $\mathbb{P}^2$ is $\kappa$-cc for every $\kappa$-cc poset $\mathbb{P}$. 
Basic facts

Fact 1. $C_\kappa$ holds for $\kappa = \aleph_0$.

We moreover show that every $\kappa$-cc poset $\langle P, \leq \rangle$ is $\kappa$-Knaster.

Given $A \in [P]^\kappa$, define a coloring $c : [A]^2 \to 2$ via $c(x, y) = 1$ iff $\bigwedge \{x, y\} \neq \emptyset$.

By Ramsey’s theorem, there exists $B \in [A]^\kappa$ which is $c$-homogeneous.

As $|B| = \kappa$ and $\langle P, \leq \rangle$ satisfies the $\kappa$-cc, there exists $X \in [B]^2$ with $\bigwedge X \neq \emptyset$. But $B$ is $c$-homogeneous, and hence, for every $X \in [B]^2$, $\bigwedge X \neq \emptyset$, so that $B$ is as sought.

Fact 2. $C_\kappa$ holds for $\kappa$ weakly compact.

$\kappa$ is weakly compact iff $\kappa > \aleph_0$ and for every $c : [\kappa]^2 \to 2$, there exists $B \in [\kappa]^\kappa$ which is homogeneous for $c$.

Fact 3. $C_\kappa$ holds for $\kappa$ singular strong limit.

- Erdős and Tarski (1943): If $\kappa$ is a singular cardinal and a poset $P$ satisfies the $\kappa$-cc, then $P$ satisfies the $\lambda$-cc for some $\lambda < \kappa$.
- Kurepa (1963): If $P$ satisfies the $\lambda^+$-cc, then $P^2$ satisfies the $(2^\lambda)^+$-cc.
The case $\kappa = \aleph_1$.

Question (Marczewski, 1947)
Is $C_{\aleph_1}$ (aka, “productivity of the ccc”) true?

Answers

▸ (Kurepa, 1952): $C_{\aleph_1}$ entails Souslin’s hypothesis.
▸ (Kunen; Rowbottom; Solovay; Hajnal-Juhász; Juhász, 1970’s) MA$_{\aleph_1}$ entails $C_{\aleph_1}$.
▸ (Todorcevic-Velickovic, 1987) MA$_{\aleph_1}$ iff every ccc poset has precaliber $\aleph_1$.
▸ (Roitman, 1979): After adding random/Cohen real, $C_{\aleph_1}$ fails;
▸ (Fleissner, 1978): After adding $\kappa$ many Cohen reals, there exists a ccc poset $\mathbb{P}$, such that $\mathbb{P}^2$ has antichain of size $\kappa$;
▸ (Galvin, 1980) after (Laver, unpublished): $c = \aleph_1$ refutes $C_{\aleph_1}$.
▸ (Todorcevic, 1988): $b = \aleph_1$ refutes $C_{\aleph_1}$.

Open problem
Is MA$_{\aleph_1}$ equivalent to $C_{\aleph_1}$?
The case $\kappa > \aleph_1$. Counterexamples in ZFC

Theorem (Todorcevic, 1985)
$\mathcal{C}_{\text{cf}(\beth_{\alpha+1})}$ fails for every limit ordinal $\alpha$.
Moreover, if $\lambda$ is a cardinal for which there exists a linear order of size $2^\lambda$ with a dense subset of size $\lambda$, then $\mathcal{C}_\kappa$ fails, for $\kappa = \text{cf}(2^\lambda)$.

Theorem (Todorcevic, 1986)
$\mathcal{C}_{\lambda^+}$ fails whenever $\lambda$ singular, and $\theta^\text{cf}(\lambda) < \lambda$ for all $\theta < \lambda$.

Theorem (Todorcevic, 1989)
$\mathcal{C}_{\lambda^+}$ fails whenever $\lambda$ singular, and $2^{\text{cf}(\lambda)} < \lambda$.

Theorem (Shelah, 1994)
$\mathcal{C}_{\lambda^+}$ fails whenever $\lambda$ singular.
More counterexamples in ZFC

Theorem (Shelah, 1990–1997)

$C_{\lambda^+}$ fails whenever $\lambda$ is a regular cardinal $\geq \aleph_1$. Specifically:
- [Sh:280]: $\lambda > 2^{\aleph_0}$;
- [Sh:327]: $\lambda > \aleph_1$;
- [Sh:572]: $\lambda = \aleph_1$.

Corollary

$C_\kappa$ fails for every successor cardinal $\kappa > \aleph_1$.

Conjecture (Todorcevic, 1980’s)

For every regular cardinal $\kappa > \aleph_1$, $C_\kappa$ iff $\kappa$ is weakly compact.

