Diamond, non-saturation, and weak square principles

*Logic Colloquium 2009*

*(European ASL summer meeting)*

02-Aug-09, Sofia

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Diamond on successor cardinals

**Definition** (Jensen, ‘72). For an infinite cardinal, $\lambda$, and a stationary set $S \subseteq \lambda^+$, $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for all $A \subseteq \lambda^+$.

**Fact.** For $S \subseteq \lambda^+$, $\diamondsuit S \Rightarrow \diamondsuit_{\lambda^+} \Rightarrow 2^\lambda = \lambda^+$.  

**Question.** Given a stationary, $S \subseteq \lambda^+$, Does $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S)$?
A related concept

Fact. ♦(S) entails that NS_{\lambda^+} \upharpoonright S is non-saturated.
That is, there exists a family of \lambda^{++} many stationary subsets of S, whose pairwise intersection nonstationary.

Proof. Let \langle A_\alpha \mid \alpha \in S \rangle witness ♦(S). Denote S_A = \{\alpha \mid A \cap \alpha = A_\alpha\}. Then \{S_A \mid A \subseteq \lambda^+\} exemplifies the non-saturation of NS_{\lambda^+} \upharpoonright S. ■

Question. Given a stationary, S \subseteq \lambda^+,
Must NS_{\lambda^+} \upharpoonright S be non-saturated?
Two negative results. $\lambda = \omega$

**Theorem** (Jensen, ‘74). It is consistent that CH holds, while $\diamondsuit(\omega_1)$ fails.

**Theorem** (Steel-Van Wesep, ’82). Suppose that $V$ is a model of “$\text{ZF} + \text{AD}_R + \Theta$ is regular”.

Then, there is a forcing extension which is a model of ZFC, in which $\text{NS}_{\omega_1}$ is saturated.

**Remark.** By Later work of Shelah and Jensen-Steel, the saturation of $\text{NS}_{\omega_1}$ is equiconsistent with the existence of a single Woodin cardinal.
Two positive results. $\lambda > \omega$

Denote $E^\lambda_\neq := \{ \delta < \lambda^+ \mid \text{cf}(\delta) \neq \kappa \}$.

**Theorem** (Shelah, ’90s). If $\lambda$ is an uncountable cardinal, and $S$ is a stationary subset of $E^\lambda_\neq = \text{cf}(\lambda)$, then $\text{NS}_{\lambda^+} \upharpoonright S$ is non-saturated.

A continuous effort of 30 years recently culminated in:

**Theorem** (Shelah, 2007). If $\lambda$ is an uncountable cardinal, and $S$ is a stationary subset of $E^\lambda_\neq = \text{cf}(\lambda)$, then $2^\lambda = \lambda^+ \Rightarrow \diamond(S)$. 


The critical cofinality. $\lambda = \text{cf}(\lambda)$

Denote $E^{\lambda^+}_\kappa := \{\delta < \lambda^+ \mid \text{cf}(\delta) = \kappa\}$.

**Theorem** (Shelah, ‘80). For every regular uncountable cardinal, $\lambda$, it is consistent that:

$$\text{GCH} + \neg \diamondsuit(E^{\lambda^+}_\lambda) .$$

**Theorem** (Woodin, ’80s). For every regular uncountable cardinal, $\lambda$, having a huge cardinal above it, in some $<\lambda$-closed forcing extension:

$$\text{NS}_{\lambda^+} \upharpoonright S \text{ saturated, for some stationary } S \subseteq E^{\lambda^+}_\lambda .$$
The critical cofinality. \( \lambda > \text{cf}(\lambda) \)

**Def.** \( S \subseteq \lambda^+ \) **reflects** iff the following set is stationary:

\[
\text{Tr}(S) := \{ \gamma < \lambda^+ \mid \text{cf}(\gamma) > \omega, S \cap \gamma \text{ is stationary} \}.
\]

**Theorem** (Shelah, ‘84). For every singular cardinal, \( \lambda \), in some cofinality-preserving forcing extension:

\[
\text{GCH} \rightarrow \neg \diamond(S) \text{ for some non-reflecting stationary set } S \subseteq E_{\text{cf}(\lambda)}^{\lambda^+}.
\]

**Theorem** (Foreman, ‘83). For every singular cardinal, \( \lambda \), having a supercompact cardinal above it, and an almost-huge cardinal above that supercompact, in some \( \lambda \)-preserving forcing extension:

\[
\text{NS}_{\lambda^+} \upharpoonright S \text{ saturated, for a non-reflecting stationary } S \subseteq E_{\text{cf}(\lambda)}^{\lambda^+}.
\]
Questions

**Question 1.** Suppose $\lambda$ is a singular cardinal.
Must $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S)$ for every $S \subseteq E_{\text{cf}(\lambda)}^{\lambda^+}$ that reflects?

