Diamond, non-saturation, and weak square principles

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Diamond on successor cardinals

Definition (Jensen, '72). For an infinite cardinal, λ, and a stationary set $S \subseteq \lambda^+$, $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for all $A \subseteq \lambda^+$.

Fact. For $S \subseteq \lambda^+$, $\diamondsuit_S \Rightarrow \diamondsuit_{\lambda^+} \Rightarrow 2^\lambda = \lambda^+$.

Question. Given a stationary, $S \subseteq \lambda^+$,
Does $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S)$?

A related concept

Fact. $\diamondsuit(S)$ entails that $\mathsf{NS}_{\lambda^+} \restriction S$ is *non-saturated*.

That is, there exists a family of λ^{++} many stationary subsets of S, whose pairwise intersection nonstationary.

Proof. Let $\langle A_\alpha \mid \alpha \in S\rangle$ witness $\diamondsuit(S)$. Denote $S_A = \{\alpha \mid A \cap \alpha = A_\alpha\}$. Then $\left\{S_A \mid A \subseteq \lambda^+\right\}$ exemplifies the non-saturation of $\mathsf{NS}_{\lambda^+} \restriction S$. ■

Question. Given a stationary, $S \subseteq \lambda^+$,

Must $\mathsf{NS}_{\lambda^+} \restriction S$ be non-saturated?

Two negative results. $\lambda = \omega$

Theorem (Jensen, '74). It is consistent that CH holds, while $\diamondsuit(\omega_1)$ fails.

Theorem (Steel-Van Wesep, '82). Suppose that V is a model of "ZF + $\mathrm{AD}_{\mathbb{R}}$ + Θ is regular".

Then, there is a forcing extension which is a model of ZFC, in which NS_{ω_1} is saturated.

Remark. By Later work of Shelah and Jensen-Steel, the saturation of NS_{ω_1} is equiconsistent with the existence of a single Woodin cardinal.

Two positive results. $\lambda > \omega$

Denote $E^{\lambda^+}_{\neq\kappa} := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \kappa\}$.

Theorem (Shelah, '90s). If λ is an uncountable cardinal, and S is a stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, then $\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated.

A continuous effort of 30 years recently culminated in:

Theorem (Shelah, 2007). If λ is an uncountable cardinal, and S is a stationary subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, then $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S)$.

The critical cofinality. $\lambda = \mathsf{cf}(\lambda)$

Denote $E^{\lambda^+}_{\kappa} := \{\delta < \lambda^+ \mid \mathsf{cf}(\delta) = \kappa\}$.

Theorem (Shelah, '80). For every regular uncountable cardinal, λ, it is consistent that:

$$\mathsf{GCH} + \neg\diamondsuit(E^{\lambda^+}_{\lambda}) \ .$$

Theorem (Woodin, '80s). For every regular uncountable cardinal, λ, having a huge cardinal above it, in some $< \lambda$-closed forcing extension:

$$\mathsf{NS}_{\lambda^+} \restriction S \text{ saturated, for some stationary } S \subseteq E^{\lambda^+}_{\lambda}.$$

The critical cofinality. $\lambda > \mathrm{cf}(\lambda)$

Def. $S \subseteq \lambda^+$ *reflects* iff the following set is stationary:

$$\mathrm{Tr}(S) := \{\gamma < \lambda^+ \mid \mathrm{cf}(\gamma) > \omega, S \cap \gamma \text{ is stationary}\}.$$

Theorem (Shelah, '84). For every singular cardinal, λ, in some cofinality-preserving forcing extension:

GCH+$\neg\diamondsuit(S)$ for some non-reflecting stationary set $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

Theorem (Foreman, '83). For every singular cardinal, λ, having a supercompact cardinal above it, and an almost-huge cardinal above that supercompact, in some λ-preserving forcing extension:

$\mathrm{NS}_{\lambda^+} \restriction S$ saturated, for a non-reflecting stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

Questions

Question 1. Suppose λ is a singular cardinal.

Must $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects?

Question 2. Suppose λ is a singular cardinal.

Must $\mathrm{NS}_{\lambda^+} \restriction S$ be non-saturated for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects?

Question 3. Can $\mathrm{NS}_{\omega_2} \restriction E^{\omega_2}_{\omega_1}$ be saturated?

Some answers

Diamond and reflecting sets

A partial affirmative answer to Question 1 is provided by Shelah and Zeman, as follows.