Theorem (2014)

For every regular cardinal $\kappa > \aleph_1$, $C_\kappa$ entails ($\kappa$ is weakly compact)$^L$.

In fact, $C_\kappa$ entails $\neg \Box(\kappa)$ and that every stationary subset of $\kappa$ reflects.
Longer products and stronger chain conditions

Shortly after our work on Todorcevic’s conjecture, Lücke and his colleagues addressed analogous questions involving stronger variations of the $\kappa$-cc. We mention three results:

Characterization theorem (Cox and Lücke, 2016)

For every regular uncountable cardinal $\kappa$:
$\kappa$ is weakly compact iff every $\kappa$-cc poset is moreover $\kappa$-stationarily layered.

Non-characterization theorem (Cox and Lücke, 2016)

Suppose $\kappa$ is weakly compact. In some cofinality-preserving forcing extension:
For every $\theta < \kappa$, the class of $\kappa$-Knaster posets is closed under $\theta$-support products, yet, $\kappa$ is not weakly compact.

Theorem (Lambie-Hanson and Lücke, 2018)

Suppose $\theta < \kappa$ are infinite and regular. If the class of $\kappa$-Knaster posets is closed under $\theta$-support products, then $\neg \square(\kappa)$, so that $(\kappa$ is weakly compact)$^L$. 
How to cook up a counterexample

Hereafter, $\kappa$ denotes a regular uncountable cardinal.

Galvin (1980) gave a consistent construction of an anti-Ramsey coloring $c : [\kappa]^2 \to 2$ from which he derived a $\kappa$-cc poset whose square is not $\kappa$-cc. In 1997, Shelah constructed a ZFC example of such a coloring for $\kappa = \aleph_2$.

Lambie-Hanson and Lücke (2018) gave a consistent construction of non-special $\kappa$-tree from which they derived a $\kappa$-Knaster poset whose infinite power is not $\kappa$-cc. They proved that such a tree exists, assuming $\Box(\kappa)$.

We would like to obtain the conclusions of Lambie-Hanson and Lücke from ZFC, e.g., getting a ZFC example of an $\aleph_2$-Knaster poset whose $\omega^{th}$-power is not $\aleph_2$-cc.

For this, let us revisit Galvin’s approach.
From a coloring $c : [\kappa]^2 \to \theta$ with $\theta \in \text{Reg}(\kappa)$, we derive posets:

- $\mathbb{P} := \{ (x, i) \mid x \in [\kappa]^{<\omega}, c``[x]^2 \subseteq \{i\} \}$. 
Colorings

From a coloring $c : [\kappa]^2 \to \theta$ with $\theta \in \text{Reg}(\kappa)$, we derive posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]<\omega, c''[x]^2 \subseteq \{i\}\}$;
- $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]<\omega, c''[x]^2 \cap i = \emptyset\}$. 
Colorings

From a coloring $c : [\kappa]^2 \to \theta$ with $\theta \in \text{Reg}(\kappa)$, we derive posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{< \omega}, c"[x]^2 \subseteq \{i\}\}$;
- $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{< \omega}, c"[x]^2 \cap i = \emptyset\}$.

Ordering: $(x, i)$ extends $(y, j)$ iff $x \supseteq y$ and $i = j$. 
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Key feature

- $\mathbb{P}^2$ fails to have the $\kappa$-cc;
- $\mathbb{Q}^{\theta}$ fails to have the $\kappa$-cc.
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Key feature

- $\mathbb{P}^2$ fails to have the $\kappa$-cc, e.g., $\{(\{\alpha\}, 0), (\{\alpha\}, 1) \mid \alpha < \kappa\}$.
- $\mathbb{Q}^\theta$ fails to have the $\kappa$-cc.

About $\mathbb{P}^2$.

For $\alpha < \beta < \kappa$ and $i := c(\alpha, \beta)$, $(\{\alpha\}, 1 - i)$ and $(\{\beta\}, 1 - i)$ are $\mathbb{P}$-incompatible. □
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From a coloring \( c : [\kappa]^2 \to \theta \) with \( \theta \in \text{Reg}(\kappa) \), we derive posets:

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- \( \mathbb{Q} := \{ (x, i) \mid x \in [\kappa]^{<\omega}, \, c^"{[x]^2} \cap i = \emptyset \} \).

Ordering: \((x, i)\) extends \((y, j)\) iff \(x \supseteq y\) and \(i = j\).

Key feature

- \( \mathbb{P}^2 \) fails to have the \( \kappa \)-cc, e.g., \( \{ (\{\alpha\}, i) \mid i < 2 \mid \alpha < \kappa \} \).
- \( \mathbb{Q}^\theta \) fails to have the \( \kappa \)-cc, e.g., \( \{ (\{\alpha\}, i) \mid i < \theta \mid \alpha < \kappa \} \).

