On the consistency strength of the Milner-Sauer Conjecture

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Assaf Rinot
Tel-Aviv University

Joint work with Moti Gitik
Tel-Aviv University

Definitions

Definition. Suppose $\langle P, \leq \rangle$ is a poset.
$A \subseteq P$ is said to be *cofinal* in P iff for each $x \in P$ there exists $y \in A$ such that $x \leq y$.

Definition. The *cofinality* of $\langle P, \leq \rangle$, denoted:
$\mathrm{cf}(P, \leq) := \min\{|A| \mid A \subseteq P \text{ is cofinal in } P\}$.

Definition. $A \subseteq P$ is said to be a (weak) *antichain* iff for all $\{x, y\} \in [A]^2$, $x \not\leq y$ and $y \not\leq x$.

Motivation

Theorem (Hausdorff, 1908). If $\langle P, \leq\rangle$ is a linearly ordered set, then $\mathrm{cf}(P, \leq)$ is a regular cardinal.

Theorem (Erdös-Tarski, 1943). If $\langle P, \leq\rangle$ is a poset with no infinite antichain, then P is a finite union of updirected posets.

Theorem (Pouzet, 1979). If $\mathrm{cf}(P, \leq) \neq 1$ for an updirected poset with no infinite antichain, then there exists a cofinal subset $P' \subseteq P$ such that $P' \cong \bigotimes_{i<n} \kappa_n$ for some finite sequence of infinite regular cardinals $\langle \kappa_i \mid i < n\rangle$.

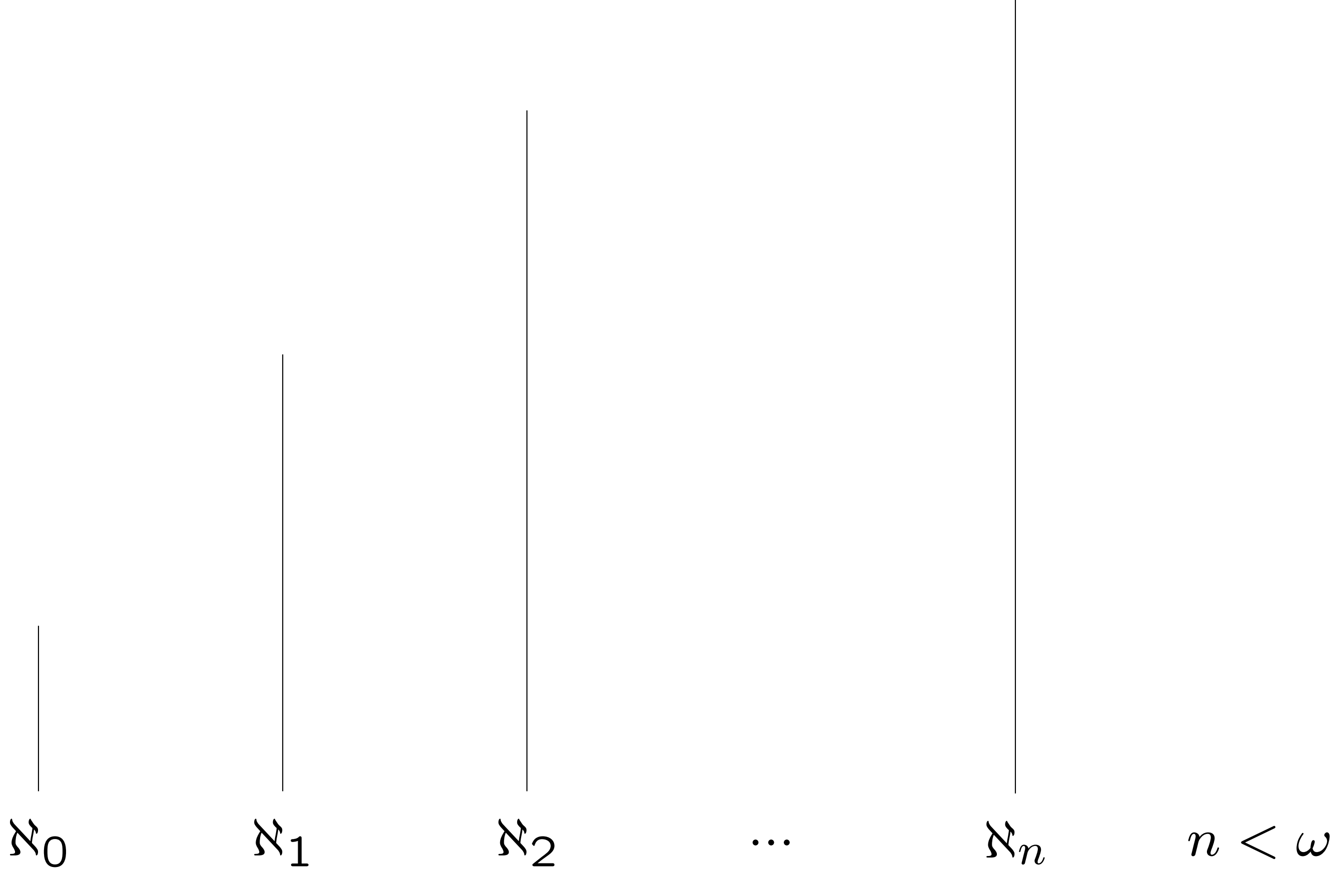
Corollary. If $\langle P, \leq\rangle$ is a poset and $\mathrm{cf}(P, \leq)$ is a singular cardinal, then P contains an infinite antichain.

