Same Graph, Different Universe

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Partial bibliography

This talk will center around the following works:

[Rin1] Hedetniemi's conjecture for uncountable graphs,
to appear in J. Eur. Math. Soc.

[Rin2] Incompactness from Martin's Axiom,
submitted to the Baumgartner memorial issue.

[Rin3] Same Graph, Different Universe,
work in progress.

Motivating Graph Theory

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Let T denote the set of all possible timeslots. Our goal, then, is to find a mapping $\chi : G \to T$ so that $\chi(a) \neq \chi(b)$ whenever aEb. To save resources, we may also want to minimize $|\operatorname{Im}(\chi)|$.

Graphs and chromatic numbers

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Equivalently, it is the least cardinal κ such that $G = \bigcup_{i<\kappa} A_i$, where A_i is E-independent for each $i < \kappa$.

An example

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$$Sym(\lhd) := \{\{a, b\} \in [T]^2 \mid a \lhd b \text{ or } b \lhd a\}.$$

Then $\mathcal{T}$ is special iff $\mathcal{T}$ is the countable union of antichains iff $\mathrm{Chr}(\mathcal{G}_{\mathcal{T}}) = \aleph_0$.

An example (cont.)

If $\mathcal{T} = (\omega_1, \lhd)$ is a Souslin tree, then it cannot be the union of countably many antichains. So, $\mathrm{Chr}(\mathcal{G}_{\mathcal{T}}) = \aleph_1$.
However:

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However:

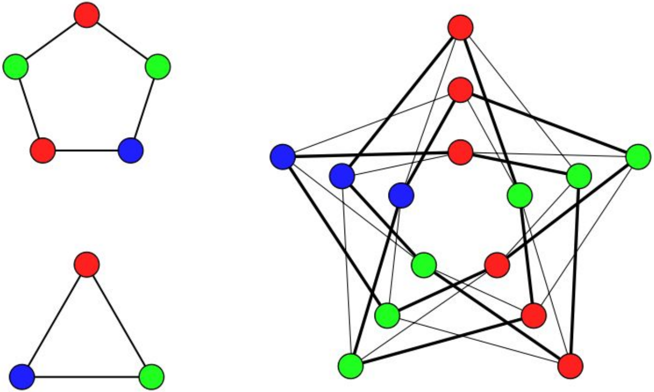
Theorem (Baumgartner-Malitz-Reinhardt, 1970)

There is a *ccc* notion of forcing, $\mathbb{P}$, such that $\Vdash_{\mathbb{P}} \mathsf{Chr}(\mathcal{G}_{\mathcal{T}}) = \aleph_0$.

Theorem (Shelah, 1980's)

There is a σ-distributive notion of forcing (of size $\mathfrak{c}$), $\mathbb{Q}$, such that $\Vdash_{\mathbb{Q}} \mathsf{Chr}(\mathcal{G}_{\mathcal{T}}) = \aleph_0$.

Hedetniemi's conjecture



The tensor product of graphs

Definition

Given graphs $\mathcal{G} = (G, E), \mathcal{H} = (H, F)$, let
$\mathcal{G} \times \mathcal{H} := (G \times H, E * F)$, where:

- $G \times H := \{(g, h) \mid g \in G, h \in H\}$;
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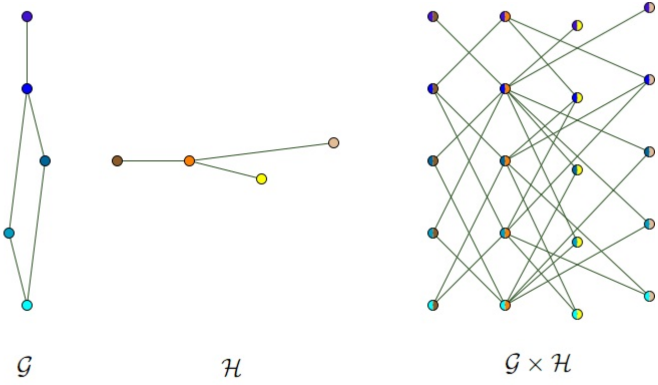

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For an E-chromatic coloring $\chi : G \to \kappa$, define a coloring $\chi^{\otimes H} : G \times H \to \kappa$ by letting $\chi^{\otimes H}(g, h) := \chi(g)$ for all $(g, h) \in G \times H$.

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$$\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) \leq \min\{\mathrm{Chr}(\mathcal{G}), \mathrm{Chr}(\mathcal{H})\}.$$

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Conjecture (Hedetniemi, 1966)

For every pair of (finite) graphs $\mathcal{G}, \mathcal{H}$:

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Not only that the above conjecture is still standing, but even the following Ramsey-type consequence of it is still unknown to hold.

Weak Hedetniemi Conjecture

For every positive integer k, there exists an integer $\varphi(k)$, such that if $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \varphi(k)$, then $\mathrm{Chr}(\mathcal{G} \times H) \geq k$.