About \( \mathbb{P}^2 \).
For \( \alpha < \beta < \kappa \) and \( i := c(\alpha, \beta) \), \( (\{\alpha\}, 1 - i) \) and \( (\{\beta\}, 1 - i) \) are \( \mathbb{P} \)-incompatible.

About \( \mathbb{Q}^\theta \).
For \( \alpha < \beta < \kappa \) and \( i := c(\alpha, \beta) \), \( (\{\alpha\}, i + 1) \) and \( (\{\beta\}, i + 1) \) are \( \mathbb{Q} \)-incompatible.
Colorings

From a coloring $c : [\kappa]^2 \to \theta$ with $\theta \in \text{Reg}(\kappa)$, we derive posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{\omega}, c"[x]^2 \subseteq \{i\}\}$;
- $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{\omega}, c"[x]^2 \cap i = \emptyset\}$.

Ordering: $(x, i)$ extends $(y, j)$ iff $x \supseteq y$ and $i = j$.

Key feature

- $\mathbb{P}^2$ fails to have the $\kappa$-cc;
- $\mathbb{Q}^\theta$ fails to have the $\kappa$-cc.

The heart of the matter is to construct $c$ for which the corresponding $\mathbb{P}$ be $\kappa$-cc, or $\mathbb{Q}^\tau$ be $\kappa$-Knaster for all $\tau < \theta$.  

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By a simple reverse-engineering process, one arrives at a reformulation of these features in the language of the coloring $c$. 

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By a simple reverse-engineering process, one arrives at a reformulation of these features in the language of the coloring $c$.

The poset $\mathbb{P}$ was analyzed by Galvin. Today, we shall focus on the poset $\mathbb{Q}$.
Suppose \( Q := \{(x, i) \mid x \in [\kappa]^{<\omega}, c^{``[x]^2 \cap i = \emptyset}\} \) is derived from \( c : [\kappa]^2 \to \theta \).

Assuming \( \theta \in \text{Reg}(\kappa) \), \( Q \) is \( \kappa \)-Knaster iff it has precaliber \( \kappa \) iff \( c \) witnesses \( U(\kappa, \theta) \):

**Definition**

\( U(\kappa, \theta) \) asserts that there exists a coloring \( c : [\kappa]^2 \to \theta \) such that for every family \( A \subseteq [\kappa]^{<\omega} \) consisting of \( \kappa \)-many pairwise disjoint sets, and every \( i < \theta \), there is \( B \in [A]^\kappa \) such that \( \min(c[a \times b]) \geq i \) for every pair \( a < b \) from \( B \).

There is also a \( \chi \)-closed variation: \( \{(x, i) \mid x \in [\kappa]^{<\chi}, c^{``[x]^2 \cap i = \emptyset}\} \). For this, we need:

**Definition**

\( U(\kappa, \theta, \chi) \) asserts there is a coloring \( c : [\kappa]^2 \to \theta \) such that for every \( \chi' < \chi \), every family \( A \subseteq [\kappa]^\chi' \) consisting of \( \kappa \)-many pairwise disjoint sets, and every \( i < \theta \), there is \( B \in [A]^\kappa \) such that \( \min(c[a \times b]) \geq i \) for every pair \( a < b \) from \( B \).
The coloring axiom

Definition

$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \rightarrow \theta$ such that for every $\chi' < \chi$, every family $A \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [A]^{\mu}$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B.$

Note that $\text{Pr}_1(\kappa, \kappa, \theta, \chi)$ entails $U(\kappa, 2, \theta, \chi).$
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\(U(\kappa, \mu, \theta, \chi)\) asserts there is a coloring \(c : [\kappa]^2 \to \theta\) such that for every \(\chi' < \chi\), every family \(A \subseteq [\kappa]^\chi'\) consisting of \(\kappa\)-many pairwise disjoint sets, and every \(i < \theta\), there is \(B \in [A]^{\mu}\) such that \(\min(c[a \times b]) \geq i\) for every pair \(a < b\) from \(B\).