**Question 2.** Suppose $\lambda$ is a singular cardinal.
Must $\text{NS}_{\lambda^+} \upharpoonright S$ be non-saturated for every $S \subseteq E_{\text{cf}(\lambda)}^{\lambda^+}$ that reflects?

**Question 3.** Can $\text{NS}_{\omega_2} \upharpoonright E_{\omega_1}^{\omega_2}$ be saturated?
Some answers
Diamond and reflecting sets

A partial \textcolor{red}{affirmative} answer to Question 1 is provided by Shelah and Zeman, as follows.

\textbf{Theorem} (Shelah, ‘84). If $2^\lambda = \lambda^+$ for a strong limit singular cardinal $\lambda$, and $\Box^*_\lambda$ holds, then $\Diamond(S)$ for every $S \subseteq E_{\text{cf}(\lambda)}^{\lambda+}$ that reflects.

\textbf{Theorem} (Zeman, 2008). If $2^\lambda = \lambda^+$ for a singular cardinal $\lambda$, and $\Box^*_\lambda$ holds, then $\Diamond(S)$ for every $S \subseteq E_{\text{cf}(\lambda)}^{\lambda+}$ that reflects.
Weak Square

Definition (Jensen ’72). $\square^*_\lambda$ asserts the existence of a sequence $\langle P_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

1. $P_\alpha \subseteq [\alpha]^{<\lambda}$ and $|P_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2. for every limit $\gamma < \lambda^+$, there exists a club $C_\gamma \subseteq \gamma$ satisfying:

   $C_\gamma \cap \alpha \in P_\alpha$ for all $\alpha < \gamma$. 


**The approachability ideal**

**Definition** (Shelah). A set $T$ is in $I[\lambda^+]$ iff:

1. $T \subseteq \lambda^+$;

2. there exists a sequence $\langle P_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

   2.1. $P_\alpha \subseteq [\alpha]^{<\lambda}$ and $|P_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

   2.2. for almost all $\gamma \in T$, there exists an unbounded $A_\gamma \subseteq \gamma$ satisfying:

       $$A_\gamma \cap \alpha \in \bigcup_{\beta < \gamma} P_\beta \text{ for all } \gamma < \alpha.$$
A relative of approachability ideal

**Definition.** Given $S \subseteq E^\lambda_\text{cf}(\lambda)$, a set $T$ is in $I[S; \lambda]$ iff:

1. $T \subseteq \text{Tr}(S)$;

2. there exists a sequence $\langle P_\alpha \mid \alpha < \lambda^+ \rangle$ such that:
   
   2.1. $P_\alpha \subseteq [\alpha]^{\lambda}$ and $|P_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

   2.2. for almost all $\gamma \in T$, there exists a stationary $S_\gamma \subseteq S \cap \gamma$ satisfying:

   $$S_\gamma \cap \alpha \in \bigcup \{P(X) \mid X \in P_\alpha\} \text{ for all } \alpha < \gamma$$

**Remark.** If $\lambda$ is SSL, then $I[S; \lambda] \subseteq I[\lambda^+]$. 
A comparison with weak square

Let $\lambda$ denote a singular cardinal, and let $S \subseteq E^{\lambda^+}_{\text{cf}(\lambda)}$.

**Observation.** If $I[S; \lambda]$ contains a stationary set, then $S$ reflects.

**Proposition.** Assume $\Box^*_\lambda$. If $S$ reflects, then $I[S; \lambda]$ contains a stationary set.

**Theorem.** It is relatively consistent with the existence of a supercompact cardinal that $\Box^*_\lambda$ fails, while $I[S; \lambda]$ contains a stationary set for every stationary $S \subseteq E^{\lambda^+}_{\text{cf}(\lambda)}$. 
Answering question 1

Improving the Shelah-Zeman theorem, we have:

**Theorem.** Suppose $\lambda$ is a singular cardinal, $S \subseteq E^\lambda_{\text{cf}(\lambda)}$;

If $I[S; \lambda]$ contains a stat. set, then $2^\lambda = \lambda^+ \Rightarrow \diamond(S)$.