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Remarks

1. Hedetniemi's conjecture is equivalent to "$\varphi(k) = k$ for all positive integer k";
2. Hedetniemi (1966) proved $\varphi(k) = k$ for all $k \in \{1, 2, 3\}$;
3. El-Zahar and Sauer (1985) proved that $\varphi(4) = 4$.

The infinite counterpart

Theorem (Hajnal, 1985)

For every infinite cardinal κ, there exist graphs $\mathcal{G}, \mathcal{H}$ such that

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This shows that a gap 1 is possible. In his paper, Hajnal asked about the possibility of realizing an infinite gap, but the best known result is that of gap 2:

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Theorem (Soukup, 1988)

It is consistent with ZFC + GCH that there exist graphs $\mathcal{G}, \mathcal{H}$ of size and chromatic number $\aleph_2$ such that $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_0$.

Hajnal's question and the weak conjecture

Hajnal's question (1985)

Is it consistent with ZFC+GCH that there are graphs $\mathcal{G}, \mathcal{H}$ such that $\text{Chr}(\mathcal{G}) = \text{Chr}(\mathcal{H}) \geq \aleph_\omega$, while $\text{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_n$?

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Observation (building on Hajnal)

If there exists a proper class of strongly-compact cardinals, then the Infinite Weak Hedetniemi Conjecture holds.

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Theorem [Rin1]

Suppose that $V = L$.
For every infinite cardinal λ, there exist graphs $\mathcal{G}, \mathcal{H}$ such that $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) > \lambda$, while $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_0$.

The main ingredient of the solution

Theorem

If $\boxed{\diamondsuit}_\lambda$ holds, then there exist graphs $\mathcal{G}_0 = (G_0, E_0), \mathcal{G}_1 = (G_1, E_1)$ of size λ^+ and $(< \lambda^+)$-distributive notions of forcing $\mathbb{P}_0, \mathbb{P}_1$ s.t.:

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$\mathrm{Chr}(H_0 \times H_1, F_0 * F_1) = \aleph_0.$

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Recall:

- $G_i = \{\alpha < \lambda^+ \mid (\alpha \mod 2) = i\}$;
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Claim

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Proof.

Define $c : H_0 \times H_1 \to \omega \times 2$, by letting $c(\chi_0, \chi_1) = (\chi_0(\alpha_{\chi_1}), 0)$ if $\alpha_{\chi_0} > \alpha_{\chi_1}$, and $c(\chi_0, \chi_1) = (\chi_1(\alpha_{\chi_0}), 1)$, otherwise.

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Towards a contradiction, suppose $c(\chi_0, \chi_1) = c(\chi'_0, \chi'_1) = (n, i)$, while $\{(\chi_0, \chi_1), (\chi'_0, \chi'_1)\} \in F_0 * F_1$.

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Towards a contradiction, suppose $c(\chi_0, \chi_1) = c(\chi_0', \chi_1') = (n, i)$, while $\{(\chi_0, \chi_1), (\chi_0', \chi_1')\} \in F_0 * F_1$. Say, $i = 0$.

The main ingredient of the solution (cont.)

Recall:

- $G_i = \{\alpha < \lambda^+ \mid (\alpha \mod 2) = i\}$;
- $H_i := \{\chi : G_{(1-i)} \cap \alpha \to \omega \mid \alpha \in G_i, \chi$ is $E_{(1-i)}$-chromatic$\}$;
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The spectrum of chromatic numbers

Disclaimer: This is work in progress. At present, we have more questions than answers!

The spectrum of chromatic numbers

Definition

For a graph $\mathcal{G}$, and a class of cardinals-preserving notions of forcing $\mathcal{P}$, let

$$\mathsf{Chr}_{\mathcal{P}}(\mathcal{G}) := \{\kappa \mid \text{exists } \mathbb{P} \in \mathcal{P} \text{ with } V^{\mathbb{P}} \models \mathsf{Chr}(\mathcal{G}) = \kappa\}.$$

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If there exists a nonspecial Aronszajn tree, then there exists a graph $\mathcal{G}$ of size $\aleph_1$ for which $\mathrm{Chr}_{ccc}(\mathcal{G}) = \{\aleph_0, \aleph_1\}$.

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For every regular cardinal λ, there exists a graph $\mathcal{G}$ of size λ^+, such that $\mathrm{Chr}_{(<\lambda)\text{-directed-closed},\lambda^+\text{-cc}}(\mathcal{G}) = \{\lambda, \lambda^+\}$.

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New rule: no cheating allowed!

Suppose that a graph (G, E) of size $\lambda > \kappa$ satisfies $\mathrm{Chr}_{\mathcal{P}}(G, E) = \{\kappa, \lambda\}$. Maybe one is cheating somehow, and in fact $\mathrm{Chr}(G', E) = \kappa$ for some key subset $G' \subseteq G$?