Proposition
Suppose \(\chi, \theta \in \text{Reg}(\kappa)\) and that \(\kappa\) is \((<\chi)\)-inaccessible. For every coloring \(c : [\kappa]^2 \to \theta\) witnessing \(U(\kappa, \mu, \theta, \chi)\), the corresponding poset \(Q\) satisfies the following:
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Suppose $\chi, \theta \in \text{Reg}(\kappa)$ and that $\kappa$ is $(<\chi)$-inaccessible. For every coloring $c : [\kappa]^2 \to \theta$ witnessing $U(\kappa, \mu, \theta, \chi)$, the corresponding poset $\mathbb{Q}$ satisfies the following:

- $\mathbb{Q}^\theta$ is not $\kappa$-cc;
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Definition

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Proposition

Suppose \( \chi, \theta \in \text{Reg}(\kappa) \) and that \( \kappa \) is \( (\vartriangleleft \chi) \)-inaccessible. For every coloring \( c : [\kappa]^2 \to \theta \) witnessing \( U(\kappa, \mu, \theta, \chi) \), the corresponding poset \( Q \) satisfies the following:

\( \uparrow \) \( Q^\theta \) is not \( \kappa \)-cc;

\( \uparrow \) if \( \mu = 2 \), then \( Q^\tau \) is \( \kappa \)-cc for all \( \tau < \min\{\chi, \theta\} \);
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$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

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Suppose $\chi, \theta \in \text{Reg}(\kappa)$ and that $\kappa$ is ($<\chi$)-inaccessible. For every coloring $c : [\kappa]^2 \to \theta$ witnessing $U(\kappa, \mu, \theta, \chi)$, the corresponding poset $Q$ satisfies the following:
- $Q^\theta$ is not $\kappa$-cc;
- if $\mu = 2$, then $Q^\tau$ is $\kappa$-cc for all $\tau < \min\{\chi, \theta\}$;
- if $\mu = \kappa$, then $Q^\tau$ has precaliber $\kappa$ for all $\tau < \min\{\chi, \theta\}$;
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\( U(\kappa, \mu, \theta, \chi) \) asserts there is a coloring \( c : [\kappa]^2 \to \theta \) such that for every \( \chi' < \chi \), every family \( A \subseteq [\kappa]^{\chi'} \) consisting of \( \kappa \)-many pairwise disjoint sets, and every \( i < \theta \), there is \( B \in [A]^\mu \) such that \( \min(c[a \times b]) \geq i \) for every pair \( a < b \) from \( B \).

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Suppose \( \chi, \theta \in \text{Reg}(\kappa) \) and that \( \kappa \) is \((<\chi)\)-inaccessible. For every coloring \( c : [\kappa]^2 \to \theta \) witnessing \( U(\kappa, \mu, \theta, \chi) \), the corresponding poset \( Q \) satisfies the following:

- \( Q^\theta \) is not \( \kappa \)-cc;
- if \( \mu = 2 \), then \( Q^\tau \) is \( \kappa \)-cc for all \( \tau < \min\{\chi, \theta\} \);
- if \( \mu = \kappa \), then \( Q^\tau \) has precaliber \( \kappa \) for all \( \tau < \min\{\chi, \theta\} \);
- \( Q \) is well-met and \( \chi \)-directed-closed with greatest lower bounds.
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$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $A \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [A]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

Conjecture

For $\kappa$ regular uncountable, $\kappa$ is weakly compact iff $U(\kappa, 2, \omega, 2)$ fails.
The coloring axiom

**Definition**

\( U(\kappa, \mu, \theta, \chi) \) asserts there is a coloring \( c : [\kappa]^2 \rightarrow \theta \) such that for every \( \chi' < \chi \), every family \( \mathcal{A} \subseteq [\kappa]^{\chi'} \) consisting of \( \kappa \)-many pairwise disjoint sets, and every \( i < \theta \), there is \( B \in [\mathcal{A}]^\mu \) such that \( \min(c[a \times b]) \geq i \) for every pair \( a < b \) from \( B \).

**Conjecture**

For \( \kappa \) regular uncountable, \( \kappa \) is weakly compact iff \( U(\kappa, 2, \omega, 2) \) fails.

In other words, we ask whether the existence of a \( \kappa \)-Aronszajn tree gives rise to a coloring \( c : [\kappa]^2 \rightarrow \omega \) with the property that \( \sup(c"[A]^2) = \omega \) for every \( A \in [\kappa]^\kappa \).
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**Definition**

$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

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In other words, we ask whether the existence of a $\kappa$-Aronszajn tree gives rise to a coloring $c : [\kappa]^2 \to \omega$ with the property that $\sup(c''[A]^2) = \omega$ for every $A \in [\kappa]^\kappa$.

**Partial answer 1**

The existence of a $\kappa$-Aronszajn tree with an $\omega$-ascent path entails $U(\kappa, 2, \omega, \omega)$. 
The coloring axiom

Definition
$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $A \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [A]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

Conjecture
For $\kappa$ regular uncountable, $\kappa$ is weakly compact iff $U(\kappa, 2, \omega, 2)$ fails.
In other words, we ask whether the existence of a $\kappa$-Aronszajn tree gives rise to a coloring $c : [\kappa]^2 \to \omega$ with the property that $\sup(c''[A]^2) = \omega$ for every $A \in [\kappa]^\kappa$.

