A unified approach to higher Souslin trees constructions

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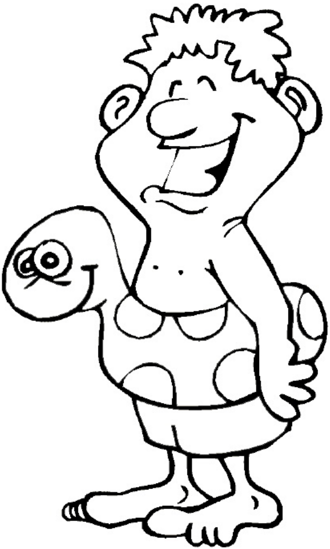
Introduction



Preliminaries: Combinatorial principles

Definition (Jensen, 1960's)

$\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for every $A \subseteq \bigcup S$.

Definition (Jensen, 1960's)

$\square_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for all limit $\alpha < \lambda^+$:

- C_α is a club in α of order-type $\leq \lambda$;
- if $\beta \in \mathrm{acc}(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.

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Definition (Schimmerling, 1995)

$\square_{\lambda,<\mu}$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$ such that for all limit $\alpha < \lambda^+$:

- $0 < |\mathcal{C}_\alpha| < \mu$;
- C is a club in α of order-type $\leq \lambda$, for all $C \in \mathcal{C}_\alpha$;
- if $C \in \mathcal{C}_\alpha$ and $\beta \in \mathrm{acc}(C)$, then $C \cap \beta \in \mathcal{C}_\beta$.

It is convenient to write $\square_{\lambda,\mu}$ for $\square_{\lambda,<\mu^+}$. So, $\square_\lambda \equiv \square_{\lambda,1}$.

Preliminaries: λ^+-trees

Definition

- A λ^+-tree is a tree of height λ^+ whose levels are of size $\leq \lambda$;
- A λ^+-Aronszajn tree is a λ^+-tree having no branches of size λ^+;
- A λ^+-Souslin tree is a λ^+-Aronszajn tree having no antichains of size λ^+;

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- A λ^+-Souslin tree is a λ^+-Aronszajn tree having no antichains of size λ^+;
- A λ^+-tree is special if it is the union of λ many antichains.

Thus, a special tree is Aronszajn, and a Souslin tree is non-special.

The role of λ

Fact

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- There exists an ω_1-Aronszajn tree;
- if GCH holds, then for every regular cardinal λ, there exists a special λ^+-Aronszajn tree;
- GCH is consistent with the nonexistence of any λ^+-Aronszajn tree at some singular cardinal λ (modulo large cardinals);

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- if GCH holds, then for every regular cardinal λ, there exists a special λ^+-Aronszajn tree;
- GCH is consistent with the nonexistence of any λ^+-Aronszajn tree at some singular cardinal λ (modulo large cardinals);
- The existence of an ω_1-Souslin tree is independent of GCH;
- Any ω_1-Aronszajn tree can be made special in some cofinalities-preserving extension;
- If $V = L$, then for every uncountable cardinal λ, there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.

The role of λ (cont.)

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Let us give two examples..

Example 1: Jensen's classical theorems

Theorem (Jensen, late 1960's)

Suppose that λ is a regular cardinal.
If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds, then there exists a λ^+-Souslin tree.

Theorem (Jensen, early 1970's)

Suppose that λ is a singular cardinal.
If GCH is valid and $\square_\lambda$ holds, then there exists a λ^+-Souslin tree.

Example 2: Souslin trees which are hard to specialize

Theorem (Baumgartner, 1980's, building on Laver)

$GCH + \square_{\aleph_1}$ *implies the existence of an* $\aleph_2$*-Souslin tree which remains non-special in any cofinalities-preserving extension.*

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Theorem (Cummings, 1997)

Suppose that λ *is a singular cardinal of countable cofinality.*
If $\mathsf{CH}_\lambda + \square_\lambda$ *holds, and* $\mu^{\aleph_1} < \lambda$ *for all* $\mu < \lambda$*, then there exists a* λ^+*-Souslin tree which remains non-special in any c.p.e.*

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Theorem (Cummings, 1997)

Suppose that λ *is a singular cardinal of* uncountable cofinality.
If $\mathrm{CH}_\lambda + \square_\lambda$ *holds, and* $\mu^{\aleph_0} < \lambda$ *for all* $\mu < \lambda$*, then there exists a* λ^+*-Souslin tree which remains non-special in any c.p.e.*

This raises the following..

