Analytic quasi-orders and two forms of diamond

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50 Years of Set Theory in Toronto
Fields Institute for Research in Mathematical Sciences

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This is a joint work with Gabriel Fernandes and Miguel Moreno at BIU
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The generalized Baire space

$\kappa^{\kappa}$ consists of functions from $\kappa$ to $\kappa$, and a basic open set takes the form

$$N_\eta := \{ f \in \kappa^{\kappa} \mid \eta \subseteq f \},$$

where $\eta \in {}^{<\kappa}\kappa$, i.e., $\eta$ is a function from an ordinal $<\kappa$ to $\kappa$. 
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$2^{\kappa}$ is the topological subspace of $\kappa^{\kappa}$ consisting of all functions from $\kappa$ to 2.
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### The generalized Cantor space (The $\kappa$-antor space)

$2^\kappa$ is the topological subspace of $\kappa^\kappa$ consisting of all functions from $\kappa$ to 2.
The class of $\kappa$-Borel sets

The least collection of sets in $\kappa^\kappa$ (resp., $2^\kappa$) containing all open sets that is closed under unions and intersections of length $\leq \kappa$, and complements.
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\( \kappa \)-Borel functions

Let \( X_1, X_2 \in \{2^\kappa, \kappa^\kappa\} \). A function \( f : X_1 \to X_2 \) is \( \kappa \)-Borel iff for every open set \( U \subseteq X_2 \), the inverse image \( f^{-1}U \) is a \( \kappa \)-Borel subset of \( X_1 \).
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Borel reductions and continuous reductions

Let $E_1$ and $E_2$ be equivalence relations on $X_1, X_2 \in \{2^\kappa, \kappa^\kappa\}$, respectively.

- We say that $E_1$ is $\kappa$-Borel reducible to $E_2$ and denote this by $E_1 \hookrightarrow_B E_2$ iff there is a $\kappa$-Borel function $f : X_1 \to X_2$ s.t. $\forall \eta, \xi \in X_1, (\eta, \xi) \in E_1 \iff (f(\eta), f(\xi)) \in E_2$. 
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- If $f$ is moreover continuous, then we say that $E_1$ is continuously reducible to $E_2$ and denote it by $E_1 \leftrightarrow^c E_2$. 

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\( \kappa \)-Borel

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- If \( f \) is moreover continuous, then we say that \( E_1 \) is continuously reducible to \( E_2 \) and denote it by \( E_1 \hookrightarrow_c E_2 \).
Comparing theories

Let $T$, $T'$ denote complete theories over a countable first-order language.

**Counting number of non-isomorphic models of cardinality $\kappa$**

Say that $T'$ is more complex than $T$ iff $I(T', \kappa)$ is bigger than $I(T, \kappa)$. 
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Say that $T'$ is more complex than $T$ iff $I(T', \kappa)$ is bigger than $I(T, \kappa)$.

Shelah’s Main Gap Theorem implies that this notion is not very informative.

**Theorem (Shelah, 1990)**

One of the following holds:

1. $I(T, \aleph_\alpha) < \beth_1(|\alpha|)$ for every nonzero ordinal $\alpha$.
2. $I(T, \kappa) = 2^\kappa$ for every uncountable cardinal $\kappa$. 
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**Theorem (Shelah, 1990)**

One of the following holds:

1. $T$ is shallow superstable without DOP and without OTOP. In this case, $I(T, \aleph_\alpha) < \beth_1(|\alpha|)$ for every nonzero ordinal $\alpha$.
2. $T$ is not superstable, or superstable and deep or with DOP or with OTOP. In this case, $I(T, \kappa) = 2^\kappa$ for every uncountable cardinal $\kappa$. 

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Comparing theories via GDST

At the outset, fix a bijection $\pi : \kappa^{<\omega} \leftrightarrow \kappa$. For each $m$, let $n_m$ denote the arity of $P_m$. Every element $\xi$ of $2^\kappa$ gives rise to a model $M_\xi = (\kappa, ...)$ over $L$ via:

$$(a_1, ..., a_{n_m}) \in P_{M_\xi m} \iff n = n_m \land \xi(\pi(m, a_1, ..., a_{n_m})) = 1.$$

Conversely, for every model $M = (\kappa, ...)$ over $L$, there is $\xi \in 2^\kappa$ such that $M = M_\xi$. Every theory determines an equivalence relation on $2^\kappa$ for a theory $T$ over $L$, and $\eta, \xi \in 2^\kappa$, let $\eta \sim_T = \xi$ iff one of the two holds:

- $M_\eta | = T$ and $M_\xi | = T$ and $M_\xi \sim = M_\eta$,
- $M_\eta \neq | = T$ and $M_\xi \neq | = T$. 

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For each $m$, let $n_m$ denote the arity of $P_m$. 
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Every element $\xi$ of $2^\kappa$ gives rise to a model $M_\xi = (\kappa, \ldots)$ over $L$ via:

$$(a_1, \ldots, a_n) \in P_m^{M_\xi} \text{ iff } n = n_m \& \xi(\pi(m, a_1, \ldots, a_n)) = 1.$$
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Every element $\xi$ of $2^\kappa$ gives rise to a model $\mathcal{M}_\xi = (\kappa, \ldots)$ over $\mathcal{L}$ via:

$$(a_1, \ldots, a_n) \in P^\mathcal{M}_\xi_m \text{ iff } n = n_m \& \xi(\pi(m, a_1, \ldots, a_n)) = 1.$$  

Conversely, for every model $\mathcal{M} = (\kappa, \ldots)$ over $\mathcal{L}$, there is $\xi \in 2^\kappa$ such that $\mathcal{M} = \mathcal{M}_\xi$.

Every theory determines an equivalence relation on $2^\kappa$

For a theory $T$ over $\mathcal{L}$, and $\eta, \xi \in 2^\kappa$, let $\eta \equiv_T \xi$ iff one of the two holds:
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For a theory $T$ over $\mathcal{L}$, and $\eta, \xi \in 2^\kappa$, let $\eta \equiv_T \xi$ iff one of the two holds:

- $\mathcal{M}_\eta \models T$ and $\mathcal{M}_\xi \models T$ and $\mathcal{M}_\xi \cong \mathcal{M}_\eta$, or
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For a theory $T$ over $\mathcal{L}$, and $\eta, \xi \in 2^\kappa$, let $\eta \equiv_T \xi$ iff one of the two holds:

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- $\mathcal{M}_\eta \not\models T$ and $\mathcal{M}_\xi \not\models T$.  

Comparing theories via GDST (cont.)

Recall

For a countable first-order relational $T$ and $\eta, \xi \in 2^\kappa$, let $\eta \equiv_T \xi$ iff:

- $M_\eta \models T$ and $M_\xi \models T$ and $M_\xi \cong M_\eta$, or
- $M_\eta \not\models T$ and $M_\xi \not\models T$.

Define a quasi-order $\leq_\kappa$ on complete countable relational theories, letting

$$T \leq_\kappa T' \iff \equiv_T \rightarrow B \equiv_{T'}.$$
Comparing theories via GDST (cont.)

Recall

For a countable first-order relational $T$ and $\eta, \xi \in 2^\kappa$, let $\eta \equiv_T \xi$ iff:

- $M_\eta \models T$ and $M_\xi \models T$ and $M_\xi \equiv M_\eta$, or
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A successful approach

- (Friedman-Hyttinen-Kulikov, 2014) If $T$ is unstable and $T'$ is classifiable, then $T \not\leq_\kappa T'$.
- (Hyttinen-Kulikov-Moreno, 2017) Consistently (e.g., under $V = L$), if $T$ is classifiable but $T'$ is not, then $T \leq_\kappa T'$ and $T' \not\leq_\kappa T$.
- (Asperó-Hyttinen-Kulikov-Moreno, 2019) If $\kappa$ is $\Pi^1_2$-indescribable, for every theory $T$, $T \leq_\kappa \text{ DLO}$ (dense lin. orders without endpoints).
A natural equivalence relation

Let $S$ denote an arbitrary stationary subset of $\kappa$.