Partial answer 1
The existence of a $\kappa$-Aronszajn tree with an $\omega$-ascent path entails $U(\kappa, 2, \omega, \omega)$.

Partial answer 2 (with Todorcevic)
The existence of a coherent $\kappa$-Aronszajn tree entails $U(\kappa, 2, \omega, \omega)$ but not $U(\kappa, \kappa, \omega, \omega)$. 
Inspecting the parameters

Definition

\( U(\kappa, \mu, \theta, \chi) \) asserts there is a coloring \( c : [\kappa]^2 \to \theta \) such that for every \( \chi' < \chi \), every family \( A \subseteq [\kappa]^\chi' \) consisting of \( \kappa \)-many pairwise disjoint sets, and every \( i < \theta \), there is \( B \in [A]^\mu \) such that \( \min(c[a \times b]) \geq i \) for every pair \( a < b \) from \( B \).

About the second parameter

- \( U(\kappa, 2, \theta, \chi) \) iff \( U(\kappa, \omega, \theta, \chi) \);
- Suppose \( c \models U(\kappa, 2, \theta, \chi) \). If \( c \) is closed, then \( c \models U(\kappa, \kappa, \theta, \chi) \).

Definition

\( c : [\kappa]^2 \to \theta \) is closed iff \( \{ \alpha < \beta \mid c(\alpha, \beta) \leq i \} \) is closed below \( \beta \) for all \( \beta < \kappa \), \( i < \theta \).
Inspecting the parameters

Definition

$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $A \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [A]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

About the third parameter

- $U(\kappa, \kappa, \kappa, \kappa)$ holds;
- $U(\kappa, \mu, \theta, \chi)$ iff $U(\kappa, \mu, \text{cf}(\theta), \chi)$;

Therefore, hereafter, we shall focus on $\theta \in \text{Reg}(\kappa)$. 
Inspecting the parameters

Definition
$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \rightarrow \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

About the third parameter
- $U(\kappa, \kappa, \kappa, \kappa)$ holds;
- $U(\kappa, \mu, \theta, \chi)$ iff $U(\kappa, \mu, \text{cf}(\theta), \chi)$;
- Lack of monotonicity: If $\lambda$ is the singular limit of strongly compact cardinals, then, for every $\theta \leq \lambda$, $U(\lambda^+, \lambda^+, \theta, \lambda)$ iff $\text{cf}(\theta) = \text{cf}(\lambda)$. 

Inspecting the parameters

Definition
\[ U(\kappa, \mu, \theta, \chi) \] asserts there is a coloring \( c : [\kappa]^2 \to \theta \) such that for every \( \chi' < \chi \), every family \( A \subseteq [\kappa]^{\chi'} \) consisting of \( \kappa \)-many pairwise disjoint sets, and every \( i < \theta \), there is \( B \in [A]^\mu \) such that \( \min(c[a \times b]) \geq i \) for every pair \( a < b \) from \( B \).

About the fourth parameter
- \( U(\kappa, \kappa, \theta, 3) \) iff \( U(\kappa, \kappa, \theta, \omega) \);
- \( U(\lambda^+, 2, \theta, 2) \) iff \( U(\lambda^+, 2, \theta, \text{cf}(\lambda)) \);

The above is optimal: If \( \lambda \) is the limit of strongly compact cardinals, \( \theta \in \text{Reg}(\lambda) \) with \( \theta \neq \text{cf}(\lambda) \), then \( U(\lambda^+, 2, \theta, \chi) \) holds for \( \chi := \text{cf}(\lambda) \), but fails for \( \chi := \text{cf}(\lambda)^+ \).
Inspecting the parameters

Definition
$U(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of $\kappa$-many pairwise disjoint sets, and every $i < \theta$, there is $B \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $B$.

About the fourth parameter
- $U(\kappa, \kappa, \theta, 3)$ iff $U(\kappa, \kappa, \theta, \omega)$;
- $U(\lambda^+, 2, \theta, 2)$ iff $U(\lambda^+, 2, \theta, \text{cf}(\lambda))$;
- There are $\kappa, \theta$ and colorings $c$, $c \models U(\kappa, \kappa, \theta, 2)$, but $c \not\models U(\kappa, 2, \theta, 3)$;
- If there is a closed witness to $U(\lambda^+, \lambda^+, \theta, 2)$, then there is for $U(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda))$. 
Further findings

Theorem

For every regular \( \lambda \) and \( \theta \in \text{Reg}(\lambda^+) \), there is \( c : [\lambda^+]^2 \rightarrow \theta \) witnessing \( U(\lambda^+, \lambda^+, \theta, \lambda) \) which is moreover closed.
Further findings

Theorem
For every regular $\lambda$ and $\theta \in \text{Reg}(\lambda^+)$, there is $c : [\lambda^+]^2 \rightarrow \theta$ witnessing $U(\lambda^+, \lambda^+, \theta, \lambda)$ which is moreover closed.

