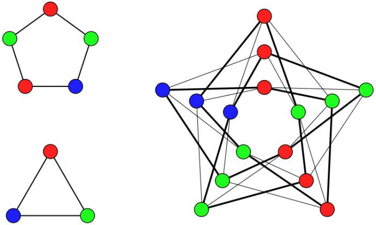
Hedetniemi's conjecture for uncountable graphs



Forcing and Large Cardinals
Erwin Schrödinger Institute, Vienna
26-September-2013

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Bar-Ilan University, Israel

Partial bibliography

This talk will center around the following works:

[Rin1] The Ostaszewski square, and homogeneous Souslin trees,
to appear in Isr. J. Math. http://www.assafrinot.com/paper/11

[Rin2] Chromatic number of graphs - large gaps,
to appear in Combinatorica http://www.assafrinot.com/paper/12

[Rin3] Hedetniemi's conjecture for uncountable graphs,
submitted for publication http://www.assafrinot.com/paper/16

[Rin4] The fragility of chromatic numbers,
work in progress

Graphs and chromatic numbers

Definition

A graph is a structure $\mathcal{G} = (G, E)$ with $E \subseteq [G]^2 := \{\{x, y\} \mid x, y \in V\}$.

We sometime write $V(\mathcal{G})$ for the set of vertices G, and $E(\mathcal{G})$ for the set of edges E.

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A coloring $\chi : G \to \kappa$ is <u>E-chromatic</u> if gEh entails $\chi(g) \neq \chi(h)$.

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$\mathrm{Chr}(G, E)$ is the least cardinal κ for which there exists an E-chromatic coloring $\chi : G \to \kappa$.
Equivalently, it is the least cardinal κ such that $G = \bigcup_{i<\kappa} A_i$, where A_i is E-independent for each $i < \kappa$.

An example

Suppose that $\mathcal{T} = \langle T, \lhd \rangle$ is an ω_1-tree. Recall that $\mathcal{T}$ is said to be special if there exists an order-preserving mapping from $\mathcal{T}$ to the rationals $\langle \mathbb{Q}, < \rangle$.

Observation 1

Consider the corresponding comparability graph $\mathcal{G}_{\mathcal{T}}$, where $V(\mathcal{G}_{\mathcal{T}}) := T$ and $E(\mathcal{G}_{\mathcal{T}}) := \{\{x, y\} \mid x \lhd y \text{ or } y \lhd x\}$.
Then $\mathcal{T}$ is special iff $\mathrm{Chr}(\mathcal{G}_{\mathcal{T}}) = \aleph_0$.

An example (cont.)

Observation 2

If $\mathcal{T}$ is a Souslin tree, then $\mathcal{T}$ is nonspecial.

Proof.

If $\mathcal{T}$ were special, then $\mathrm{Chr}(\mathcal{G}_{\mathcal{T}}) = \aleph_0$, and $T = \bigcup_{i<\omega} A_i$, where A_i is an antichain for all $i < \omega$. This is a contradiction to the fact that $\mathcal{T}$ has no uncountable antichains. □

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Alternative proof.

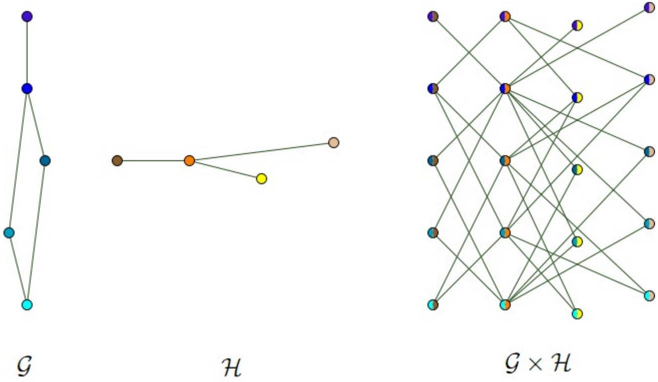
Forcing with $\mathcal{T}$ yields an extended universe W with $\omega_1^V = \omega_1^W$, in which $\mathcal{T}$ admits a cofinal branch b. Now, given a coloring $\chi : T \to \aleph_0$ from V, we may find (in W) distinct elements s, t of b in with $\chi(s) = \chi(t)$. This means that χ is not E-chromatic. □

The tensor product of graphs

Definition

Given graphs $\mathcal{G} = (G, E), \mathcal{H} = (H, F)$, let $\mathcal{G} \times \mathcal{H} := (G \times H, E * F)$, where:

- $G \times H := \{(g, h) \mid g \in G, h \in H\}$;
- $E * F = \{\{(g_0, h_0), (g_1, h_1)\} \mid g_0 E g_1 \ \& \ h_0 F h_1\}$.