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We shall propose a solution..

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1. Introduce a combinatorial principle from which many constructions can be carried out uniformly;
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2. Prove that this operational principle is a consequence of the "usual" hypotheses. Typically, this proof is divided into two or three independent subcases. However, this is done only once.

On clause 1

Ideally, the proposed principle would squeeze the most out of the prospective hypotheses (i.e., be logically equivalent to them).

The proposed proxy

For cardinals λ, μ, and a nonempty set of regular cardinals $\Gamma \subseteq \lambda^+$, we introduce the principle $\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$, which combines $\square_{\lambda,<\mu}$ together with a reminiscent of $\diamondsuit(\lambda^+ \cap \mathrm{cof}(\Gamma))$. We then infer a λ^+-Souslin tree already from the weakest among these principles:

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Proposition

Suppose that λ is an uncountable cardinal.
If $\boxtimes^{\Gamma}_{\lambda,\lambda}$ holds, then there exists a λ^+-Souslin tree.

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Proposition

Suppose that λ is an uncountable cardinal.
If $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ holds, then there exists a λ^+-Souslin tree.

Remarks

- The construction of the above tree is indeed uniform. That is, it does not depend on the identity of λ;
- Let κ denote the least cardinal such that $\lambda^\kappa > \lambda$. If $\Gamma \setminus \kappa \neq \emptyset$, then the resulting tree is moreover $(<\kappa)$-complete.

The principle $\boxed{\Diamond}^{\Gamma}_{\lambda,<\mu}$

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Fact (GCH)

In many cases, $\boxed{\diamondsuit}^{\{\theta\}}_{\lambda,<\mu}$ happens to be equivalent to the existence of a $\square_{\lambda,<\mu}$-sequence, $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$, with the additional property:

- for every unbounded $A \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_\theta$ such that $\mathrm{nacc}(C) \subseteq A$ for all $C \in \mathcal{C}_\alpha$.

Is the proposed principle $\boxed{\Diamond}^{\Gamma}_{\lambda,<\mu}$ any useful?

"Okay your father
managed to get a mouse.
Now how do we use it?"

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$

Let λ denote a regular uncountable cardinal.

Theorem 1
If $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds, then $\boxtimes^{\{\lambda\}}_{\lambda,\lambda}$ holds.

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Corollary (Jensen, 1960's)

If $\lambda^{<\lambda} = \lambda$ *and* $\diamondsuit(E^{\lambda^+}_{\lambda})$ *holds, then there exists a* $(<\lambda)$*-complete* λ^+*-Souslin tree.*

Getting $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ (cont.)

Let λ denote a regular uncountable cardinal.

Theorem 2

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ holds.

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$ (cont.)

Let λ denote a regular uncountable cardinal.

Theorem 2

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then $\boxtimes^{\Gamma}_{\lambda,\lambda}$ holds.

Corollary (Gregory, 1976)

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^{+}-Souslin tree.

Getting $\boxed{\Diamond}^{\Gamma}_{\lambda,\lambda}$ (cont.)

Theorem 3

If $2^{\aleph_0} = \aleph_1$ and NS_{ω_1} is saturated, then $\boxed{\Diamond}^{\Gamma}_{\aleph_1,\aleph_1}$ holds.

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$ (cont.)

Theorem 3

If $2^{\aleph_0} = \aleph_1$ and NS_{ω_1} is saturated, then $\boxtimes^{\Gamma}_{\aleph_1,\aleph_1}$ holds.

Corollary (Shelah, 1984)

If $2^{\aleph_0} = \aleph_1$ and NS_{ω_1} is saturated, then there exists an $\aleph_2$-Souslin tree.

Getting $\boxtimes^{\Gamma}_{\lambda,\lambda}$ (cont.)