In case you wondered
The corresponding tree $T(c) := \{ c(\cdot, \gamma) \upharpoonright \beta \mid \beta \leq \gamma < \lambda^+ \}$ may consistently be special $\lambda^+$-Aronszajn tree / almost Souslin $\lambda^+$-Aronszajn tree.
Further findings

**Theorem**
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**Corollary**
There exists an $\aleph_2$-Knaster poset whose $\omega^{th}$-power is not $\aleph_2$-cc.
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More generally
Suppose that $\theta \leq \chi \leq \lambda$ are regular, with $\lambda^{<\chi} = \lambda$. Then $\exists \chi$-directed-closed poset $\mathbb{Q}$:
- $\mathbb{Q}^\tau$ has precaliber $\lambda^+$ for all $\tau < \theta$;
- $\mathbb{Q}^\theta$ is not $\lambda^+$-cc.
Further findings

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There exists an $\aleph_2$-Knaster poset whose $\omega^{th}$-power is not $\aleph_2$-cc.

CH entails a $\sigma$-directed-closed $\aleph_2$-Knaster poset whose $\omega^{th}$-power is not $\aleph_2$-cc.
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For every regular $\lambda$ and $\theta \in \text{Reg}(\lambda^+)$, there is $c : [\lambda^+]^2 \to \theta$ witnessing $U(\lambda^+, \lambda^+, \theta, \lambda)$ which is moreover closed.

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There exists an $\aleph_2$-Knaster poset whose $\omega^{\text{th}}$-power is not $\aleph_2$-cc.

CH entails a $\sigma$-directed-closed $\aleph_2$-Knaster poset whose $\omega^{\text{th}}$-power is not $\aleph_2$-cc.

Open problem
Does CH entail a $\sigma$-closed $\aleph_2$-cc poset whose square is not $\aleph_2$-cc?
Further findings (cont.)

Theorem

For every singular $\lambda$ and $\theta \in \text{Reg}(\lambda)$, any of the following entail the existence of a closed witness to $U(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda))$:

- $2\lambda = \lambda^+$;
- $\text{Refl}(< \text{cf}(\lambda), \lambda^+)$ fails;
- $\theta = \omega$ or $\theta = \text{cf}(\lambda)$;
- $\theta < \nu < \nu^+ = \text{cf}(\lambda)$;
- $\theta < \text{cf}(\lambda)$ and $\text{cf}(\text{NS}_{\text{cf}(\lambda)}, \subseteq) < \lambda$. 


Further findings (cont.)

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For every singular $\lambda$ and $\theta \in \text{Reg}(\lambda)$, any of the following entail the existence of a closed witness to $U(\lambda^+, \lambda^+, \theta, \text{cf}(\lambda))$:

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Corollary
If the class of $\kappa$-Knaster posets is closed under $\omega$ powers, then $\kappa$ is inaccessible.
Further findings (cont.)

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Theorem
For every $\theta, \chi \in \text{Reg}(\kappa)$, any of the following entails a closed witness to $U(\kappa, \kappa, \theta, \chi)$:

- $\Box(\kappa, < \omega)$ or $\Box^{\text{ind}}(\kappa, \theta)$;
- $\exists$ stationary $S \subseteq E^{\kappa}_{\geq \chi}$ with $S \cap \alpha$ nonstationary for all $\alpha \in E^{\kappa}_{> \omega}$;
- $\exists$ stationary $S \subseteq E^{\kappa}_{\geq \chi}$ with $S \cap \alpha$ nonstationary for all $\alpha \in \text{Reg}(\kappa)$, and $\kappa$ is inacc.
A new cardinal invariant

Theorem (Todorcevic, 1987)
For every strongly inaccessible cardinal $\kappa$, the following are equivalent:

1. $\kappa$ is weakly compact;
2. For every $C$-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \rightarrow \kappa$ such that $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for every $\alpha < \kappa$.

Recall
$\langle C_\beta \mid \beta < \kappa \rangle$ is a $C$-sequence iff each $C_\beta$ is closed subset of $\beta$ with $\sup(C_\beta) = \sup(\beta)$. 
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The cardinal invariant that we introduce suggests a way to measure how far an inaccessible cardinal $\kappa$ is from being weakly compact, though, as we will see, it is of interest for successor cardinals as well.
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   \[ \Delta \cap \alpha = C_{b(\alpha)} \cap \alpha \]
   for every \( \alpha < \kappa \).