**An equivalence relation on $\kappa^\kappa$**

For $\eta, \xi \in \kappa^\kappa$, let $\eta =_S \xi$ iff $\{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}$ is nonstationary.
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**An equivalence relation on $2^\kappa$**

For $\eta, \xi \in 2^\kappa$, let $\eta =_S^2 \xi$ iff $\{ \alpha \in S \mid \eta(\alpha) \neq \xi(\alpha) \}$ is nonstationary.
A natural equivalence relation

Let $S$ denote an arbitrary stationary subset of $\kappa$.

**An equivalence relation on $\kappa^\kappa$**

For $\eta, \xi \in \kappa^\kappa$, let $\eta \approx_S \xi$ iff $\{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}$ is nonstationary.

**Theorem (Hyttinen-Moreno, 2017)**

*If $T$ is a complete classifiable theory, then $\approx_T \hookrightarrow_c \simeq_S$.*

**An equivalence relation on $2^\kappa$**

For $\eta, \xi \in 2^\kappa$, let $\eta \approx_2 \xi$ iff $\{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}$ is nonstationary.

**Theorem (Friedman-Hyttinen-Kulikov, 2014)**

*If $T$ is a complete classifiable theory, then $\approx_2 \not\hookrightarrow_B \simeq_T$.***
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Let $S$ denote an arbitrary stationary subset of $\kappa$.

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*If $T$ is a complete classifiable theory, then $\equiv_T \rightarrow_c \equiv = S$.***

**Theorem (Friedman-Hyttinen-Kulikov, 2014)**

*Consistently, if $T$ is a complete non-classifiable theory, then there exists a stationary $S \subseteq \kappa$ for which $\equiv_2 S \leftrightarrow B \equiv = T$.***

**Theorem (Friedman-Hyttinen-Kulikov, 2014)**

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### Theorem (Hyttinen-Moreno, 2017)

If $T$ is a complete classifiable theory, then $\cong_T \hookrightarrow_c =_S$.

### Theorem (Friedman-Hyttinen-Kulikov, 2014)

Consistently, if $T$ is a complete non-classifiable theory, then there exists a stationary $S \subseteq \kappa$ for which $=^2_S \hookrightarrow_B =_T$.

### Problem

Does $=_S \hookrightarrow_B =^2_S$?
A natural equivalence relation

Let $S$ denote an arbitrary stationary subset of $\kappa$.

**An equivalence relation on $\kappa^\kappa$**

For $\eta, \xi \in \kappa^\kappa$, let $\eta \equiv_S \xi$ iff $\{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}$ is nonstationary.

**Theorem (Hyttinen-Kulikov-Moreno, 2018)**

*If $T$ is a complete classifiable theory and $\diamondsuit_S$, then $\cong_T \hookrightarrow c \equiv \kappa^2_S$.*

**Theorem (Friedman-Hyttinen-Kulikov, 2014)**

*Consistently, if $T$ is a complete non-classifiable theory, then there exists a stationary $S \subseteq \kappa$ for which $\equiv_S^2 \hookrightarrow B \cong_T$.*

**Problem**

Does $\equiv_S \hookrightarrow B \equiv_S^2$?
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An equivalence relation on $\kappa^\kappa$

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Theorem (Hyttinen-Kulikov-Moreno, 2018)

If $T$ is a complete classifiable theory and $\diamondsuit_S$, then $\models_T \leftrightarrow_c 2^S$.

Theorem (Friedman-Hyttinen-Kulikov, 2014)

Consistently, if $T$ is a complete non-classifiable theory, then there exists a stationary $S \subseteq \kappa$ for which $\models_{2^S} \leftrightarrow_B \models_T$.

So, consistently, for $T$ classifiable and $T'$ non-classifiable, for some $S \subseteq \kappa$:

$\models_T \leftrightarrow_c 2^S \leftrightarrow_B \models_{T'}$
Comparing the natural equivalence relations

A large antichain (Friedman-Hyttinen-Kulikov, 2014)

Consistently, there exists a collection $S$ of $2^\kappa$ many stationary subsets of $\kappa$ such that, for all $S \neq S'$ from $S$, $=^{2^\kappa}_S \leftrightarrow_B =^{2^\kappa}_{S'}$.

Comparability is possible as well

If $V = L$, then for all two stationary $S$, $S' \subseteq \kappa$, $=^{2^\kappa}_S \leftrightarrow c =^{2^\kappa}_{S'}$.

This improves a result from [Hyttinen-Kulikov-Moreno, 2018]. Also, the proof is different. The proof works locally as an application of two strong diamond principles at level $\kappa$. In particular, it is not limited to $L$.

Why two? One for ineffable sets and one for stationary non-ineffable.

We'll say more about this later on.
Comparing the natural equivalence relations

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Comparability is possible as well

If $V = L$, then for all two stationary $S, S' \subseteq \kappa$, $\mathrel{\lambda_{S}} \mathrel{\leftrightarrow_{\mathcal{C}}} \mathrel{\lambda_{S'}}$.
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Comparability is possible as well

If \( V = L \), then for all two stationary \( S, S' \subseteq \kappa \), \( \frac{2^\kappa}{S} \leftrightarrow_c \frac{2^\kappa}{S'} \).

This improves a result from [Hyttinen-Kulikov-Moreno, 2018]. Also, the proof is different.

- The proof works locally as an application of two strong diamond principles at level \( \kappa \). In particular, it is not limited to \( L \).
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Consistently, there exists a collection $S$ of $2^\kappa$ many stationary subsets of $\kappa$ such that, for all $S \neq S'$ from $S$, $=^2_S \leftrightarrow_B =^2_{S'}$.

Comparability is possible as well

If $V = L$, then for all two stationary $S, S' \subseteq \kappa$, $=^c_S \rightarrow =^2_{S'}$.

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Comparability is possible as well

If $V = L$, then for all two stationary $S, S' \subseteq \kappa$, $=_S \leftrightarrow_c =^2_{S'}$.

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- The proof works locally as an application of two strong diamond principles at level $\kappa$. In particular, it is not limited to $L$.
- Why two? One for ineffable sets and one for stationary non-ineffable.

We’ll say more about this later on.
A simple test case

Let $S^2_0 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega \}$
and $S^2_1 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1 \}$. 
A simple test case

Let $S^2_0 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega \}$
and $S^2_1 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1 \}$.

What can be said about the following statements?

1. $S^2_0 \rightarrow B = S^2_1$
A simple test case

Let $S_0^2 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega \}$
and $S_1^2 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1 \}$.

What can be said about the following statements?

1. $= S_0^2 \hookrightarrow B = S_1^2$
2. $= S_1^2 \hookrightarrow B = S_0^2$
A simple test case

Let $S_0^2 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega \}$
and $S_1^2 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1 \}$.

What can be said about the following statements?

1. $S_0^2 \hookrightarrow B = S_1^2$
2. $S_1^2 \hookrightarrow B = S_0^2$
3. $2^{S_0^2} \hookrightarrow B = 2^{S_1^2}$
4. $2^{S_1^2} \hookrightarrow B = 2^{S_0^2}$
A simple test case

Let $S_0^2 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega \}$ and $S_1^2 := \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \omega_1 \}$.

What can be said about the following statements?

1. $S_0^2 \hookrightarrow B = S_1^2$
2. $S_1^2 \hookrightarrow B = S_0^2$
3. $\mathbb{2} S_0^2 \hookrightarrow B = \mathbb{2} S_1^2$
4. $\mathbb{2} S_0^2 \hookrightarrow B = \mathbb{2} S_1^2$
5. $S_0^2 \hookrightarrow B = \mathbb{2} S_1^2$
6. $S_1^2 \hookrightarrow B = \mathbb{2} S_0^2$
Theorem (Friedman-Hyttinen-Kulikov, 2014)
There is a model of ZFC in which clauses (3) and (4) fail.

What can be said about the following statements?

1. $S_0 \rightarrow B \rightarrow S_1$
2. $S_1 \rightarrow B \rightarrow S_0$
3. $2_{S_0} \rightarrow B \rightarrow 2_{S_1}$
4. $2_{S_1} \rightarrow B \rightarrow 2_{S_0}$
5. $S_0 \rightarrow B \rightarrow 2_{S_1}$
6. $S_1 \rightarrow B \rightarrow 2_{S_0}$

Note that if Clause (1) holds in the F-H-K model, then $S_2 \rightarrow B \rightarrow 2_{S_2}$.
A simple test case

Theorem (Friedman-Hyttinen-Kulikov, 2014)
There is a model of ZFC in which clauses (3) and (4) fail.