Let λ denote a singular cardinal.

Theorem 4

If GCH $+\square_{\lambda,<\mathrm{cf}(\lambda)}$ holds, then $\boxtimes^{\Gamma}_{\lambda,\lambda}$ is valid.

Getting $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ (cont.)

Let λ denote a singular cardinal.

Theorem 4

If GCH $+\square_{\lambda,<\mathrm{cf}(\lambda)}$ holds, then $\boxed{\diamondsuit}^{\Gamma}_{\lambda,\lambda}$ is valid.

Corollary (Jensen, 1970's)

If GCH $+\square_{\lambda}$ holds, then there exists a λ^+-Souslin tree.

Corollary (Schimmerling, 2004)

If GCH $+\square_{\lambda,<\omega}$ holds, then there exists a λ^+-Souslin tree.

Nota bene

We have just seen four alternative proofs of the classical theorems concerning the existence of Souslin tree. Yet, the actual part of the construction was identical in all of them.

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Let us exemplify..

1. Souslin trees with a trivial automorphism group

Let λ denote an arbitrary uncountable cardinal.

Proposition

If $\boxtimes^{\Gamma}_{\lambda,\lambda}$ holds, then there exists a rigid λ^+-Souslin tree.

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Immediate corollary

If any of the following is valid:

1. $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(E^{\lambda^+}_{\lambda})$ holds
2. $\lambda^{<\lambda} = \lambda < \lambda^{\lambda} = \lambda^+$, and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$
3. $\lambda^{<\lambda} = \lambda = \kappa^+$ and $\mathsf{NS}_{\kappa^+} \restriction E^{\kappa^+}_{\kappa}$ is saturated
4. GCH $+\square_{\lambda,<\mathrm{cf}(\lambda)}$ holds

then $\exists\ 2^{\lambda^+}$ many isomorphism types of rigid λ^+-Souslin trees.

2. Souslin trees which are hard to specialize

Let λ denote an arbitrary uncountable cardinal.

Proposition

If $\boxtimes^{\Gamma}_{\lambda,1}$ holds, then there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.

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Proposition

If $\boxtimes^{\Gamma}_{\lambda,1}$ holds, then there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.
As always, the construction is uniform and we get as much completeness as possible.

Moreover

Suppose that $\kappa < \lambda = \lambda^{<\mu}$ are given infinite cardinals.
If $\boxtimes^{\Gamma}_{\lambda,1}$ holds with $\Gamma \setminus (\kappa \cup \mu) \neq \emptyset$, then $\exists$ a $(< \mu)$-complete λ^+-Souslin tree with a θ-ascent path for all regular $\theta \leq \kappa$.

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Let λ denote an arbitrary uncountable cardinal.

Proposition

If $\boxed{\Diamond}^{\Gamma}_{\lambda,1}$ holds, then there exists a λ^+-Souslin tree which remains non-special in any cofinalities-preserving extension.
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Theorem 5

The following are equivalent:

1. $\square_\lambda + \mathsf{CH}_\lambda$
2. $\boxed{\Diamond}^{\Gamma}_{\lambda,1}$

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Corollary

The four Baumgartner and Cummings theorems.

3. Your contribution!

Pick your favorite $\square$-based/$\diamondsuit$-based construction, and see if you can base it on $\boxtimes^{\Gamma}_{\lambda,1}, \boxtimes^{\Gamma}_{\lambda,<\omega}, \boxtimes^{\Gamma}_{\lambda,\mathrm{cf}(\lambda)}$ or $\boxtimes^{\Gamma}_{\lambda,\lambda}$.
An affirmative answer would make your construction portable in-between (successors of) regular and singular cardinals.

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Pick your favorite $\square$-based/$\diamondsuit$-based construction, and see if you can base it on $\boxdiamond^{\Gamma}_{\lambda,1}, \boxdiamond^{\Gamma}_{\lambda,<\omega}, \boxdiamond^{\Gamma}_{\lambda,\mathrm{cf}(\lambda)}$ or $\boxdiamond^{\Gamma}_{\lambda,\lambda}$.
An affirmative answer would make your construction portable in-between (successors of) regular and singular cardinals.