No cheating

Definition

Say that a graph (G, E) has everywhere chromatic number λ, if $\mathrm{Chr}(G', E) = \lambda$ for all $G' \subseteq G$ with $|G'| = |G|$.

No cheating

Proposition [Rin2]

If $\langle f_\alpha \mid \alpha < \lambda \rangle$ is a $<^*$-increasing and unbounded sequence of reals ${}^\omega\omega$, then there exists a graph $\mathcal{G}$ of size and everywhere chromatic number λ, such that $\aleph_0 \in \mathrm{Chr}_{ccc}(\mathcal{G})$.

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To which λ's do the proposition apply? Recall Hechler's theorem:

Theorem (Hechler, 1974)

If $\mathbb{P}$ is a partially ordered set in which every countable subset has an upper bound, then $\mathbb{P}$ can consistently be isomorphic to a cofinal subset of $\langle {}^\omega\omega, <^* \rangle$.

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Another application:

Corollary [Rin2]

Suppose that Martin's Axiom holds.
Then there exists an edge relation $E \subseteq [\mathfrak{c}]^2$, such that for all $G \subseteq \mathfrak{c}$:

$$\aleph_0 + \mathrm{Chr}(G, E) = \begin{cases} \mathfrak{c}, & |G| = \mathfrak{c} \\ \aleph_0, & |G| < \mathfrak{c} \end{cases}.$$

This appears to be the simplest construction of incompacntess graphs with arbitrarily large gaps.

Everywhere chromatic graphs from strong colorings

Definition [Rin3]

$\mathsf{Pr}^U(\lambda, \kappa) = \mathsf{Pr}^U(\lambda, \kappa^+, 2, \kappa)$ asserts the existence of a coloring $c : [\lambda]^2 \to 2$ satisfying the two:

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Remark

$\mathrm{Pr}^U(\lambda, \kappa, \theta, \sigma)$ is an unbalanced form of Shelah's $\mathrm{Pr}_1(\lambda, \kappa, \theta, \sigma)$.

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Conjecture

GCH $+\neg\mathrm{Pr}^U(\aleph_2, \aleph_1)$ is consistent (modulo large cardinals).

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Wild guess

CH $+\neg\mathrm{Pr}^{U}(\aleph_2, \aleph_1)$ is equiconsistent with the existence of a weakly-compact cardinal.

The infinitary generalization of chromatic numbers

Question

We have seen examples of graphs $\mathcal{G}$ with $|\operatorname{Chr}_{\mathcal{P}}(\mathcal{G})| > 1$.
So, what does $\operatorname{Chr}(\mathcal{G})$ really tell us?

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Testcase: higher Aronszajn trees

Laver, Baumgartner, Devlin, Shelah-Stanley, Todorcevic, R. David, Cummings, and more recently, Lücke, gave examples of peculiar nonspecial $\aleph_2$-Aronszajn trees.

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For instance, if $V = L$, then there exist $\aleph_2$-Aronszajn trees $\mathcal{T}_1, \mathcal{T}_2$ such that

- $\mathrm{Chr}_{\text{cofinality-preserving}}(\mathcal{G}_{\mathcal{T}_1}) = \{\aleph_2\}$;
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- $\mathrm{Chr}_{\text{cofinality-preserving}}(\mathcal{G}_{\mathcal{T}_1}) = \{\aleph_2\}$;
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Testcase: higher Aronszajn trees

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Question

What about $|\mathrm{Chr}_{\mathcal{P}}(\mathcal{G})| > 2$?

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Question

What about $\mathrm{Chr}_{\mathcal{P}}(\mathcal{G})$ infinite?

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Question

What about $\mathrm{Chr}_{\mathcal{P}}(\mathcal{G})$ uncountable?

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Question

What about $|\mathrm{Chr}_{\mathcal{P}}(\mathcal{G})|$ = fixed-point of the $\aleph$-function?

Realizable sets

Main Theorem [Rin3]

Suppose that $V = L$ and ϕ is the least to satisfy $\phi = \aleph_\phi$.
Then for every infinite cardinal $\mu < \aleph_\phi$, there exists a graph $\mathcal{G}$ of size μ such that:

$$\mathsf{Chr}_{\text{cofinality-preserving}}(\mathcal{G}) = \{\aleph_0, \aleph_1, \aleph_2, \ldots, \mu\}.$$

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Conjecture

By a more careful construction of $\boxed{\diamondsuit}_\lambda$-sequences in L, the restriction "$\mu < \aleph_\phi$" in the above theorem may be waived.

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Proposed project

Characterize all sets $\mathcal{K}$ of cardinals for which there exists a graph $\mathcal{G}$ with $\mathrm{Chr}_{\text{cofinality-preserving}}(\mathcal{G}) = \mathcal{K}$.

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Basic question

Is $\mathrm{Chr}_{\text{cofinality-preserving}}(\mathcal{G})$ provably/consistently a closed set?