The conjecture

Conjecture (Milner-Sauer, 1981). If $\langle P, \leq \rangle$ is a poset and $\mathrm{cf}(P, \leq) = \lambda > \mathrm{cf}(\lambda) = \kappa$, then P must contain an antichain of size κ.

Canonical example :

$$\bigoplus_{n<\omega}\aleph_n$$



Consistency results

The conjecture is consistent and known to follow from GCH-type assumptions, e.g., for $\lambda > \mathrm{cf}(\lambda) = \kappa$:

Theorem (Milner-Prikry '81). If $\mu^{<\kappa} < \lambda$ for all $\mu < \lambda$, then any poset of cofinality λ contains an antichain of size κ.

Theorem (Milner-Pouzet '82). If $\lambda^{<\kappa} = \lambda$, then any poset of cofinality λ contains an antichain of size κ.

Theorem (Hajnal-Sauer '86). If λ is a strong limit, then any poset of cofinality λ contains λ^κ antichains of size κ.

Most recent result

For $\lambda > \mathrm{cf}(\lambda) = \kappa$,

Theorem (Milner-Pouzet '97, Gorelic 2005). If $\lambda^{<\kappa} = \lambda$, then any poset of cofinality λ contains λ^κ antichains of size κ.

Problem

GCH-type assumption can easily be violated by forcing.

Notation

For simplicity, from now on, we fix a singular cardinal λ and let $\kappa := \mathrm{cf}(\lambda)$.

We denote $[\lambda]^{<\kappa} := \{X \subseteq \lambda \mid |X| < \kappa\}$.

A related problem

For a topological space $\langle X, O\rangle$, put :
$d(X) :=$ min$\{|D| \mid D \subseteq X$ is dense in $X\} + \aleph_0$.
$w(X) :=$ min$\{|B| \mid B \subseteq O$ is a base for $X\} + \aleph_0$.
$hC(X) :=$ min$\{\mu \in$ ICN $\mid \forall Y \subseteq X$, every open cover of Y has a subcover of cardinality $< \mu\}$.

Theorem 1. If there exists a poset of cofinality λ with no antichain of size κ, then there exists a T_0 topological space $\langle X, O\rangle$ such that for all $U \in O \setminus \{\emptyset\}$:
1. $|U| = d(U) = w(U) = \lambda$,
2. $hC(U) \leq \kappa$.

Topological result

Theorem 2. If there exists a space $\langle X, O \rangle$ such that $d(X) = w(X) = \lambda$ and $hC(X) \leq \kappa$, then $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) > \lambda$.

Main result

Theorem 3. Suppose $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$.
If $\langle P, \leq \rangle$ is a poset and $\mathrm{cf}(P, \leq) = \lambda$, then P contains λ^κ antichains of size κ.

Proof. If $\lambda^{<\kappa} = \lambda$, the result is already known. If $\lambda^{<\kappa} > \lambda$, apply Theorems 1 and 2 to yield an antichain $A \in [P]^\kappa$ and notice that $[A]^\kappa$ is a family of λ^κ antichains of size κ.