What can be said about the following statements?

1. $s_0^2 \iff B = s_1^2$
2. $s_1^2 \iff B = s_0^2$
3. $2^{s_0} \iff B = 2^{s_1}$
4. $2^{s_0} \iff B = 2^{s_1}$
5. $s_0^2 \iff B = 2^{s_1}$
6. $s_1^2 \iff B = 2^{s_0}$

Note that if Clause (1) holds in the F-H-K model, then $=s_1^2 \not\iff B = 2^{s_1}$. 
Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$. 

**Observation**

Friedman’s problem implies $2^\kappa \rightarrow 2^\omega$. 

**Proof.**

Define a reduction $f : 2^{\omega_2} \rightarrow 2^{\omega_2}$, as follows. Given $\xi : \omega_2 \rightarrow 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff $\delta \in S$ and $\{\alpha < \delta \mid \xi(\alpha) = 0\}$ is nonstationary in $\delta$. 

Note that $f(\xi)(\delta)$ depends only on $\xi|\delta$, so that $f$ is continuous.
Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

_Friedman’s problem implies $\kappa_0 \cL c \iff \kappa_1 \cL c$._

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.
Friedman’s problem

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

Observation

**Friedman’s problem implies** $\mathfrak{c} = \mathfrak{s}_0^2 \hookrightarrow \mathfrak{s}_1^2$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S_1^2 \text{ and } \{ \alpha < \delta \mid \xi(\alpha) = 0 \} \text{ is nonstationary in } \delta.$$
Stańtary refełction principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

Friedman’s problem implies $\Rightarrow S_0^2 \arrows c \Rightarrow S_1^2$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

- $\delta \in S_1^2$ and \{ $\alpha < \delta \mid \xi(\alpha) = 0$ \} is nonstationary in $\delta$.

Note that $f(\xi)(\delta)$ depends only on $\xi \restriction \delta$, so that $f$ is continuous.
Friedman’s problem

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

Observation

Friedman’s problem implies $\mathcal{S}_0^2 \hookrightarrow c \mathcal{S}_1^2$.

Proof. Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.
Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff
$$\delta \in \mathcal{S}_1^2 \text{ and } \{\alpha < \delta \mid \xi(\alpha) = 0\} \text{ is nonstationary in } \delta.$$ Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

Suppose $\xi \mathcal{S}_0^2 \circlearrowleft \eta$. Pick a club $C \subseteq \omega_2$ s.t. $\forall \alpha \in C \cap \mathcal{S}_0^2[\xi(\alpha) = \eta(\alpha)]$. 

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Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

*Friedman’s problem implies* $\{2_{S_0^2}\} \hookrightarrow c = 2_{S_1^2}$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$\delta \in S_1^2$ and $\{\alpha < \delta \mid \xi(\alpha) = 0\}$ is nonstationary in $\delta$.

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

- Suppose $\xi = \stackrel{2}{S_0^2} \eta$. Pick a club $C \subseteq \omega_2$ s.t. $\forall \alpha \in C \cap S_0^2[\xi(\alpha) = \eta(\alpha)]$.

Consider the club $D$ of accumulation points of $C$. For any $\delta \in D \cap S_1^2$:
Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \rightarrow S$.

**Observation**

*Friedman’s problem implies* $2^{S_0} \cdot c = 2^{S_1}$.

**Proof.** Define a reduction $f : 2^{\omega_2} \rightarrow 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \rightarrow 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S_1^2 \text{ and } \{ \alpha < \delta \mid \xi(\alpha) = 0 \} \text{ is nonstationary in } \delta.$$  

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

► Suppose $\xi = \frac{2}{S_0} \cdot \eta$. Pick a club $C \subseteq \omega_2$ s.t. $\forall \alpha \in C \cap S_0^2 [\xi(\alpha) = \eta(\alpha)]$.

Consider the club $D$ of accumulation points of $C$. For any $\delta \in D \cap S_1^2$:

$f(\xi)(\delta) = 1$ iff $\{ \alpha \in C \cap \delta \mid \xi(\alpha) = 0 \}$ is nonstationary in $\delta$ iff

$\{ \alpha \in C \cap \delta \mid \eta(\alpha) = 0 \}$ is nonstationary in $\delta$ iff $f(\eta)(\delta) = 1$. 
Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

Friedman’s problem implies $\models 2^\omega_0 \leftrightarrow_\kappa 2^\omega_1$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S_1^2 \text{ and } \{\alpha < \delta \mid \xi(\alpha) = 0\} \text{ is nonstationary in } \delta.$$  

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

- Suppose $\xi = 2^\omega_0 \eta$. Pick a club $C \subseteq \omega_2$ s.t. $\forall \alpha \in C \cap S_0^2[\xi(\alpha) = \eta(\alpha)]$.

Consider the club $D$ of accumulation points of $C$. For any $\delta \in D \cap S_1^2$:

$f(\xi)(\delta) = 1$ iff $\{\alpha \in C \cap \delta \mid \xi(\alpha) = 0\}$ is nonstationary in $\delta$ iff $\{\alpha \in C \cap \delta \mid \eta(\alpha) = 0\}$ is nonstationary in $\delta$ iff $f(\eta)(\delta) = 1$.

Consequently, $f(\xi) = 2^\omega_0 \leftarrow_\kappa f(\eta)$. 
Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

*Friedman’s problem implies* $=^2_{S_0^2} \leftrightarrow_c =^2_{S_1^2}$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows. Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S_1^2 \text{ and } \{\alpha < \delta \mid \xi(\alpha) = 0\} \text{ is nonstationary in } \delta.$$

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

Suppose $\xi \not\equiv_{S_0^2} \eta$. Say, $S := \{\alpha \in S_0^2 \mid \xi(\alpha) = 1 \& \eta(\alpha) = 0\}$ is stationary.
Friedman’s problem

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

Observation

\textit{Friedman’s problem implies $\vdash_{S_0^2} \varphi \hookrightarrow c = _{S_1^2}^2$.}

Proof. Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$\delta \in S_1^2$ and $\{\alpha < \delta \mid \xi(\alpha) = 0\}$ is nonstationary in $\delta$.

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

Suppose $\xi \not\in_{S_0^2} \eta$. Say, $S := \{\alpha \in S_0^2 \mid \xi(\alpha) = 1 \& \eta(\alpha) = 0\}$ is stationary.

Let $D$ be an arbitrary club in $\omega_2$. 

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**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

*Friedman’s problem implies* $\mathcal{P}^{2}_{S_0} \iff c = \mathcal{P}^{2}_{S_1}$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows. Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

\[ \delta \in S_1^2 \text{ and } \{ \alpha < \delta \mid \xi(\alpha) = 0 \} \text{ is nonstationary in } \delta. \]

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

▶ Suppose $\not\mathcal{P}^{2}_{S_0}\eta$. Say, $S := \{ \alpha \in S_0^2 \mid \xi(\alpha) = 1 \& \eta(\alpha) = 0 \}$ is stationary.

Let $D$ be an arbitrary club in $\omega_2$. Fix a strictly increasing and continuous map $\varphi : \omega_1 \to S \cap D$. 

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**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

*Friedman’s problem implies* $\models 2_{S_0^2} \leftrightarrow c \models 2_{S_1^2}$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows. Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S_1^2 \text{ and } \{ \alpha < \delta \mid \xi(\alpha) = 0 \} \text{ is nonstationary in } \delta.$$  

Note that $f(\xi)(\delta)$ depends only on $\xi \upharpoonright \delta$, so that $f$ is continuous.

- Suppose $\xi \not\models_{S_0^2} \eta$. Say, $S := \{ \alpha \in S_0^2 \mid \xi(\alpha) = 1 \& \eta(\alpha) = 0 \}$ is stationary.

Let $D$ be an arbitrary club in $\omega_2$. Fix a strictly increasing and continuous map $\varphi : \omega_1 \to S \cap D$. Set $d := \text{Im}(\varphi)$, noting that $\delta := \text{sup}(d)$ is in $D \cap S_1^2$. 