For example, the proof of the following theorem is implicitly based on $\boxdiamond^{\Gamma}_{\lambda,1}$:

Theorem (Farah-Veličković, 2006)

Assume that $\square_\lambda + \mathsf{CH}_\lambda$ holds for a singular strong limit cardinal of uncountable cofinality λ.
Then there exists a complete Boolean algebra of size λ^+ which is ccc and weakly distributive, but is not a Maharam algebra.

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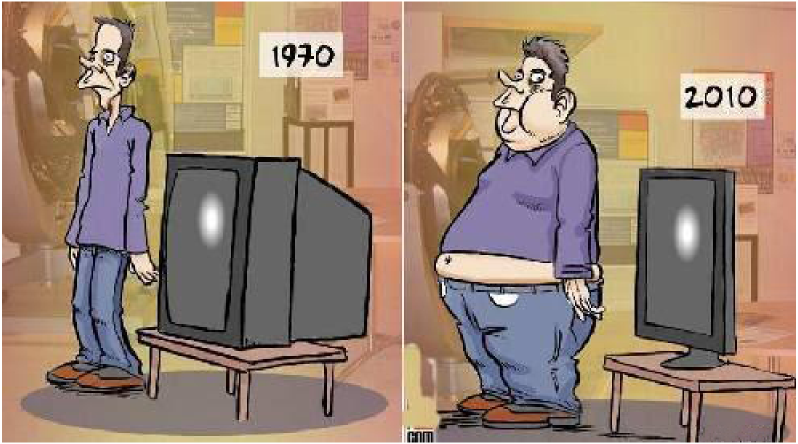
Theorem (Farah-Veličković), ported through Theorem 5

Assume that $\Box_\lambda + \mathsf{CH}_\lambda$ holds for ~~a singular strong limit cardinal of uncountable cofinality λ~~ a cardinal $\lambda \geq \mathfrak{d}$.
Then there exists a complete Boolean algebra of size λ^+ which is ccc and weakly distributive, but is not a Maharam algebra.

Covering more recent trees constructions



Guessing of generalized clubs: $\lambda^*(\kappa, S)$

Let λ denote an uncountable cardinal.

Shelah's Club Guessing Theorem

If $S \subseteq E^{\lambda^+}_{<\lambda}$, then there exists a sequence $\langle C_\alpha \mid \alpha \in S\rangle$ such that:

1. C_α is a club in α for all $\alpha \in S$;
2. $\{\alpha \in S \mid C_\alpha \subseteq D\} \neq \emptyset$ for every club $D \subseteq \lambda^+$.

Guessing of generalized clubs: $\curlywedge^*(\kappa, S)$

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König, Larson and Yoshinobu introduced a principle for guessing generalized clubs, denoted $\curlywedge^*(\kappa, S)$. They proved that it follows from $\diamondsuit^*(S)$, and showed how to derive a Souslin tree from it.

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König, Larson and Yoshinobu introduced a principle for guessing generalized clubs, denoted $⅄^*(\kappa, S)$. They proved that it follows from $\diamondsuit^*(S)$, and showed how to derive a Souslin tree from it. Here is a weakening of their principle (that follows already from $\diamondsuit$):

Definition of $⅄^-(\kappa, S)$, for $S \subseteq \lambda^+$

There exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in S\rangle$ such that:

1. for all $\alpha \in S$, $\mathcal{C}_\alpha$ is a collection of $\leq \lambda$ many clubs in $[\alpha]^{<\kappa}$;
2. $\{\alpha \in S \mid \exists C \in \mathcal{C}_\alpha(C \subseteq \mathcal{D})\} \neq \emptyset$ for every club $\mathcal{D} \subseteq [\lambda^+]^{<\kappa}$.

Guessing of generalized clubs: $\curlywedge^{-}(\kappa, S)$

Let λ denote a regular uncountable cardinal.

Theorem 6

If $\lambda = 2^{<\lambda} < 2^{\lambda} = \lambda^{+}$ and $\curlywedge^{-}(\lambda, E^{\lambda^{+}}_{\lambda})$ holds, then $\boxed{\diamondsuit}^{\{\lambda\}}_{\lambda,\lambda}$ is valid.