Theorem (Shelah, '84). If $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ, and $\square^*_\lambda$ holds, then $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Theorem (Zeman, 2008). If $2^\lambda = \lambda^+$ for a singular cardinal λ, and $\square^*_\lambda$ holds, then $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Weak Square

Definition (Jensen '72). $\square^*_\lambda$ asserts the existence of a sequence $\langle \mathcal{P}_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

1. $\mathcal{P}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{P}_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2. for every limit $\gamma < \lambda^+$, there exists a club $C_\gamma \subseteq \gamma$ satisfying:

$$C_\gamma \cap \alpha \in \mathcal{P}_\alpha \text{ for all } \alpha < \gamma.$$

The approachability ideal

Definition (Shelah). A set T is in $I[\lambda^+]$ iff:

1. $T \subseteq \lambda^+$;

2. there exists a sequence $\langle \mathcal{P}_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

2.1. $\mathcal{P}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{P}_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2.2. for almost all $\gamma \in T$, there exists
an unbounded $A_\gamma \subseteq \gamma$ satisfying:

$$A_\gamma \cap \alpha \in \bigcup_{\beta<\gamma} \mathcal{P}_\beta \text{ for all } \gamma < \alpha.$$

A relative of approachability ideal

Definition. Given $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$, a set T is in $I[S;\lambda]$ iff:

1. $T \subseteq \mathrm{Tr}(S)$;

2. there exists a sequence $\langle \mathcal{P}_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

2.1. $\mathcal{P}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{P}_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2.2. for almost all $\gamma \in T$, there exists a stationary $S_\gamma \subseteq S \cap \gamma$ satisfying:

$$S_\gamma \cap \alpha \in \bigcup\{\mathcal{P}(X) \mid X \in \mathcal{P}_\alpha\} \text{ for all } \alpha < \gamma$$

Remark. If λ is SSL, then $I[S;\lambda] \subseteq I[\lambda^+]$.

A comparison with weak square

Let λ denote a singular cardinal, and let $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

Observation. If $I[S;\lambda]$ contains a stationary set, then S reflects.

Proposition. Assume $\square^*_\lambda$. If S reflects, then $I[S;\lambda]$ contains a stationary set.

Theorem. It is relatively consistent with the existence of a supercompact cardinal that $\square^*_\lambda$ fails, while $I[S;\lambda]$ contains a stationary set for every stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

Answering question 1

Improving the Shelah-Zeman theorem, we have:

Theorem. Suppose λ is a singular cardinal, $S \subseteq E^{\lambda^+}_{\text{cf}(\lambda)}$;
If $I[S;\lambda]$ contains a stat. set, then $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S)$.

Answering Question 1 in the negative, while establishing that the above improvement is optimal, we have:

Theorem (Gitik-R.). It is relatively consistent with the existence of a supercompact cardinal that:
(1) GCH holds;
(2) $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$;
(3) Every stationary subset of $E^{\aleph_{\omega+1}}_{\omega}$ reflects;
(4) $\diamondsuit(S)$ fails, for some (reflecting) $S \subseteq E^{\aleph_{\omega+1}}_{\omega}$.

Stationary Approachability Property

Let λ denote a singular cardinal.

Definition. SAP_λ denote the assertion that $I[S;\lambda]$ contains a stationary set for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Thus, $\square^*_\lambda \Rightarrow \mathsf{SAP}_\lambda$, $\mathsf{SAP}_\lambda \not\Rightarrow \square^*_\lambda$, and:

Corollary. Suppose SAP_λ holds and $2^\lambda = \lambda^+$.
Then $\diamondsuit(S)$ is valid for every $S \subseteq \lambda^+$ that reflects.

Stronger Diamond

Theorem (Shelah, '84). If $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ, and $\Box^*_\lambda$ holds, then $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Theorem. If $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ, and $\Box^*_\lambda$ holds, and every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects, then, moreover, $\diamondsuit^*(\lambda^+)$ holds.

Theorem. Replacing $\Box^*_\lambda$ with SAP_λ is impossible, in the sense that the conclusion would fail to hold. (obtained by forcing over a model with a supercompact.)

Summary: Square vs. Diamond

Let Refl_λ denote the assertion that every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects.

Then, for λ singular, we have:

1. $\mathsf{GCH} + \square^*_\lambda \not\Rightarrow \diamondsuit^*(\lambda^+)$;
2. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \square^*_\lambda \Rightarrow \diamondsuit^*(\lambda^+)$;
3. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \mathsf{SAP}_\lambda \not\Rightarrow \diamondsuit^*(\lambda^+)$;
4. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \mathsf{SAP}_\lambda \Rightarrow \diamondsuit(S)$ for every stat. $S \subseteq \lambda^+$;
5. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \mathsf{AP}_\lambda \not\Rightarrow \diamondsuit(S)$ for every stat. $S \subseteq \lambda^+$.

Remark. AP_λ asserts that $\lambda^+ \in I[\lambda^+]$.

Around question 2

Let λ denote a singular cardinal, and $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

Theorem (Gitik-Shelah, '97).
$\mathrm{NS}_{\lambda^+} \restriction E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is non-saturated.