Stationary reflection principles and $2^\kappa$

**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \rightarrow S$.

**Observation**

*Friedman’s problem implies $=^2_{S^1_0} \subsetneq c =^2_{S^2_1}$.*

**Proof.** Define a reduction $f : 2^{\omega_2} \rightarrow 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \rightarrow 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S^2_1 \text{ and } \{\alpha < \delta \mid \xi(\alpha) = 0\} \text{ is nonstationary in } \delta.$$  

Note that $f(\xi)(\delta)$ depends only on $\xi \restriction \delta$, so that $f$ is continuous.

$\blacktriangleright$ Suppose $\xi \not\equiv^2_{S^2_0} \eta$. Say, $S := \{\alpha \in S^2_0 \mid \xi(\alpha) = 1 \& \eta(\alpha) = 0\}$ is stationary.

Let $D$ be an arbitrary club in $\omega_2$. Fix a strictly increasing and continuous map $\varphi : \omega_1 \rightarrow S \cap D$. Set $d := \text{Im}(\varphi)$, noting that $\delta := \sup(d)$ is in $D \cap S^2_1$.

Then $d$ is a club in $\delta$ disjoint from $\{\alpha < \delta \mid \xi(\alpha) = 0\}$, so that $f(\xi)(\delta) = 1$. 

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**Friedman’s problem**

For every stationary $S \subseteq \kappa \cap \text{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

**Observation**

*Friedman’s problem implies* $2^{S_0} \prec_c 2^{S_1}$.

**Proof.** Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.

Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ if

$$\delta \in S_1^2 \quad \text{and} \quad \{\alpha < \delta \mid \xi(\alpha) = 0\} \text{ is nonstationary in } \delta.$$ 

Note that $f(\xi)(\delta)$ depends only on $\xi \restriction \delta$, so that $f$ is continuous.

$\blacktriangleright$ Suppose $\xi \not\in^2 S_0^2 \eta$. Say, $S := \{\alpha \in S_0^2 \mid \xi(\alpha) = 1 \& \eta(\alpha) = 0\}$ is stationary.

Let $D$ be an arbitrary club in $\omega_2$. Fix a strictly increasing and continuous map $\varphi : \omega_1 \to S \cap D$. Set $d := \text{Im}(\varphi)$, noting that $\delta := \text{sup}(d)$ is in $D \cap S_1^2$. Then $d$ is a club in $\delta$ disjoint from $\{\alpha < \delta \mid \xi(\alpha) = 0\}$, so that $f(\xi)(\delta) = 1$.

In contrast, $\{\alpha < \delta \mid \eta(\alpha) = 0\}$ contains the club $d$, so that $f(\eta)(\delta) = 0$. 

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Friedman’s problem yields a reduction on the $\kappa$-antor space, $2^\kappa$. Moreno proved that, for $\kappa$ accessible, $=_{S} \leftrightarrow_c =_{S'}$, entails $=_{S} \leftrightarrow_c =_{S'}$. The latter (i.e., reduction in $\kappa^\kappa$) already follows from vanilla reflection:
Stationary reflection principles and $\kappa^\kappa$

Friedman’s problem yields a reduction on the $\kappa$-antor space, $2^\kappa$.
Moreno proved that, for $\kappa$ accessible, $=^2_S \leftrightarrow_c =^2_{S'}$, entails $=^S \leftrightarrow_c =^S'$. The latter (i.e., reduction in $\kappa^\kappa$) already follows from vanilla reflection:

If every stationary subset of $S_0^2$ reflects in $S_1^2$, then $=^{S_0^2} \leftrightarrow_c =^{S_1^2}$.
Stationary reflection principles and $\kappa^\kappa$

Friedman’s problem yields a reduction on the $\kappa$-antor space, $2^\kappa$.
Moreno proved that, for $\kappa$ accessible, $=_{2^S} \hookrightarrow_c =_{2^{S'}}$, entails $=_{2^S} \hookrightarrow_c =_{2^{S'}}$.
The latter (i.e., reduction in $\kappa^\kappa$) already follows from vanilla reflection:

*If every stationary subset of $S_0^2$ reflects in $S_1^2$, then $=_{S_0^2} \hookrightarrow_c =_{S_1^2}$.***

More generally:

**Theorem (Asperó-Hyttinen-Kulikov-Moreno, 2019)**

*Suppose $X, S$ are stationary subsets of $\kappa$, with $S \subseteq \text{cof}(>\omega)$.
If every stationary subset of $X$ reflects in $S$, then $=_{X} \hookrightarrow_c =_{S}$.***
Stationary reflection principles and $\kappa^\kappa$

Friedman’s problem yields a reduction on the $\kappa$-antor space, $2^\kappa$.
Moreno proved that, for $\kappa$ accessible, $=_{S_0}^2 \rightarrow_c =_{S_1}^2$, entails $=_{S_0} \rightarrow_c =_{S_1}$.
The latter (i.e., reduction in $\kappa^\kappa$) already follows from vanilla reflection:

If every stationary subset of $S_0^2$ reflects in $S_1^2$, then $=_{S_0}^2 \rightarrow_c =_{S_1}^2$.

More generally:

**Theorem (Asperó-Hyttinen-Kulikov-Moreno, 2019)**

Suppose $X, S$ are stationary subsets of $\kappa$, with $S \subseteq \text{cof}(>\omega)$.
If every stationary subset of $X$ reflects in $S$, then $=_X \rightarrow_c =_S$.

Bear with me, as I overcomplicate their proof...
Reflection principles and reductions on $\kappa^{\kappa}$

Suppose $X, S$ are stationary subsets of $\kappa$, with $S \subseteq \text{cof}(\omega^2)$. For any ordinal $\delta$, let $\mathcal{F}_\delta$ denote the club filter on $\delta$. 
Reflection principles and reductions on $\kappa^\kappa$

Suppose $X, S$ are stationary subsets of $\kappa$, with $S \subseteq \text{cof}(>\omega)$. For any ordinal $\delta$, let $\mathcal{F}_\delta$ denote the club filter on $\delta$.

The following are equivalent

- Every stationary subset of $X$ reflects in $S$. 

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Reflection principles and reductions on $\kappa^\kappa$

Suppose $X, S$ are stationary subsets of $\kappa$, with $S \subseteq \text{cof}(>\omega)$. For any ordinal $\delta$, let $\mathcal{F}_\delta$ denote the club filter on $\delta$.

The following are equivalent

- Every stationary subset of $X$ reflects in $S$.
- For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$
Reflection principles and reductions on $\kappa^{\kappa}$

Suppose $X, S$ are stationary subsets of $\kappa$, with $S \subseteq \text{cof}(\omega)$. For any ordinal $\delta$, let $\mathcal{F}_\delta$ denote the club filter on $\delta$.

The following are equivalent

- Every stationary subset of $X$ reflects in $S$.
- For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$ and, for every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$. 
Reflection principles and reductions on $\kappa^\kappa$

Suppose $X, S$ are stationary subsets of $\kappa$, and
$$\vec{\mathcal{F}} = \langle \mathcal{F}_\delta \mid \delta \in S \rangle$$
is a sequence such that each $\mathcal{F}_\delta$ is a filter on $\delta$.

**Filter reflection (aka, Fake reflection)**

We say that $X \vec{\mathcal{F}}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$. 
Reflection principles and reductions on $\kappa^\kappa$

Suppose $X, S$ are stationary subsets of $\kappa$, and $\vec{F} = \langle F_\delta \mid \delta \in S \rangle$ is a sequence such that each $F_\delta$ is a filter on $\delta$.

Filter reflection (aka, Fake reflection)

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2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in F_\delta$.

Proposition

If there exists $\vec{F}$ such that $X \vec{F}$-reflects to $S$, then $= X \leftrightarrow_c = S$. 
Reflection principles and reductions on $\kappa^\kappa$

Suppose $X, S$ are stationary subsets of $\kappa$, and $\vec{F} = \langle F_\delta \mid \delta \in S \rangle$ is a sequence such that each $F_\delta$ is a filter on $\delta$.