Answering Question 1 *in the negative*, while establishing that the above improvement is optimal, we have:

**Theorem (Gitik-R.).** It is relatively consistent with the existence of a supercompact cardinal that:

1. GCH holds;
2. $\mathbb{N}_{\omega+1} \in I[\mathbb{N}_{\omega+1}]$;
3. Every stationary subset of $E^\mathbb{N}_{\omega+1}$ reflects;
4. $\diamond(S')$ fails, for some (reflecting) $S \subseteq E^\mathbb{N}_{\omega+1}$.
Stationary Approachability Property

Let $\lambda$ denote a singular cardinal.

**Definition.** $\text{SAP}_\lambda$ denote the assertion that $I[S; \lambda]$ contains a stationary set for every $S \subseteq E_{\text{cf}(\lambda)}^{\lambda^+}$ that reflects.

Thus, $\square_\lambda^* \Rightarrow \text{SAP}_\lambda$, $\text{SAP}_\lambda \not\Rightarrow \square_\lambda^*$, and:

**Corollary.** Suppose $\text{SAP}_\lambda$ holds and $2^\lambda = \lambda^+$. Then $\Diamond(S)$ is valid for every $S \subseteq \lambda^+$ that reflects.
Theorem (Shelah, ‘84). If $2^\lambda = \lambda^+$ for a strong limit singular cardinal $\lambda$, and $\Box^*_\lambda$ holds, then $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\text{cf}(\lambda)}$ that reflects.

Theorem. If $2^\lambda = \lambda^+$ for a strong limit singular cardinal $\lambda$, and $\Box^*_\lambda$ holds, and every stationary subset of $E^{\lambda^+}_{\text{cf}(\lambda)}$ reflects, then, moreover, $\Diamond^*(\lambda^+)$ holds.

Theorem. Replacing $\Box^*_\lambda$ with SAP$\lambda$ is impossible, in the sense that the conclusion would fail to hold. (obtained by forcing over a model with a supercompact.)
Summary: Square vs. Diamond

Let $\text{Refl}_\lambda$ denote the assertion that every stationary subset of $E_{\text{cf}(\lambda)}^{\lambda^+}$ reflects.

Then, for $\lambda$ singular, we have:
1. $\text{GCH} + \square^*_\lambda \nRightarrow \diamond^*(\lambda^+);$ 
2. $\text{GCH} + \text{Refl}_\lambda + \square^*_\lambda \Rightarrow \diamond^*(\lambda^+);$ 
3. $\text{GCH} + \text{Refl}_\lambda + \text{SAP}_\lambda \nRightarrow \diamond^*(\lambda^+);$ 
4. $\text{GCH} + \text{Refl}_\lambda + \text{SAP}_\lambda \Rightarrow \diamond(S)$ for every stat. $S \subseteq \lambda^+;$ 
5. $\text{GCH} + \text{Refl}_\lambda + \text{AP}_\lambda \nRightarrow \diamond(S)$ for every stat. $S \subseteq \lambda^+.$

Remark. $\text{AP}_\lambda$ asserts that $\lambda^+ \in I[\lambda^+].$
Around question 2

Let $\lambda$ denote a singular cardinal, and $S \subseteq E_{\text{cf}(\lambda)}^{\lambda^+}$.

**Theorem** (Gitik-Shelah, ’97).

$\text{NS}_{\lambda^+} \upharpoonright E_{\text{cf}(\lambda)}^{\lambda^+}$ is non-saturated.

**Theorem** (Krueger, 2003).

If $\text{NS}_{\lambda^+} \upharpoonright S$ is saturated, then $S$ is co-fat.

**Theorem.** If $\text{NS}_{\lambda^+} \upharpoonright S$ is saturated, then $I[S; \lambda]$ does not contain a stationary set.

In particular, $\text{SAP}_\lambda$ (and hence $\Box^{\ast}_\lambda$) imposes a positive answer to Question 2.
The effect of smaller cardinals
A shift in focus

Instead of studying the validity of $\diamond (S)$ (or saturation), we now focus on finding sufficient conditions for $I[S; \lambda]$ to contain a stationary set.

This yields a linkage between virtually unrelated objects.

**Theorem.** Assume GCH and that $\kappa$ is an uncoutable cardinal with no $\kappa^+$-Souslin trees.

Then $\diamond (E_{\text{cf}(\lambda)}^{\lambda^+})$ holds for the class of singular cardinals $\lambda$ of cofinality $\kappa$.

let us explain how small cardinals effects $\lambda$..
The effect of smaller cardinals, I

Definition. Assume $\theta > \kappa > \omega$ are regular cardinals.