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Definition (The \( C \)-sequence number of \( \kappa \))

If \( \kappa \) is weakly compact, then let \( \chi(\kappa) := 0 \).
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The cardinal invariant that we introduce suggests a way to measure how far an inaccessible cardinal $\kappa$ is from being weakly compact, though, as we will see, it is of interest for successor cardinals as well.

Definition (The $C$-sequence number of $\kappa$)
If $\kappa$ is weakly compact, then let $\chi(\kappa) := 0$.
Otherwise, let $\chi(\kappa)$ denote the least $\chi \leq \kappa$ s.t., for every $C$-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^{\kappa}$ and $b : \kappa \rightarrow [\kappa]^{\chi}$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for all $\alpha < \kappa$. 

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Note that $\chi(\kappa)$ is well-defined. In fact, $\chi(\kappa) \leq \sup(\text{Reg}(\kappa))$.

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Todorcevic’s analysis of the number of steps function readily establishes the following.

The $C$-sequence number and $yOU$

$U(\kappa, \kappa, \omega, \chi(\kappa))$ holds, as witnessed by the closed function $\rho_2$.

However, it is consistent that $U(\kappa, \kappa, \omega, \chi)$ holds with $\chi \gg \chi(\kappa)$.

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$U(\kappa, \kappa, \omega, \chi(\kappa))$ holds, as witnessed by the closed function $\rho_2$.

However, it is consistent that $U(\kappa, \kappa, \omega, \chi)$ holds with $\chi \gg \chi(\kappa)$.

Corollary

If the class of $\kappa$-Knaster posets is closed under taking $\omega$ powers, then $\chi(\kappa) < \omega$.

Definition (The C-sequence number of $\kappa$)

If $\kappa$ is weakly compact, then let $\chi(\kappa) := 0$.

Otherwise, let $\chi(\kappa)$ denote the least $\chi \leq \kappa$ s.t., for every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for all $\alpha < \kappa$.
A new cardinal invariant

Questions

• Is \( \chi(\kappa) < \omega \) a large cardinal property?
• How about \( \chi(\kappa) < \sup(\text{Reg}(\kappa)) \)?
• Could \( \chi(\kappa) \) be singular?

Corollary

If the class of \( \kappa \)-Knaster posets is closed under taking \( \omega \) powers, then \( \chi(\kappa) < \omega \).

Definition (The \( C \)-sequence number of \( \kappa \))

If \( \kappa \) is weakly compact, then let \( \chi(\kappa) := 0 \).
Otherwise, let \( \chi(\kappa) \) denote the least \( \chi \leq \kappa \) s.t., for every \( C \)-sequence \( \langle C_\beta \mid \beta < \kappa \rangle \), there exist \( \Delta \in [\kappa]^\kappa \) and \( b : \kappa \to [\kappa]^\chi \) with \( \Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta \) for all \( \alpha < \kappa \).
**Increasing the C-sequence number**

Kunen (1978) showed that by forcing over a model with a weakly compact cardinal $\kappa$, one obtains a model $V$ having a $\kappa$-Souslin tree $S$ such that $V^S \models \kappa$ is weakly compact.

**Proposition**

*In Kunen’s model, $\chi(\kappa) = 1$.*

**Proof.** The $\kappa$-Souslin tree witnesses that $\kappa$ is not weakly compact, so $\chi(\kappa) \neq 0$.

Now, let $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ be an arbitrary $C$-sequence.

In $V^S$, $\vec{C}$ is a $C$-sequence over a weakly compact cardinal $\kappa$, and hence there is $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to \kappa$ such that $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$.

Clearly, $\Delta$ is a club. As $S$ is $\kappa$-cc, there is a club $D \subseteq \kappa$ in $V$, with $D \subseteq \Delta$.

Then $D \cap \alpha \subseteq C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$. \qed

**Theorem**

*Suppose $\chi(\kappa) = 0$. For every $\theta \in \text{Reg}(\kappa^+)$, there is a cofinality-preserving forcing extension in which $\kappa$ remains strongly inaccessible, and $\chi(\kappa) = \theta$.**
Increasing the $C$-sequence number (cont.)

Observation
$\text{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.

Theorem
If $\lambda$ is a singular limit of supercompact cardinals, then $\chi(\lambda^+) = \text{cf}(\lambda)$.

Theorem
If $\lambda$ is a singular limit of supercompact cardinals, and $\theta \in \text{Reg}(\lambda)$ with $\theta \geq \text{cf}(\lambda)$, then, in some cofinality-preserving forcing extension, $\chi(\lambda^+) = \theta$.