Theorem (Krueger, 2003).
If $\mathrm{NS}_{\lambda^+} \restriction S$ is saturated, then S is co-fat.

Theorem. If $\mathrm{NS}_{\lambda^+} \restriction S$ is saturated, then $I[S;\lambda]$ does not contain a stationary set.

In particular, SAP_λ (and hence $\square^*_\lambda$) imposes a positive answer to Quetsion 2.

The effect of smaller cardinals

A shift in focus

Instead of studying the validity of $\diamondsuit(S)$ (or saturation), we now focus on finding sufficient conditions for $I[S;\lambda]$ to contain a stationary set.

This yields a linkage between virtually unrelated objects.

Theorem. Assume GCH and that κ is an uncoutable cardinal with no κ^+-Souslin trees.

Then $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds for the class of singular cardinals λ of cofinality κ.

let us explain how small cardinals effects λ..

The effect of smaller cardinals, I

Definition. Assume $\theta > \kappa > \omega$ are regular cardinals.

$R_1(\theta, \kappa)$ asserts that for every function $f : E^\theta_{<\kappa} \to \kappa$, there exists some $j < \kappa$ such that:

$$\{\delta \in E^\theta_\kappa \mid f^{-1}[j] \cap \delta \text{ is stationary}\} \text{ is stationary.}$$

Facts. 1. $\square_\kappa \Rightarrow \neg R_1(\kappa^+, \kappa)$;
2. every stationary subset of $E^{\kappa^{++}}_\kappa$ reflects $\Rightarrow R_1(\kappa^{++}, \kappa^+)$;
3. By Harrington-Shelah '85, $R_1(\aleph_2, \aleph_1)$ is equiconsistent with the existence of a Mahlo cardinal.

The effect of smaller cardinals, II

Theorem. Suppose $\lambda > \mathrm{cf}(\lambda) = \kappa > \omega$;
If there exists a regular $\theta \in (\kappa, \lambda)$ such that $R_1(\theta, \kappa)$ holds, then $I[E^{\lambda^+}_{\mathrm{cf}(\lambda)}; \lambda]$ contains a stationary set.

Corollary. Suppose κ is a regular cardinal and every stationary subset of $E^{\kappa^{++}}_{\kappa}$ reflects.

Then $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ for the class of singular cardinals λ of cofinality κ^+.

Corollary. Assume PFA$^+$;
$\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds for every λ strong limit of cofinality ω_1.

The effect of smaller cardinals, III

Definition. Assume $\theta > \kappa > \omega$ are regular cardinals.

$R_2(\theta, \kappa)$ asserts that for every function $f : E^{\theta}_{<\kappa} \to \kappa$, there exists some $j < \kappa$ such that:

$\{\delta \in E^{\theta}_{\kappa} \mid f^{-1}[j] \cap \delta$ is non-stationary$\}$ is non-stationary.

Facts. 1. $R_2(\theta, \kappa) \Rightarrow R_1(\theta, \kappa)$ and hence the strength of $R_2(\kappa^+, \kappa)$ is at least of a Mahlo cardinal.
2. By Magidor '82, $R_2(\aleph_2, \aleph_1)$ is relatively consistent with the existence of a weakly compact cardinal.

Remark. The exact strength of $R_2(\aleph_2, \aleph_1)$ is unknown.

The effect of smaller cardinals, IV

Theorem. Suppose $\lambda > \mathrm{cf}(\lambda) = \kappa > \omega$;
If there exists a regular $\theta \in (\kappa, \lambda)$ such that $R_2(\theta, \kappa)$ holds, then $\mathrm{Tr}(S) \cap E^{\lambda^+}_\theta \in I[S;\lambda]$ for every $S \subseteq \lambda^+$.

Corollary. Suppose $R_2(\theta, \kappa)$ holds.
For every sing. cardinal λ of cofinality κ with $2^\lambda = \lambda^+$:

$$\diamondsuit(S) \text{ holds whenever } \mathrm{Tr}(S) \cap E^{\lambda^+}_\theta \text{ is stationary.}$$

Remark. The $R_2(\cdot,\cdot)$ proof resembles the one of an analogous theorem by Viale-Sharon concerning the weak approachability ideal. The $R_1(\cdot,\cdot)$ proof builds on a fundamental fact from Shelah's *pcf* theory.

Generalized stationary sets

The sup function, I

Definition. A set $\mathcal{X} \subseteq \mathcal{P}(\lambda^+)$ is *stationary* (in the generalized sense) iff for every $f : [\lambda^+]^{<\omega} \to \lambda^+$, there exists some $X \in \mathcal{X}$ such that $f : [X]^{<\omega} \subseteq X$.