$R_1(\theta, \kappa)$ asserts that for every function $f : E^\theta_{<\kappa} \to \kappa$, there exists some $j < \kappa$ such that:

$$\{ \delta \in E^\theta_\kappa \mid f^{-1}[j] \cap \delta \text{ is stationary} \} \text{ is stationary.}$$

Facts. 1. $\square_\kappa \Rightarrow \neg R_1(\kappa^+, \kappa)$;
2. every stationary subset of $E^\kappa^{++}_\kappa$ reflects $\Rightarrow R_1(\kappa^{++}, \kappa^+)$;
3. By Harrington-Shelah '85, $R_1(\aleph_2, \aleph_1)$ is equiconsistent with the existence of a Mahlo cardinal.
**The effect of smaller cardinals, II**

**Theorem.** Suppose $\lambda > \text{cf}(\lambda) = \kappa > \omega$; If there exists a regular $\theta \in (\kappa, \lambda)$ such that $R_1(\theta, \kappa)$ holds, then $I[E^\lambda_{\text{cf}(\lambda)}; \lambda]$ contains a stationary set.

**Corollary.** Suppose $\kappa$ is a regular cardinal and every stationary subset of $E^{\kappa}_{\kappa}$ reflects. Then $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(E^\lambda_{\text{cf}(\lambda)})$ for the class of singular cardinals $\lambda$ of cofinality $\kappa^+$.

**Corollary.** Assume $\text{PFA}^+$; $\diamondsuit(E^\lambda_{\text{cf}(\lambda)})$ holds for every $\lambda$ strong limit of cofinality $\omega_1$. 

23
The effect of smaller cardinals, III

Definition. Assume $\theta > \kappa > \omega$ are regular cardinals.

$R_2(\theta, \kappa)$ asserts that for every function $f : E^{\theta}_{<\kappa} \to \kappa$, there exists some $j < \kappa$ such that:

$$\{ \delta \in E^\theta_\kappa \mid f^{-1}[j] \cap \delta \text{ is non-stationary} \} \text{ is non-stationary.}$$

Facts. 1. $R_2(\theta, \kappa) \Rightarrow R_1(\theta, \kappa)$ and hence the strength of $R_2(\kappa^+, \kappa)$ is at least of a Mahlo cardinal.
2. By Magidor ’82, $R_2(\aleph_2, \aleph_1)$ is relatively consistent with the existence of a weakly compact cardinal.

Remark. The exact strength of $R_2(\aleph_2, \aleph_1)$ is unknown.
The effect of smaller cardinals, IV

**Theorem.** Suppose $\lambda > \text{cf}(\lambda) = \kappa > \omega$;
If there exists a regular $\theta \in (\kappa, \lambda)$ such that $R_2(\theta, \kappa)$
holds, then $\text{Tr}(S) \cap E^\lambda_\theta \in I[S; \lambda]$ for every $S \subseteq \lambda^+$.

**Corollary.** Suppose $R_2(\theta, \kappa)$ holds.
For every sing. cardinal $\lambda$ of cofinality $\kappa$ with $2^\lambda = \lambda^+$:

$$\diamond(S) \text{ holds whenever } \text{Tr}(S) \cap E^\lambda_\theta \text{ is stationary.}$$

**Remark.** The $R_2(\cdot, \cdot)$ proof resembles the one of an analogous theorem by Viale-Sharon concerning the weak approachability ideal. The $R_1(\cdot, \cdot)$ proof builds on a fundamental fact from Shelah’s $pcf$ theory.
Generalized stationary sets
The sup function, I

**Definition.** A set $\mathcal{X} \subseteq P(\lambda^+)$ is *stationary* (in the generalized sense) iff for every $f : [\lambda^+]^{<\omega} \to \lambda^+$, there exists some $X \in \mathcal{X}$ such that $f : [X]^{<\omega} \subseteq X$.

**Question** (König-Larson-Yoshinobu). Let $\lambda$ denote an infinite cardinal. Is it possible to prove in ZFC that every stationary $\mathcal{B} \subseteq [\lambda^+]^\omega$ can be thinned out to a stationary $\mathcal{A} \subseteq \mathcal{B}$ on which the sup-function is injective?
The sup function, II

Question (König-Larson-Yoshinobu). Let $\lambda$ denote an infinite cardinal. Is it possible to prove in ZFC that every stationary $B \subseteq [\lambda^+]^\omega$ can be thinned out to a stationary $A \subseteq B$ on which the sup-function is injective?