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Given an E-chromatic coloring $\chi : G \to \kappa$, define a coloring $\chi' : G \times H \to \kappa$ by stipulating $\chi'(g, h) := \chi(g)$. Then χ' is $E * F$-chromatic.

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It follows that $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) \leq \mathrm{Chr}(\mathcal{G})$.
By symmetry, $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) \leq \mathrm{Chr}(\mathcal{H})$.
Thus,

$$\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) \leq \min\{\mathrm{Chr}(\mathcal{G}), \mathrm{Chr}(\mathcal{H})\}.$$

Hedetniemi's conjecture

Conjecture (Hedetniemi, 1966)

For every pair of (finite) graphs $\mathcal{G}, \mathcal{H}$:

$$\mathsf{Chr}(\mathcal{G} \times \mathcal{H}) = \min\{\mathsf{Chr}(\mathcal{G}), \mathsf{Chr}(\mathcal{H})\}.$$

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Not only that the above conjecture is still standing, but even the following Ramsey-type consequence of it is still unknown to hold.

Weak Hedetniemi Conjecture (1970's)

For every positive integer k, there exists an integer $\varphi(k)$, such that if $\mathsf{Chr}(\mathcal{G}) = \mathsf{Chr}(\mathcal{H}) = \varphi(k)$, then $\mathsf{Chr}(\mathcal{G} \times H) \geq k$.

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Remarks

1. Hedetniemi's conjecture is equivalent to "$\varphi(k) = k$ for all positive integer k";
2. Hedetniemi (1966) proved that $\varphi(k) = k$ for $k \in \{1, 2, 3\}$;
3. El-Zahar and Sauer (1985) proved that $\varphi(4) = 4$.

The infinite counterpart

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> *The corresponding statement for infinite graphs is not true [Haj85], but little is known about the relationship between the cardinalities of the chromatic number of A and B, and the chromatic number of $A \times B$.*

N.W. Sauer, Hedetniemi Conjecture,
Encyclopedia of Mathematics,
page 287, Kluwer, 1997.

The infinite counterpart (cont.)

Theorem (Hajnal, 1985)

For every infinite cardinal κ, there exist graphs $\mathcal{G}, \mathcal{H}$ such that

- $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \kappa^{+}$;
- $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \kappa$.

The infinite counterpart (cont.)

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- $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \kappa^+$;
- $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \kappa$.

Theorem (Soukup, 1988)

It is consistent with ZFC + GCH that there exist graphs $\mathcal{G}, \mathcal{H}$ of size and chromatic number $\aleph_2$ such that $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_0$.

Effective versions

Theorem (Kechris-Solecki-Todorcevic, 1999)

If $\aleph_1^{L[a]} = \aleph_1$ for some $a \subseteq \mathbb{N}$, then there exist two co-analytic graphs $\mathcal{G}$ and H such that:

- $\min\{\mathrm{Chr}(\mathcal{G}), \mathrm{Chr}(\mathcal{H})\} > \aleph_0$;
- $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_0$.

Theorem (Kechris-Solecki-Todorcevic, 1999)

If $\mathcal{G}, \mathcal{H}$ are two analytic graphs with uncountable Borel chromatic number, then their product $\mathcal{G} \times \mathcal{H}$ also has uncountable Borel chromatic number.

The extent of the failure

Question 1

In his paper, Hajnal asked whether it is consistent with ZFC+GCH that there are graphs $\mathcal{G}, \mathcal{H}$ with $\text{Chr}(\mathcal{G}) = \text{Chr}(\mathcal{H}) \geq \aleph_\omega$ and $\text{Chr}(\mathcal{G} \times \mathcal{H}) < \aleph_\omega$?

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Soukup, as well as Hajnal, also asked whether it is consistent with ZFC+GCH that there exist graphs of size and chromatic number $\aleph_3$ whose tensor product is countably chromatic.

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Soukup, as well as Hajnal, also asked whether it is consistent with ZFC+GCH that there exist graphs of size and chromatic number $\aleph_3$ whose tensor product is countably chromatic.

Putting aside the GCH requirement, these questions simply address instances of the infinite version of the weak conjecture:

Infinite Weak Hedetniemi Conjecture

For every infinite cardinal κ, there exists a cardinal $\varphi(\kappa)$, such that if $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \varphi(\kappa)$, then $\mathrm{Chr}(\mathcal{G} \times H) \geq \kappa$.