**Filter reflection (aka, Fake reflection)**

We say that $X$ $\vec{F}$-reflects to $S$ iff the two hold:

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2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in F_\delta$.

**Proposition**

If there exists $\vec{F}$ such that $X$ $\vec{F}$-reflects to $S$, then $\models_X \langle \Rightarrow_c \rangle = S$.

**Proof.** For $\delta \in S$ and $\eta, \xi \in ^\delta \kappa$, let $\eta \sim_\delta \xi$ iff $\{ \alpha < \delta \mid \eta(\alpha) = \xi(\alpha) \} \in F_\delta$. 
Suppose $X, S$ are stationary subsets of $\kappa$, and
$$\vec{F} = \langle F_\delta \mid \delta \in S \rangle$$
is a sequence such that each $F_\delta$ is a filter on $\delta$.

**Filter reflection (aka, Fake reflection)**

We say that $X$ \(\vec{F}\)-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in F_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in F_\delta$.

**Proposition**

If there exists $\vec{F}$ such that $X$ \(\vec{F}\)-reflects to $S$, then $X \hookrightarrow_c = S$.

**Proof.** For $\delta \in S$ and $\eta, \xi \in ^\delta \kappa$, let $\eta \sim_\delta \xi$ iff $\{ \alpha < \delta \mid \eta(\alpha) = \xi(\alpha) \} \in F_\delta$.

Evidently, $|{}^\delta \kappa/ \sim_\delta| \leq \kappa^{<\kappa} = \kappa$, so we may identify $\bigcup_{\delta \in S} {}^\delta \kappa/ \sim_\delta$ with $\kappa$. 

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Reflection principles and reductions on $\kappa^{\kappa}$

Suppose $X, S$ are stationary subsets of $\kappa$, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\delta \mid \delta \in S \rangle$ is a sequence such that each $\mathcal{F}_\delta$ is a filter on $\delta$.

**Filter reflection (aka, Fake reflection)**

We say that $X$ $\vec{\mathcal{F}}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

**Proposition**

If there exists $\vec{\mathcal{F}}$ such that $X$ $\vec{\mathcal{F}}$-reflects to $S$, then $=X \hookrightarrow c = s$.

**Proof.** For $\delta \in S$ and $\eta, \xi \in \delta^{\kappa}$, let $\eta \sim_\delta \xi$ iff $\{ \alpha < \delta \mid \eta(\alpha) = \xi(\alpha) \} \in \mathcal{F}_\delta$. Evidently, $|^{\delta^{\kappa}}/\sim_\delta| \leq \kappa^{<\kappa} = \kappa$, so we may identify $\bigcup_{\delta \in S}^{\delta^{\kappa}}/\sim_\delta$ with $\kappa$.

Define a continuous map $f : \kappa^{\kappa} \to \kappa^{\kappa}$ by letting $f(\xi)(\delta) = [\xi \upharpoonright \delta]_{\sim_\delta}$.
Reflection principles and reductions on $\kappa^\kappa$

Suppose $X, S$ are stationary subsets of $\kappa$, and $
\vec{\mathcal{F}} = \langle \mathcal{F}_\delta \mid \delta \in S \rangle$

is a sequence such that each $\mathcal{F}_\delta$ is a filter on $\delta$.

Filter reflection (aka, Fake reflection)

We say that $X \vec{\mathcal{F}}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

Proposition

If there exists $\vec{\mathcal{F}}$ such that $X \vec{\mathcal{F}}$-reflects to $S$, then $\models_X c \leftrightarrow s$.

Proof. For $\delta \in S$ and $\eta, \xi \in \delta^\kappa$, let $\eta \sim_\delta \xi$ iff \{\(\alpha < \delta \mid \eta(\alpha) = \xi(\alpha)\}\} \in \mathcal{F}_\delta$.

Evidently, $|\delta^\kappa/\sim_\delta| \leq \kappa^{<\kappa} = \kappa$, so we may identify $\bigcup_{\delta \in S} \delta^\kappa/\sim_\delta$ with $\kappa$.

Define a continuous map $f : \kappa^\kappa \to \kappa^\kappa$ by letting $f(\xi)(\delta) = [\xi \upharpoonright \delta]_{\sim_\delta}$.

\[\text{If } \xi \models_X \eta, \text{ then, by Clause (2), } f(\xi) =_S f(\eta).\]
Reflection principles and reductions on $\kappa^{\kappa}$

Suppose $X, S$ are stationary subsets of $\kappa$, and $\vec{F} = \langle F_\delta \mid \delta \in S \rangle$ is a sequence such that each $F_\delta$ is a filter on $\delta$.

Filter reflection (aka, Fake reflection)

We say that $X \vec{F}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in F_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in F_\delta$.

Proposition

If there exists $\vec{F}$ such that $X \vec{F}$-reflects to $S$, then $=_X \leftrightarrow_c =_S$.

Proof. For $\delta \in S$ and $\eta, \xi \in \delta^{<\kappa}$, let $\eta \sim_\delta \xi$ iff $\{ \alpha < \delta \mid \eta(\alpha) = \xi(\alpha) \} \in F_\delta$.

Evidently, $|\delta^{<\kappa}/\sim_\delta| \leq \kappa = \kappa^{\kappa}$, so we may identify $\bigcup_{\delta \in S} \delta^{<\kappa}/\sim_\delta$ with $\kappa$.

Define a continuous map $f : \kappa^{\kappa} \rightarrow \kappa^{\kappa}$ by letting $f(\xi)(\delta) = [\xi \upharpoonright \delta]_{\sim_\delta}$.

If $\xi \not=_{X \kappa} \eta$, then, by Clause (1), $f(\xi) \not=_{S \kappa} f(\eta)$.

$\Box$
Filter reflection

Filter reflection (aka, Fake reflection)

We say that $X \overset{\mathcal{F}}{\rightarrow}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

Consistency strength
Filter reflection

Filter reflection (aka, Fake reflection)

We say that $X$ $\tilde{\mathcal{F}}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_{\delta}^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_{\delta}$.

Just force it

Unlike stationary reflection, fake reflection at the levels of successor cardinals $\kappa = \lambda^+$ may be forced without appealing to large cardinals! Furthermore, forcing over models of GCH preserves the cardinals structure.
**Filter reflection**

**Filter reflection (aka, Fake reflection)**

We say that $X \overset{\mathcal{F}}{\rightarrow}$-reflects to $S$ iff the two hold:

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**Just force it**

Unlike stationary reflection, fake reflection at the levels of successor cardinals $\kappa = \lambda^+$ may be forced without appealing to large cardinals! Furthermore, forcing over models of GCH preserves the cardinals structure.

Let us take a closer look at this principle...
Filter reflection

Filter reflection (aka, Fake reflection)

We say that \( X \xrightarrow{\mathcal{F}} \)-reflects to \( S \) iff the two hold:

1. For every stat. \( Y \subseteq X \), there are stat. many \( \delta \in S \) s.t. \( Y \cap \delta \in \mathcal{F}_\delta^+ \);
2. For every club \( C \subseteq \kappa \), there are club many \( \delta \in S \) s.t. \( C \cap \delta \in \mathcal{F}_\delta \).

About Clause (1):

- Sequences \( \mathcal{F} \) satisfying just Clause (1) exist in ZFC, where each \( \mathcal{F}_\delta \) is principal.
Filter reflection

Filter reflection (aka, Fake reflection)

We say that $X \overset{F}{\rightarrow} \text{reflects to } S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

About Clause (1):
- Sequences $\vec{F}$ satisfying just Clause (1) exist in ZFC, where each $\mathcal{F}_\delta$ is principal.
- A consistent non-principal example is given by Moore’s work on trace reflection in models of MRP.
Filter reflection

Filter reflection (aka, Fake reflection)

We say that $X \overset{\mathcal{F}}{\text{reflects to}} S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

About Clause (2):

- Sequences $\mathcal{F}$ satisfying just Clause (2) exist in ZFC, letting $\mathcal{F}_\delta := \text{CUB}(\delta)$. 

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Filter reflection

Filter reflection (aka, Fake reflection)

We say that $X \overset{\mathcal{F}}{\longrightarrow} S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}\mathcal{U}^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}\mathcal{U}$.