On the hypothesis cf$([\lambda]^{<\kappa}, \subseteq) = \lambda$

Theorem 4. If there exists an inner model of ZFC satisfying GCH and the covering lemma, then cf$([\lambda]^{<\kappa}, \subseteq) = \lambda$.

Corollary. If the Milner-Sauer conjecture does not hold, then there exists an inner model with a measurable cardinal.
Proof. By Dodd-Jensen, if there exists no inner model with a measurable cardinal, then the hypothesis of Theorem 4 holds for the core model K^{DJ}. Now apply Theorem 3.

The Milner-Sauer conjecture has large cardinals consistency strength

Starting with a ground model with no counter-example to the conjecture, a forcing notion that does not make use of the existence of large cardinals cannot produce a counter-example.

More on cf$([\lambda]^{<\kappa}, \subseteq)$

Theorem 5. If there exists a cardinal $\theta < \lambda$ such that $\theta^{<\theta} = \theta$ and for any regular cardinal $\mu > \theta$, $\mu^{\aleph_0} = \mu$, then cf$([\lambda]^{<\kappa}, \subseteq) = \lambda$.

Corollary. Martin's Maximum implies the Milner-Sauer conjecture.

Proof. By the preceding theorem for $\theta = \aleph_2$.

Corollary. The Milner-Sauer conjecture holds above a strongly-compact cardinal.

Proof. By Ketonen/Solovay, if θ is strongly-compact, then $\mu^{\aleph_0} = \mu$ for all regular $\mu > \theta$.

Dichotomy - A ZFC Theorem

Theorem 6. For λ, One and only one holds :
(a) For any poset $\langle P, \leq \rangle$ of cofinality λ, there exists some $n \in \mathbb{N}^+$, such that in the product order, P^n contains an antichain of size κ.

(b) There exists a cardinal $\theta < \lambda$ and a family of sets $\mathcal{J} \subseteq \mathcal{P}(\theta)$ such that $\mathrm{cf}(\mathcal{J}, \subseteq) = \lambda$ and:

(b.1) $\mathcal{J}$ is closed under finite unions and intersections. In particular, the poset $\langle \mathcal{J}, \subseteq \rangle$ is updirected and downdirected.

(b.2) For all $n \in \mathbb{N}^+$, every antichain in the product order, $\mathcal{J}^n$, is of size $< \kappa$.

Dichotomy towards a proof

If one would like to try and **prove** the conejcture, we suggest the following strategy :
(1) Eliminate the possibility of the second item of the dichotomy theorem and (2) Argue the existence of a bound for the first item. I.e. :

Conjecture. For any cardinal $\lambda > \text{cf}(\lambda) = \kappa$, there exists $n_\lambda \in \mathbb{N}^+$ such that for any poset $\langle P, \leq \rangle$ of cofinality λ, there exists a positive $n \leq n_\lambda$, such that in the product order, P^n contains an antichain of size κ.

(3) prove that $n_\lambda = 1$ for any singular λ.

More about consistency strength

Definition. For a singular cardinal μ, let $\mathrm{pp}(\mu) :=$ $\sup\{\mathrm{cf}(\prod \mathbf{a}/D) \mid \mathbf{a} \in [\mu]^{\mathrm{cf}(\mu)}$ is a family of regular cardinals cofinal in μ, D is an ultrafilter on $\mathbf{a}$ extending the filter of co-bounded sets $\}$.

Definition. *Shelah's Strong Hypothesis* states: For any singular cardinal μ, $\mathrm{pp}(\mu) = \mu^+$.

Theorem (Gitik, 1991). If there is no inner model with a cardinal δ, $o(\delta) \geq \delta^{++}$, then for any singular $\mu > 2^{\aleph_0}$, $\mathrm{pp}(\mu) = \mu^+$.

Finer consistency strength result

Theorem (Shelah, 1993). If $\mathrm{pp}(\mu) = \mu^+$ for all $\mu < \lambda$, then $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$.

Corollary. Suppose $2^{\aleph_0} < \aleph_\omega$.
If the Milner-Sauer conjecture does not hold, then there exists an inner model with a cardinal δ and $o(\delta) \geq \delta^{++}$.

References

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[2] E.C. Milner, N. Sauer, *Remarks on the cofinality of a partially ordered set, and a generalization of König's lemma*. Discrete Math., Vol. 35 (1981), 165-171. Available online: dx.doi.org/10.1016/0012-365X(81)90205-3