Necessary features

Observation

If $\mathsf{Chr}(\mathcal{G} \times \mathcal{H}) < \min\{\mathsf{Chr}(\mathcal{G}), \mathsf{Chr}(\mathcal{H})\}$, then $\mathcal{G}$ does not embed to $\mathcal{H}$, and vice versa.

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Proposition (Hajnal, 1980's)

If $\mathcal{G}, \mathcal{H}$ have infinite chromatic number, then every subgraph of $\mathcal{G}$ of size $< \mathrm{Chr}(\mathcal{H})$ has chromatic number $\leq \mathrm{Chr}(\mathcal{G} \times \mathcal{H})$.

So, if $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) < \min\{\mathrm{Chr}(\mathcal{H}), \mathrm{Chr}(\mathcal{G})\}$, then $\mathcal{G}$ or $\mathcal{H}$ exemplifies the incompactness of the chromatic number.

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A difficult problem on its own

The very existence of incompactness graphs was questioned by Erdös and Hajnal in the 1960's. At that point in time, the only known fact was the compactness result of de Bruijn and Erdös.

Compactness

Theorem (de Bruijn-Erdös, 1951)

Suppose that $\mathcal{G}$ *is a graph,* $k < \omega$*, and every finite subgraph of* $\mathcal{G}$ *has chromatic number* $\leq k$*, then* $\mathrm{Chr}(\mathcal{G}) \leq k$*.*

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Suppose that $\mathcal{G}$ is a graph, $k < \omega$, and every finite subgraph of $\mathcal{G}$ has chromatic number $\leq k$, then $\mathrm{Chr}(\mathcal{G}) \leq k$.

Proof.

First, note that for every graph (G, E) and $X \subseteq G$, the set

$$F(X) := \{f \in {}^{G}k \mid f \restriction X \text{ is } E\text{-chromatic}\}$$

is a closed subset of the Tychonoff space ${}^{G}k$.

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Now, suppose that every finite subgraphs of (G, E) is $\leq k$-chromatic. Then $\{F(X) \mid X \in [G]^{<\omega}\}$ has the finite intersection property. So, by compactness, there exists f in $\bigcap\{F(X) \mid X \in [G]^2\}$, and clearly, $f : G \to k$ is E-chromatic. □

Compactness

Theorem (de Bruijn-Erdös, 1951)

Suppose that $\mathcal{G}$ *is a graph,* $\kappa < \theta$, $\theta = \aleph_0$ *or strongly-compact, and every subgraph of* $\mathcal{G}$ *of size* $< \theta$ *has chromatic number* $\leq \kappa$*, then* $\mathrm{Chr}(\mathcal{G}) \leq \kappa$.

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Corollary

Suppose that there exist class many strongly compact cardinals. Then the Infinite Weak Hedetniemi Conjecture holds.
Moreover, for every infinite cardinal κ, there exists a cardinal $\varphi(\kappa)$, such that if $\min\{\mathrm{Chr}(\mathcal{G}), \mathrm{Chr}(\mathcal{H})\} \geq \varphi(\kappa)$, then $\mathrm{Chr}(\mathcal{G} \times H) \geq \kappa$.

Proof.

For a cardinal κ, let $\varphi(\kappa)$ be the least strongly-compact cardinal $\theta \geq \kappa$. Now, suppose that $\min\{\mathrm{Chr}(\mathcal{G}), \mathrm{Chr}(\mathcal{H})\} \geq \theta$, and $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \kappa' < \kappa$. Then, by Hajnal's proposition, subgraphs of $\mathcal{G}$ of size $< \theta$ have chromatic number $\leq \mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \kappa'$. Then, by de Bruijn-Erdös, we get $\mathrm{Chr}(\mathcal{G}) \leq \kappa' < \theta$. A contradiction. □

Summary

- The Infinite Weak Hedetniemi Conjecture is consistent, relative to large cardinals.
- Any consistent counterexample to the weak conjecture must be based on (class many) pairs of incompactness graphs, pairwise non-embeddable.
- The best known result is due to Soukup (1988), who proved the consistency of $\varphi(\aleph_0) > \aleph_2$.

Incompactness graphs

Incompactness graphs

Definition

Say that a graph $\mathcal{G}$ is $(\aleph_0, \lambda)$-chromatic, if $\mathsf{Chr}(\mathcal{G}) = \lambda$, but every subgraph of $\mathcal{G}$ of strictly smaller size has chromatic number $\leq \aleph_0$. We sometime say that $\mathcal{G}$ is almost countably chromatic.