About Clause (2):

- Sequences $\overset{\mathcal{F}}{\longrightarrow}$ satisfying just Clause (2) exist in ZFC, letting $\mathcal{F}\mathcal{U}_\delta := \text{CUB}(\delta)$.
- If $S$ is ineffable and Clause (2) holds, then $\mathcal{F}\mathcal{U}_\delta \supseteq \text{CUB}(\delta)$. 
Filter reflection

**Filter reflection (aka, Fake reflection)**

We say that $X \overset{\mathcal{F}}{\longrightarrow} S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

About Clause (2):

- Sequences $\mathcal{F}$ satisfying just Clause (2) exist in ZFC, letting $\mathcal{F}_\delta := \text{CUB}(\delta)$.
- If $S$ is ineffable and Clause (2) holds, then $\mathcal{F}_\delta \supseteq \text{CUB}(\delta)$.
- If each $\mathcal{F}_\delta$ has a small base, then Clause (2) is a strong club guessing principle.
Filter reflection

Filter reflection (aka, Fake reflection)

We say that $X$ $\tilde{F}$-reflects to $S$ iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$.

About Clause (2):

- Sequences $\tilde{F}$ satisfying just Clause (2) exist in ZFC, letting $\mathcal{F}_\delta := \text{CUB}(\delta)$.
- If $S$ is ineffable and Clause (2) holds, then $\mathcal{F}_\delta \supseteq \text{CUB}(\delta)$.
- If each $\mathcal{F}_\delta$ has a small base, then Clause (2) is a strong club guessing principle. Such a sequence is easily derivable from $\Diamond^*_S$. 

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Filter reflection

Filter reflection with diamond

We say that $X \stackrel{\mathcal{F}}{\rightsquigarrow}$-reflects to $S$ with $\Diamond$ iff there is $\langle Y_\delta \mid \delta \in S \rangle$ such that:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$, $Y_\delta = Y \cap \delta \in \mathcal{F}_\delta^+$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in \mathcal{F}_\delta$. 
Filter reflection

Filter reflection with diamond

We say that $X \vec{F}$-reflects to $S$ with ♦ iff there is $\langle Y_\delta \mid \delta \in S \rangle$ such that:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$, $Y_\delta = Y \cap \delta \in F^+_\delta$;
2. For every club $C \subseteq \kappa$, there are club many $\delta \in S$ s.t. $C \cap \delta \in F_\delta$.

Proposition

If $X \vec{F}$-reflects to $S$ with ♦, then $=_X \leftrightarrow_c =^2_S$.  

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Filter reflection

Filter reflection with diamond

We say that \( X \overset{\mathcal{F}}{\rightarrow} \) reflects to \( S \) with \( \diamond \) iff there is \( \langle Y_\delta \mid \delta \in S \rangle \) such that:

1. For every stat. \( Y \subseteq X \), there are stat. many \( \delta \in S \), \( Y_\delta = Y \cap \delta \in \mathcal{F}_\delta^+ \);
2. For every club \( C \subseteq \kappa \), there are club many \( \delta \in S \) s.t. \( C \cap \delta \in \mathcal{F}_\delta \).

Proposition

If \( X \overset{\mathcal{F}}{\rightarrow} \) reflects to \( S \) with \( \diamond \), then \( \equiv_X \rightarrow_c \equiv_S^2 \).

Fake it till you make it

Fake reflection makes sense even for \( S \) that concentrates on points of countable cofinality, so we consistently get a complete cycle of reductions:

\[
\equiv_{S_1^2} \rightarrow_c \equiv_{S_0^2} \rightarrow c \equiv_{S_0^2} \rightarrow c \equiv_{S_1^2} \rightarrow c \equiv_{S_1^2}.
\]
Authentic reflection with diamond

The special case $\mathcal{X}$ CUB-reflects to $\mathcal{S}$ with ♦ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $\mathcal{X}$ ♦-reflects to $\mathcal{S}$. 
Authentic reflection with diamond

The special case $X \text{ CUB-reflects to } S$ with $\diamond$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X \diamond$-reflects to $S$. They proved it implies $=^2_X \hookrightarrow_c =^2_S$, but we now know that it furthermore implies $=X \hookrightarrow_c =^2_S$.
Authentic reflection with diamond

The special case $X \text{ CUB-reflects to } S$ with $\diamondsuit$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X \diamondsuit$-reflects to $S$. They proved it implies $=^2_X \hookrightarrow_c =^2_S$, but we now know that it furthermore implies $=_X \hookrightarrow_c =^2_S$.

**Theorem (Sun, 1993)**

*If $\kappa$ is ineffable, then $\diamondsuit^1_\kappa$ holds. I.e., there is a sequence $\langle Y_\delta \mid \delta < \kappa \rangle$ s.t., for every $Y \subseteq \kappa$, $\{ \delta < \kappa \mid Y_\delta = Y \cap \delta \}$ is a $\Pi^1_1$-indescribable set.*
Authentic reflection with diamond

The special case $X \text{ CUB-reflects to } S$ with $\Diamond$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X \Diamond$-reflects to $S$. They proved it implies $=^2_X \mapsto_c =^2_S$, but we now know that it furthermore implies $=^1_X \mapsto_c =^2_S$.

**Theorem (Sun, 1993)**

If $\kappa$ is ineffable, then $\Diamond^{1}_{\kappa}$ holds. i.e., there is a sequence $\langle Y_\delta \mid \delta < \kappa \rangle$ s.t., for every $Y \subseteq \kappa$, $\{ \delta < \kappa \mid Y_\delta = Y \cap \delta \}$ is a $\Pi^1_1$-indescribable set. In particular, $\kappa \Diamond$-reflects to $\kappa$. 
Authentic reflection with diamond

The special case $X$ CUB-reflects to $S$ with $\diamondsuit$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X \diamondsuit$-reflects to $S$. They proved it implies $=^2_X \hookrightarrow c =^2_S$, but we now know that it furthermore implies $=_X \hookrightarrow c =^2_S$.

Theorem (Sun, 1993)

*If $S$ is ineffable, then $\diamondsuit^1_S$ holds. In particular, $\kappa \diamondsuit$-reflects to $S$.***
Authentic reflection with diamond

The special case \( X \text{ CUB-reflects to } S \) with \( \diamond \) appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name \( X \diamond \)-reflects to \( S \). They proved it implies \( \mathcal{E}_X \leftrightarrow c = \mathcal{E}_S \), but we now know that it furthermore implies \( \mathcal{E}_X \leftrightarrow c = \mathcal{E}_S \).

Theorem (Sun, 1993)

If \( S \) is ineffable, then \( \diamond^1_S \) holds. In particular, \( \kappa \diamond \)-reflects to \( S \).

Theorem

1. If \( S \) is weakly compact and \( \diamond^*_S \) holds, then so does \( \diamond^1_S \).
Authentic reflection with diamond

The special case $X$ CUB-reflects to $S$ with $\Diamond$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X \Diamond$-reflects to $S$. They proved it implies $\models^X_c =_S^2$, but we now know that it furthermore implies $\models X \leftrightarrow c = S^2$.

**Theorem (Sun, 1993)**

*If $S$ is ineffable, then $\Diamond^1_S$ holds. In particular, $\kappa \Diamond$-reflects to $S$.***

**Theorem**

1. *If $S$ is weakly compact and $\Diamond^*_S$ holds, then so does $\Diamond^1_S$.***

2. *If $\kappa$ is weakly compact, $\lambda \in \text{Reg}(\kappa)$ and GCH holds, then after forcing with $\text{Col}(\lambda, <\kappa)$, $\lambda^+ \cap \text{cof}(<\lambda) \Diamond$-reflects to $\lambda^+ \cap \text{cof}(\lambda)$.***
Authentic reflection with diamond

The special case $X$ CUB-reflects to $S$ with $\diamond$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X$ $\diamond$-reflects to $S$. They proved it implies $=_X \leftrightarrow c =_S^2$, but we now know that it furthermore implies $=X \leftrightarrow c =_S^2$.

Theorem (Sun, 1993)

If $S$ is ineffable, then $\diamond_1^S$ holds. In particular, $\kappa$ $\diamond$-reflects to $S$.