Guessing of generalized clubs: $\curlywedge^-(\kappa, S)$

Let λ denote a regular uncountable cardinal.

Theorem 6

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and $\curlywedge^-(\lambda, E^{\lambda^+}_\lambda)$ holds, then $\boxtimes\!\!\!\!\Diamond^{\{\lambda\}}_{\lambda,\lambda}$ is valid.

Corollary (König-Larson-Yoshinobu, 2007)

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and $\curlywedge^*(\lambda, E^{\lambda^+}_\lambda)$ holds, then there exists a $(<\lambda)$-complete λ^+-Souslin tree.

Schimmerling's question

Let λ denote a singular cardinal.

Question (Schimmerling, 2004)

Assuming GCH, for which μ, do $\square_{\lambda,<\mu}$ imply the existence of a λ^+-Souslin tree?

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Note

By Jensen, $\mu \geq 2$.
By Schimmerling, $\mu \geq \omega$.
By Theorem 4, $\mu \geq \mathrm{cf}(\lambda)$.

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Note

By Jensen, $\mu \geq 2$.
By Schimmerling, $\mu \geq \omega$.
By Theorem 4, $\mu \geq \mathrm{cf}(\lambda)$.
Now, how about a larger μ? Specifically, will $\mu = \lambda^+$ work?

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Theorem 7

If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, then $\square_{\lambda,\lambda}$ implies $\boxdiamond^{\Gamma}_{\lambda,\lambda}$.

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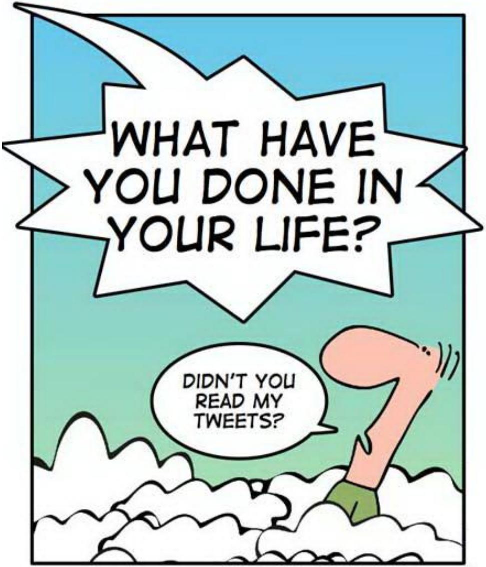
If $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, then $\square_{\lambda,\lambda}$ implies $\boxtimes^{\Gamma}_{\lambda,\lambda}$.

Partial answer (corollary)

$\mu = \lambda^+$, provided that $\exists$ non-reflecting stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$.

The extent of $\boxed{\Diamond}^{\Gamma}_{\lambda,<\mu}$



The extent of $\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$

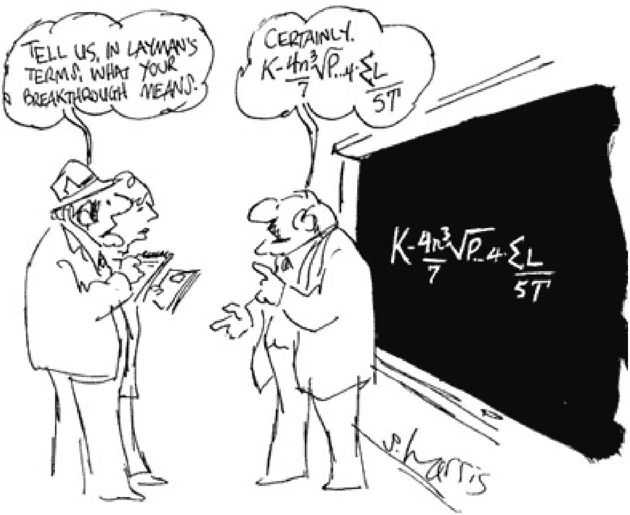
Theorem

In all the below cases, the following two are equivalent:

1. $\Box_{\lambda,<\mu} + \mathsf{CH}_\lambda$
2. $\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$

μ	λ	$\mathrm{cf}(\lambda)$	Remarks
$\mu = 2$	$\lambda \geq \aleph_1$	any	$\Gamma = \mathrm{Reg}(\lambda) := \{\theta < \lambda \mid \mathrm{cf}(\theta) = \theta\}$
$\mu \leq \mathrm{cf}(\lambda)$	$\lambda = \aleph_1$	any	$\Gamma = \mathrm{Reg}(\lambda)$, assuming CH
$\mu \leq \mathrm{cf}(\lambda)$	$\lambda > \aleph_1$	ctbl	$\Gamma = \{\theta\}$ for all large enough $\theta \in \mathrm{Reg}(\lambda)$
$\mu \leq \mathrm{cf}(\lambda)$	$\lambda > \aleph_1$	unctbl	Γ containing a final segment of $\mathrm{Reg}(\lambda)$
$\mu = \lambda^+$	$\lambda \geq \aleph_1$	any	some Γ, if: $2^{<\lambda} = \lambda$ & $\neg\mathrm{Refl}(E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)})$
$\mu = \lambda^+$	sing.	any	some Γ, if: $2^{<\lambda} = \lambda$ & $\mathrm{SNR}(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$

The definition of $\diamondsuit^{\Gamma}_{\lambda,<\mu}$



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$\boxed{\diamondsuit}^{\Gamma}_{\lambda,<\mu}$ asserts the existence of two sequences, $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ and $\langle \varphi_\theta \mid \theta \in \Gamma \rangle$, such that all of the following holds:

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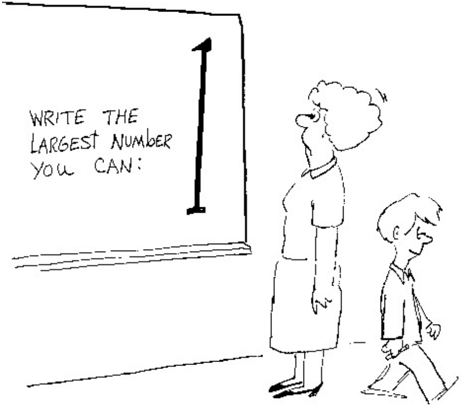
  1. $\sup(\mathrm{acc}(C)) = \alpha$ for some $C \in \mathcal{C}_\alpha$;
  2. for every $C \in \mathcal{C}_\alpha$, either $\sup(\mathrm{acc}(C)) < \alpha$, or
     $$\sup\{\delta \in \mathrm{nacc}(\mathrm{acc}(C)) \cap D \mid \varphi_\theta(C \cap \delta) = A \cap \delta\} = \alpha.$$

Open Problems



Two questions

1. Assume $\square_{\lambda,\mathrm{cf}(\lambda)}$ + every stationary subset of λ^+ reflects. Can you find a $\square_{\lambda,\lambda}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ such that for every club $D \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ with

$$\sup(\mathrm{nacc}(C) \cap D) = \alpha \text{ for all } C \in \mathcal{C}_\alpha?$$

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$$\sup(\mathrm{nacc}(C) \cap D) = \alpha \text{ for all } C \in \mathcal{C}_\alpha?$$

2. Let $\mathrm{Refl}(\lambda^+, \kappa)$ assert that for every stationary $S \subseteq E^{\lambda^+}_\kappa$, there exists some $\alpha \in E^{\lambda^+}_{>\kappa}$ for which $S \cap \alpha$ is stationary. Let $\mathrm{WRefl}(\lambda^+, \kappa)$ assert that for every $S \subseteq E^{\lambda^+}_\kappa$ and $f : S \to \lambda$, there exists some $\alpha \in E^{\lambda^+}_{>\kappa}$ such that $f \restriction C$ is non-injective for every club $C \subseteq \alpha$.
Question: Does $\mathrm{WRefl}(\lambda^+, \mathrm{cf}(\lambda))$ imply $\mathrm{Refl}(\lambda^+, \mathrm{cf}(\lambda))$?

Thank you!