Question (König-Larson-Yoshinobu). Let λ denote an infinite cardinal. Is it possible to prove in ZFC that every stationary $\mathcal{B} \subseteq [\lambda^+]^\omega$ can be thinned out to a stationary $\mathcal{A} \subseteq \mathcal{B}$ on which the sup-function is injective?

The sup function, II

Question (König-Larson-Yoshinobu). Let λ denote an infinite cardinal. Is it possible to prove in ZFC that every stationary $\mathcal{B} \subseteq [\lambda^+]^\omega$ can be thinned out to a stationary $\mathcal{A} \subseteq \mathcal{B}$ on which the sup-function is injective?

Proposition. If $\mathcal{A} \subseteq [\lambda^+]^\omega$ is a stationary set on which the sup-function is injective, then $\mathrm{cf}([\lambda^+]^\omega, \subseteq) = \lambda^+$.

In particular, if the SCH fails, then we get a counterexample to the above question. But what can one say in the context of GCH?

▶ It turns out that diamond helps..

The sup function, III

Theorem. Suppose λ is a cardinal, $2^\lambda = \lambda^+$.
For a stationary $S \subseteq E^{\lambda^+}_{<\lambda}$, TFAE:
1) $\diamondsuit(S)$;
2) there exists a stationary $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$, on which the sup-function is an injection from $\mathcal{X}$ to S.

Corollary. A negative answer to the K-L-Y question.
Proof. Work in a model of GCH and there exists $S \subseteq E^{\aleph_{\omega+1}}_{\omega}$ on which $\diamondsuit(S)$ fails.
Put $\mathcal{B} := \{X \in [\aleph_{\omega+1}]^\omega \mid \sup(X) \in S\}$. Then $\mathcal{B}$ is (a rather large) stationary set, and the sup-function is non-injective on any stationary subset of $\mathcal{B}$. ■

A related result

Theorem. Let λ denote an infinite cardinal.

Suppose $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ is a stationary set on which the sup-function is $(\leq \lambda)$-to-1. Put $S := \{\text{sup}(X) \mid X \in \mathcal{X}\}$. Then $\text{NS}_{\lambda^+} \restriction S$ is non-saturated.

λ^+-guessing

A very weak consequence of $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$

Definition. For a function $f : \lambda^+ \to \mathrm{cf}(\lambda)$, let κ_f denote the minimal cardinality of a family $\mathcal{P} \subseteq [\lambda^+]^{\mathrm{cf}(\lambda)}$ with the following property.

For all $Z \subseteq \lambda^+$ such that $\bigwedge_{\beta<\mathrm{cf}(\lambda)} |Z \cap f^{-1}\{\beta\}| = \lambda^+$, there exist some $a \in \mathcal{P}$ with $\sup(f[a \cap Z]) = \mathrm{cf}(\lambda)$.*

Definition. For a singular cardinal λ, we say that λ^+-*guessing* holds iff $\kappa_f \leq \lambda^+$ for all $f \in {}^{\lambda^+}\mathrm{cf}(\lambda)$.

*Note that if λ is SSL, then we may assume that $\mathcal{P}$ is closed under taking subsets. Thus, we may moreover demand the existence of $a \in \mathcal{P}$ such that $a \subseteq Z$ and $f \restriction a$ is injective.

the failure of λ^+-guessing

Theorem (Džamonja-Shelah, 2000). It is relatively consistent with the existence of a supercompact cardinal that there exist a strong limit singular cardinal, λ, and a function $f : \lambda^+ \to \mathrm{cf}(\lambda)$ such that $\kappa_f = 2^\lambda > \lambda^+$.

Theorem. Suppose λ is a strong limit singular; then:

$$\{\kappa_f \mid f \in {}^{\lambda^+}\mathrm{cf}(\lambda)\} = \{0, 2^\lambda\}.$$

Corollary. For a strong limit singular cardinal, λ, TFAE:
1) λ^+-guessing;
2) $\diamondsuit^+(E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)})$.

a fundamental cardinal arithmetic statement in disguise

Theorem. The following are equivalent:

1. λ^+-guessing holds for every singular cardinal, λ;

2. Shelah's Strong Hypothesis, i.e.,

   $\mathrm{cf}([\lambda]^{\mathrm{cf}(\lambda)}, \subseteq) = \lambda^+$ for every singular cardinal, λ.

3. Every first-countable topol. space whose density is a regular cardinal, κ, enjoys the following reflection: if every separable subspace is of size $\leq \kappa$, then the whole space is of size $\leq \kappa$.

Open problems

Open problems

Let λ denote a singular cardinal.

Question I. Does $2^\lambda = \lambda^+$ entail $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$?

Question II. Must there exist a stationary subset of $E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ that carries a partial (weak) square sequence?

Question III. Is "NS_{ω_1} saturated" consistent with CH?

Thank you!