Theorem

1. If $S$ is weakly compact and $\diamond^*_S$ holds, then so does $\diamond_{\frac{1}{2}}^S$.
2. If $\kappa$ is weakly compact, $\lambda \in \text{Reg}(\kappa)$ and GCH holds, then after forcing with $\text{Col}(\lambda, <\kappa)$, $\lambda^+ \cap \text{cof}(<\lambda)$ $\diamond$-reflects to $\lambda^+ \cap \text{cof}(\lambda)$.
3. Assuming MM (Martin's Maximum), if $\diamond_{\kappa \cap \text{cof}(\omega)}$ holds, then $\kappa \cap \text{cof}(\omega)$ $\diamond$-reflects to $\kappa \cap \text{cof}(\omega_1)$.
Authentic reflection with diamond

The special case $X \text{ CUB-reflects to } S$ with $\Box$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name $X \llbracket \Box \rrbracket$-reflects to $S$. They proved it implies $=^{2}_X \rightarrow c =^{2}_S$, but we now know that it furthermore implies $=X \rightarrow c =^{2}_S$.

**Theorem (Sun, 1993)**

If $S$ is ineffable, then $\llbracket \Box \rrbracket^{1}_S$ holds. In particular, $\kappa \llbracket \Box \rrbracket$-reflects to $S$.

**Theorem**

1. If $S$ is weakly compact and $\llbracket \Box \rrbracket^{*}_S$ holds, then so does $\llbracket \Box \rrbracket^{1}_S$.
2. If $\kappa$ is weakly compact, $\lambda \in \text{Reg}(\kappa)$ and GCH holds, then after forcing with $\text{Col}(\lambda, < \kappa)$, $\lambda^+ \cap \text{cof}(\lambda^+) \llbracket \Box \rrbracket$-reflects to $\lambda^+ \cap \text{cof}(\lambda)$.
3. Assuming MM (Martin’s Maximum), if $\llbracket \Box \rrbracket^{\kappa \cap \text{cof}(\omega)}$ holds, then $\kappa \cap \text{cof}(\omega) \llbracket \Box \rrbracket$-reflects to $\kappa \cap \text{cof}(\omega_1)$.
4. Whenever $\text{Add}(\kappa, 1)$ forces that every stationary subset of $X$ reflects in $S$, it moreover forces that $X \llbracket \Box \rrbracket$-reflects to $S$. 
The non-ineffable case

Definition (Devlin, 1982)

\[\diamondsuit_S \text{ asserts the existence of a sequence } \langle N_\delta \mid \delta \in S \rangle \text{ satisfying } (1)-(3):\]

1. each \( N_\delta \) is a p.r.-closed transitive set of size \(|\delta| + \aleph_0 \) & \( \delta + 1 \subseteq N_\delta; \)
The non-ineffable case

Definition (Devlin, 1982)

\(\diamondsuit_S\) asserts the existence of a sequence \(\langle N_\delta \mid \delta \in S \rangle\) satisfying (1)–(3):

1. each \(N_\delta\) is a p.r.-closed transitive set of size \(|\delta| + \aleph_0 \& \delta + 1 \subseteq N_\delta\);
2. for every \(Y \subseteq \kappa\), there is a club \(D \subseteq \kappa\) such that, for all \(\delta \in D \cap S\), \(Y \cap \delta, D \cap \delta \in N_\delta\).
The non-ineffable case

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\[\diamondsuit_S\] asserts the existence of a sequence \(\langle N_\delta \mid \delta \in S \rangle\) satisfying (1)–(3):

1. Each \(N_\delta\) is a p.r.-closed transitive set of size \(|\delta| + \aleph_0\) & \(\delta + 1 \subseteq N_\delta\);
2. For every \(Y \subseteq \kappa\), there is a club \(D \subseteq \kappa\) such that, for all \(\delta \in D \cap S\), \(Y \cap \delta, D \cap \delta \in N_\delta\);
3. Whenever \(\langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models_{H_(\kappa^+)} \phi\), with \(\phi\) a \(\Pi^1_2\)-sentence, there are stationarily many \(\delta \in S\) such that \(\langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models_{N_\delta} \phi\).
The non-ineffable case

Definition (Devlin, 1982)

\[ \diamondsuit_S^\# \] asserts the existence of a sequence \( \langle N_\delta \mid \delta \in S \rangle \) satisfying (1)–(3):

1. each \( N_\delta \) is a p.r.-closed transitive set of size \( |\delta| + \aleph_0 \) & \( \delta + 1 \subseteq N_\delta \);
2. for every \( Y \subseteq \kappa \), there is a club \( D \subseteq \kappa \) such that, for all \( \delta \in D \cap S \), \( Y \cap \delta, D \cap \delta \in N_\delta \);
3. whenever \( \langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models_{H(\kappa^+)} \phi \), with \( \phi \) a \( \Pi^1_2 \)-sentence, there are stationarily many \( \delta \in S \) such that \( \langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models_{N_\delta} \phi \).

Note: Clauses (1) and (2) amount to \( \diamondsuit_S^+ \). Unlike \( \diamondsuit_S^+ \), for every \( \tilde{N} \) witnessing \( \diamondsuit_S^\# \), there is a stationary \( T \subseteq S \) such that \( \tilde{N} \upharpoonright T \) fails to witness \( \diamondsuit_T^\# \).
The non-ineffable case

Definition (Devlin, 1982)

\( \Diamond^\#_S \) asserts the existence of a sequence \( \langle N_\delta \mid \delta \in S \rangle \) satisfying (1)–(3):

1. each \( N_\delta \) is a p.r.-closed transitive set of size \(|\delta| + \aleph_0 \& \delta + 1 \subseteq N_\delta\);
2. for every \( Y \subseteq \kappa \), there is a club \( D \subseteq \kappa \) such that, for all \( \delta \in D \cap S \), \( Y \cap \delta, D \cap \delta \in N_\delta \);
3. whenever \( \langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models_{H(\kappa^+)} \phi \), with \( \phi \) a \( \Pi^1_2 \)-sentence, there are stationarily many \( \delta \in S \) such that \( \langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models_{N_\delta} \phi \).

Theorem (Devlin, 1982)

If \( V = L \), then for every regular uncountable cardinal \( \kappa \), \( \Diamond^\#_\kappa \) holds iff \( \kappa \) is not ineffable.
The non-ineffable case

**Definition (Devlin, 1982)**

\[ \sh^S \] asserts the existence of a sequence \( \langle N_\delta \mid \delta \in S \rangle \) satisfying (1)–(3):

1. each \( N_\delta \) is a p.r.-closed transitive set of size \( |\delta| + \aleph_0 \) & \( \delta + 1 \subseteq N_\delta \);
2. for every \( Y \subseteq \kappa \), there is a club \( D \subseteq \kappa \) such that, for all \( \delta \in D \cap S \), \( Y \cap \delta, D \cap \delta \in N_\delta \);
3. whenever \( \langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models H(\kappa^+) \phi \), with \( \phi \) a \( \Pi^1_2 \)-sentence, there are stationarily many \( \delta \in S \) such that \( \langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models N_\delta \phi \).

**Theorem (Devlin, 1982)**

If \( V = L \), then for every regular uncountable cardinal \( \kappa \), \( \sh^\kappa \) holds iff \( \kappa \) is not ineffable. In fact, for every stationary subset \( S \) of a regular uncountable cardinal, \( \sh^S \) holds iff \( S \) is not ineffable.
The non-ineffable case

Definition (Devlin, 1982)

\[ \clubsuit^#_S \] asserts the existence of a sequence \( \langle N_\delta \mid \delta \in S \rangle \) satisfying (1)–(3):

1. each \( N_\delta \) is a p.r.-closed transitive set of size \( |\delta| + \aleph_0 \) & \( \delta + 1 \subseteq N_\delta \);
2. for every \( Y \subseteq \kappa \), there is a club \( D \subseteq \kappa \) such that, for all \( \delta \in D \cap S \), \( Y \cap \delta, D \cap \delta \in N_\delta \);
3. whenever \( \langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models_{H(\kappa^+)} \phi \), with \( \phi \) a \( \Pi^1_2 \)-sentence, there are stationarily many \( \delta \in S \) such that \( \langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models N_\delta \phi \).