Theorem
$\chi(\aleph_{\omega+1}) = \aleph_{\omega}$ is consistent, and so is $\chi(\aleph_{\omega+1}) = \omega$.

1The latter assumes the consistency of a supercompact.
How large

Theorem

1. \( \text{Refl}(<\omega, E^\kappa_{>\chi(\kappa)}) \);
2. If \( \chi(\kappa) < \omega \), then \( \chi(\kappa) \in \{0, 1\} \);
3. If \( \kappa \) is inaccessible and \( \chi(\kappa) < \kappa \), then \( \kappa \) is \( \omega \)-Mahlo;
4. If \( \chi(\kappa) = 1 \), then \( \Box(\kappa, <\mu) \) fails for all \( \mu < \kappa \);
5. If \( \chi(\kappa) = 1 \), then, for every sequence \( \langle S_i \mid i < \kappa \rangle \) of stationary subsets of \( \kappa \), there exists an inaccessible \( \beta < \kappa \) such that \( S_i \cap \beta \) is stationary in \( \beta \) for all \( i < \beta \).

Corollary

- In \( L \), either \( \chi(\kappa) = 0 \) or \( \chi(\kappa) = \sup(\text{Reg}(\kappa)) \);
- \( \Box(\kappa, <\omega) \) entails \( \chi(\kappa) = \sup(\text{Reg}(\kappa)) \);
- If \( \chi(\kappa) = 1 \), then \( \kappa \) is greatly Mahlo.
- If the class of \( \kappa \)-Knaster posets is closed under \( \omega \) powers, then \( \kappa \) is greatly Mahlo.
The $C$-sequence spectrum

Definition
For a $C$-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$, let $\chi(\vec{C})$ denote the least cardinal $\chi \leq \kappa$ such that there exist $\Delta \in [\kappa]^{\kappa}$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

Definition
$\text{Cspec}(\kappa) := \{ \chi(\vec{C}) \mid \vec{C} \text{ is a } C\text{-sequence over } \kappa \} \setminus \omega$.

Theorem
1. If $\text{Cspec}(\kappa) \neq \emptyset$, then $\min(\text{Cspec}(\kappa)) = \omega$ and $\chi(\kappa) = \max(\text{Cspec}(\kappa))$;
2. $\chi \in \text{Cspec}(\kappa) \implies \text{cf}(\chi) \in \text{Cspec}(\kappa)$, but not $\iff$.

Open problem
Is $\text{Cspec}(\kappa)$ an interval? Is it a closed set?
Is every limit uncountable cardinal in $\text{Cspec}(\kappa)$ an accumulation point of $\text{Cspec}(\kappa)$?
Unexpected equivalency

**Theorem**

*For every $\theta \in \text{Reg}(\kappa)$, the following are equivalent:*

- $\theta \in \text{Cspec}(\kappa)$;
- *There exists a closed witness to $U(\kappa, \kappa, \theta, \theta)$.*

The forward implication also works for $\theta$ singular; the backward does not.

**Corollary**

- *If $\kappa$ is a successor of a regular cardinal, then $\text{Reg}(\kappa) \subseteq \text{Cspec}(\kappa)$;*
- *If $\kappa$ is a non-Mahlo inaccessible, then $\text{Reg}(\kappa) \subseteq \text{Cspec}(\kappa)$;*
- *If $\Box(\kappa, <\omega)$ holds, then $\text{Reg}(\kappa) \subseteq \text{Cspec}(\kappa)$;*
- *If $E^\kappa_{\geq \chi}$ admits a non-reflecting stationary subset, then $\text{Reg}(\chi^+) \subseteq \text{Cspec}(\kappa)$.***
Conjectures

1. If $\kappa$ is inaccessible and $1 < \chi(\kappa) < \kappa$, $\exists \kappa$-Aronszajn tree with a $\chi(\kappa)$-ascent path.
2. Any instance $U(\kappa, \kappa, \ldots)$ may be witnessed by a closed coloring.
3. If $\chi(\kappa) = 1$, then, there exists a coherent $\kappa$-Aronszajn tree.
4. If $\chi(\kappa) = 1$, then, in some set-forcing extension, $\chi(\kappa) = 0$.
5. If $\chi(\kappa)$ is singular, then $\text{cf}(\chi(\kappa)) = \text{cf}((\text{sup} (\text{Reg}(\kappa))))$.
6. $\text{Reg}(\text{cf}(\lambda)^+) \subseteq \text{Cspec}(\lambda^+)$ for every singular $\lambda$.
7. For all $\theta, \chi \in \text{Cspec}(\kappa)$, $U(\kappa, \kappa, \theta, \chi)$ holds.