Proposition. If $A \subseteq [\lambda^+]^\omega$ is a stationary set on which the sup-function is injective, then $\text{cf}([\lambda^+]^\omega, \subseteq) = \lambda^+$. In particular, if the SCH fails, then we get a counterexample to the above question. But what can one say in the context of GCH?

▶ It turns out that diamond helps..
The sup function, III

**Theorem.** Suppose $\lambda$ is a cardinal, $2^\lambda = \lambda^+$. For a stationary $S \subseteq E^{\lambda^+_\lambda}$, TFAE:

1) $\Diamond(S)$;
2) there exists a stationary $X \subseteq [\lambda^+]^{<\lambda}$, on which the sup-function is an injection from $X$ to $S$.

**Corollary.** A negative answer to the K-L-Y question.

**Proof.** Work in a model of GCH and there exists $S \subseteq E^{\aleph_\omega+1}$ on which $\Diamond(S)$ fails.

Put $\mathcal{B} := \{X \in [\aleph_\omega+1]^{\omega} \mid \text{sup}(X) \in S\}$. Then $\mathcal{B}$ is (a rather large) stationary set, and the sup-function is non-injective on any stationary subset of $\mathcal{B}$. ■
A related result

Theorem. Let $\lambda$ denote an infinite cardinal.

Suppose $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ is a stationary set on which the sup-function is $(\leq \lambda)$-to-1. Put $S := \{\sup(X) \mid X \in \mathcal{X}\}$. Then $\text{NS}_{\lambda^+} \upharpoonright S$ is non-saturated.
$\lambda^+\text{-guessing}$
A very weak consequence of $\Diamond\left( E^\lambda_{\text{cf}(\lambda)} \right)$

**Definition.** For a function $f : \lambda^+ \rightarrow \text{cf}(\lambda)$, let $\kappa_f$ denote the minimal cardinality of a family $\mathcal{P} \subseteq [\lambda^+]^{\text{cf}(\lambda)}$ with the following property.

For all $Z \subseteq \lambda^+$ such that $\bigwedge_{\beta < \text{cf}(\lambda)} |Z \cap f^{-1}\{\beta\}| = \lambda^+$, there exist some $a \in \mathcal{P}$ with $\text{sup}(f[a \cap Z]) = \text{cf}(\lambda)$.

**Definition.** For a singular cardinal $\lambda$, we say that $\lambda^+-\text{guessing}$ holds iff $\kappa_f \leq \lambda^+$ for all $f \in \lambda^+ \text{cf}(\lambda)$.

*Note that if $\lambda$ is SSL, then we may assume that $\mathcal{P}$ is closed under taking subsets. Thus, we may moreover demand the existence of $a \in \mathcal{P}$ such that $a \subseteq Z$ and $f \upharpoonright a$ is injective.
the failure of $\lambda^+$-guessing

**Theorem** (Džamonja-Shelah, 2000). It is relatively consistent with the existence of a supercompact cardinal that there exist a strong limit singular cardinal, $\lambda$, and a function $f : \lambda^+ \to \text{cf}(\lambda)$ such that $\kappa_f = 2^\lambda > \lambda^+$.

**Theorem.** Suppose $\lambda$ is a strong limit singular; then:

$$\{\kappa_f \mid f \in \lambda^+ \text{ cf}(\lambda)\} = \{0, 2^\lambda\}.$$

**Corollary.** For a strong limit singular cardinal, $\lambda$, TFAE:
1) $\lambda^+$-guessing;
2) $\diamondsuit^+ (E_{\lambda^+}^{\lambda^+} \neq \text{cf}(\lambda))$. 

33
Theorem. The following are equivalent:

1. $\lambda^+$-guessing holds for every singular cardinal, $\lambda$;

2. Shelah’s Strong Hypothesis, i.e.,
   
   \[ \text{cf}(\text{cf}(\lambda), \subseteq) = \lambda^+ \]  
   for every singular cardinal, $\lambda$.

3. Every first-countable topological space whose density is a regular cardinal, $\kappa$, enjoys the following reflection:
   if every \textbf{separable subspace} is of size $\leq \kappa$, then the whole space is of size $\leq \kappa$. 

Open problems
Open problems

Let $\lambda$ denote a singular cardinal.

**Question I.** Does $2^\lambda = \lambda^+$ entail $\Diamond(E^\lambda_{\text{cf}(\lambda)}^+)$?

**Question II.** Must there exist a stationary subset of $E^\lambda_{>\text{cf}(\lambda)}^+$ that carries a partial (weak) square sequence?

**Question III.** Is "NS$_{\omega_1}$ saturated" consistent with CH?
Thank you!