Proposition

If \( \clubsuit^#_S \) holds, then, \( \kappa \) \( \vec{\mathcal{F}} \)-reflects to \( S \) with \( \diamond \) (for some choice of \( \vec{\mathcal{F}} \)).
The non-ineffable case

**Definition (Devlin, 1982)**

\[ S \]

\[ S \] asserts the existence of a sequence \( \langle N_\delta \mid \delta \in S \rangle \) satisfying (1)–(3):

1. each \( N_\delta \) is a p.r.-closed transitive set of size \( |\delta| + \aleph_0 \) & \( \delta + 1 \subseteq N_\delta \);
2. for every \( Y \subseteq \kappa \), there is a club \( D \subseteq \kappa \) such that, for all \( \delta \in D \cap S \), \( Y \cap \delta, D \cap \delta \in N_\delta \);
3. whenever \( \langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models_{H(\kappa^+)} \phi \), with \( \phi \) a \( \Pi^1_2 \)-sentence, there are stationarily many \( \delta \in S \) such that \( \langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models N_\delta \phi \).

**Proposition**

If \( \#_S \) holds, then, \( \kappa \vec{F} \)-reflects to \( S \) with \( \Diamond \) (for some choice of \( \vec{F} \)).

Recalling Sun's theorem about ineffable sets, we infer:

In \( L \), for every regular uncountable cardinal \( \kappa \), and every stationary \( S \subseteq \kappa \), there is a sequence of filters \( \vec{F}_S \) for which \( \kappa \vec{F}_S \)-reflects to \( S \) with \( \Diamond \).
The non-ineffable case

**Definition (Devlin, 1982)**

\[\diamondsuit_S\] asserts the existence of a sequence \(\langle N_\delta \mid \delta \in S \rangle\) satisfying (1)–(3):

1. each \(N_\delta\) is a p.r.-closed transitive set of size \(|\delta| + \aleph_0 \& \delta + 1 \subseteq N_\delta\);
2. for every \(Y \subseteq \kappa\), there is a club \(D \subseteq \kappa\) such that, for all \(\delta \in D \cap S\), \(Y \cap \delta, D \cap \delta \in N_\delta\);
3. whenever \(\langle \kappa, \in, (A_n)_{n \in \omega} \rangle \models_{H(\kappa^+)} \phi\), with \(\phi\) a \(\Pi^1_2\)-sentence, there are stationarily many \(\delta \in S\) such that \(\langle \delta, \in, (A_n \upharpoonright \delta)_{n \in \omega} \rangle \models N_\delta \phi\).

**Corollary**

In \(L\), for all two stationary \(S, S' \subseteq \kappa\), \(=S \leftrightarrow_c =_{2S'}\).

Recalling Sun’s theorem about ineffable sets, we infer:

In \(L\), for every regular uncountable cardinal \(\kappa\), and every stationary \(S \subseteq \kappa\), there is a sequence of filters \(\vec{F}_S\) for which \(\kappa\ \vec{F}_S\)-reflects to \(S\) with \(\diamondsuit\).
◊ # is a GDST’s best friend

It takes more work, yet, all of the above generalize to arbitrary $\Sigma^1_1$-equivalence relations (projections of closed sets in $(\kappa^{<\kappa})^3$ or $(2^{<\kappa})^3$), and to $\Sigma^1_1$-quasi-orders (reflexive+transitive).
is a GDST’s best friend

It takes more work, yet, all of the above generalize to arbitrary \( \Sigma_1 \)-equivalence relations (projections of closed sets in \((\kappa^\kappa)^3 \) or \((2^\kappa)^3 \)), and to \( \Sigma_1 \)-quasi-orders (reflexive+transitive).

### Natural quasi-orders on \( \kappa^\kappa \) and \( 2^\kappa \)

- For \( \eta, \xi \in \kappa^\kappa \), let \( \eta \leq_s \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.
◊# is a GDST’s best friend

It takes more work, yet, all of the above generalize to arbitrary \( \Sigma_1 \)-equivalence relations (projections of closed sets in \((\kappa^\kappa)^3\) or \((2^\kappa)^3\)), and to \( \Sigma_1 \)-quasi-orders (reflexive + transitive).

**Natural quasi-orders on \( \kappa^\kappa \) and \( 2^\kappa \)**

- For \( \eta, \xi \in \kappa^\kappa \), let \( \eta \leq_S \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.
- For \( \eta, \xi \in 2^\kappa \), let \( \eta \subseteq_S \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.
◊ ♦ is a GDST’s best friend

It takes more work, yet, all of the above generalize to arbitrary \( \Sigma_1^- \)-equivalence relations (projections of closed sets in \((\kappa^\kappa)^3\) or \((2^\kappa)^3\)), and to \( \Sigma_1^- \)-quasi-orders (reflexive+transitive).

Natural quasi-orders on \( \kappa^\kappa \) and \( 2^\kappa \)

- For \( \eta, \xi \in \kappa^\kappa \), let \( \eta \leq_S \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.
- For \( \eta, \xi \in 2^\kappa \), let \( \eta \subseteq_S \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.

Theorem

◊ ♦ \( S \) implies that \( Q \hookrightarrow_c \subseteq_S \) for every \( \Sigma_1^- \)-quasi-order \( Q \) on \( \kappa^\kappa \).

In particular, ◊ ♦ \( S \) implies that \( =_{2^S} \) is a \( \Sigma_1^- \)-complete equivalence relation.


\[ \text{\# is a GDST's best friend} \]

It takes more work, yet, all of the above generalize to arbitrary \( \Sigma^1_1 \)-equivalence relations (projections of closed sets in \((\kappa^\kappa)^3\) or \((2^\kappa)^3\)), and to \( \Sigma^1_1 \)-quasi-orders (reflexive + transitive).

### Natural quasi-orders on \( \kappa^\kappa \) and \( 2^\kappa \)

- For \( \eta, \xi \in \kappa^\kappa \), let \( \eta \leq_S \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.
- For \( \eta, \xi \in 2^\kappa \), let \( \eta \subseteq_S \xi \) iff \( \{ \alpha \in S \mid \eta(\alpha) > \xi(\alpha) \} \) is nonstationary.

### Theorem

\[ \text{\#}_S \text{ implies that } Q \hookrightarrow_c \subseteq_S \text{ for every } \Sigma^1_1 \text{-quasi-order } Q \text{ on } \kappa^\kappa. \]

In particular, \( \text{\#}_S \) implies that \( =_{2^S} \) is a \( \Sigma^1_1 \)-complete equivalence relation.

By [Friedman-Wu-Zdomskyy, 2015], it is consistent with GCH that for a successor cardinal \( \kappa \), for all stationary \( S \subseteq \kappa \), \( =_{2^S} \) is \( \Delta^1_1 \).
is a GDST’s best friend

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Natural quasi-orders on \( \kappa^\kappa \) and \( 2^\kappa \)

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Theorem

\( \diamondsuit_S \) implies that \( Q \hookrightarrow c \subseteq_S \) for every \( \Sigma^1_1 \)-quasi-order \( Q \) on \( \kappa^\kappa \).

In particular, \( \diamondsuit_S \) implies that \( \equiv^2_S \) is a \( \Sigma^1_1 \)-complete equivalence relation, while \( \diamondsuit^+_S \) does not.

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Comparing theories, revisited

Devlin’s $\diamondsuit_S$ may become an essential tool for the model theorist...

**Theorem**

Let $T$ be a complete first-order countable relational theory. In any of the following cases, $\equiv_T$ is $\Sigma^1_1$-complete:

- $\kappa = \lambda^+$, $\lambda^\kappa = \lambda$, $\diamondsuit_{\kappa \cap \text{cof}(\lambda)}$ holds and $T$ is unstable;
- $\kappa$ is inaccessible, $\diamondsuit_{\kappa \cap \text{cof}(2^{\aleph_0}^+)}$ and $T$ is superstable with $S$-DOP;
- $\kappa$ is $\aleph_0$-inaccessible, $\diamondsuit_{\kappa \cap \text{cof}(\aleph_0)}$ holds, and $T$ is stable unsuperstable.