- (Erdös-Hajnal, 1966) if $2^{\aleph_0} = \aleph_1$, then there exists an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_2$;
- (Komjáth, 1988) it is consistent with $2^{\aleph_0} = \aleph_{\omega_1+1}$ that there exists an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_{\omega_1}$;
- (Shelah, 1990) it is consistent with GCH that there exists an $(\aleph_0, \aleph_1)$-chromatic graph of size $\aleph_{\omega_1}$;

Constructions of almost countably chromatic graphs

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Baumgartner proved that CH entails a poset of size $\aleph_2$ which is σ-closed, has the $\aleph_2$-c.c., and adds an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$. A whole page in Baumgartner's original paper is dedicated to motivating the definition of his complicated poset.

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Soukup's model of GCH $+\varphi(\aleph_0) > \aleph_2$ is a further sophistication of Baumgartner's forcing.

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Unfortunately, Baumgartner's approach does not seem to generalize to yield $(\aleph_0, \aleph_3)$-chromatic graphs.

Almost countably chromatic and highly chromatic graphs

- (Shelah, 1990) if $V = L$, then (GCH holds, and) for every uncountable not weakly-compact cardinal λ, there exists an $(\aleph_0, \lambda)$-chromatic graph of size λ;
- (Soukup, 1990) If Martin's Axiom holds, then there is a $(< \mathfrak{c})$-distributive, $\mathfrak{c}^+$-c.c., notion of forcing of size $\mathfrak{c}^+$ that adds an $(\aleph_0, \mathfrak{c}^+)$-chromatic graph of size $\mathfrak{c}^+$;
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Note that Shelah's 1990 result is the only one to provide a model with class many incompactness graphs.

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Note that Shelah's 1990 result is the only one to provide a model with class many incompactness graphs. However, as Hajnal mentioned in his 2004 PIMS Distinguished Chair Lecture, there was no success in generalizing Shelah's result.

The C-sequence graph

Simple way to derive graphs

Definition ([Rin2])

Given a sequence of local clubs $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \theta \text{ limit}\rangle$, and a subset $G \subseteq \theta$, we define an edge relation E as follows:

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Key Feature ([Rin2])

If $\overrightarrow{C}$ is end-segment-coherent (e.g., a $\square(\theta)$-sequence), and $G \subseteq \theta$ is a non-reflecting set, then $G(\overrightarrow{C})$ is an $(\aleph_0, \kappa)$-graph, for some cardinal κ.

Simple way to derive graphs - application #1

Definition (Ostaszewski square, [Rin1])

$\boxed{\clubsuit}_\lambda$ asserts the existence of a $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ with the following additional feature.

For every sequence $\langle A_i \mid i < \lambda \rangle$ of cofinal subsets of λ^+, and every limit $\theta < \lambda$, there exists some $\alpha < \lambda^+$ such that $\mathrm{otp}(C_\alpha) = \theta$, and $C_\alpha(i+1) \in A_i$ for all $i < \theta$.

Simple way to derive graphs - application #1

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Theorem ([Rin2])

If $\overrightarrow{C}$ is an $\boxed{\clubsuit}_\lambda$-sequence, then for every infinite cardinal $\kappa < \lambda$, there exists a non-reflecting stationary set $G_\kappa \subseteq \lambda^+$ such that $G_\kappa(\overrightarrow{C})$ is $(\aleph_0, \kappa^+)$-chromatic.

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Corollary ([Rin2])

If $\square_\lambda + \mathrm{CH}_\lambda$ holds, then for every cardinal $\kappa \leq \lambda$, there exists an $(\aleph_0, \kappa)$-chromatic graph of size λ^+.

Simple way to derive graphs - application #2

Theorem (Baumgartner, 1984)

Assume CH. Then there exists a poset of size $\aleph_2$ which is σ-closed, has the $\aleph_2$-c.c., and adds an $(\aleph_0, \aleph_2)$-chromatic graph of size $\aleph_2$.

Theorem ([Rin2])

Assume CH. If $\square_{\aleph_1}$ holds, then in $V^{\mathrm{Add}(\omega_1,1)}$, there exists an $(\aleph_0, \aleph_2)$-graph of size $\aleph_2$.

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Theorem ([Rin2])

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This does generalize:

Theorem ([Rin2])

Suppose $\lambda^{<\lambda} = \lambda$ and $\square_\lambda$ holds. Then in $V^{\mathrm{Add}(\lambda,1)}$, there exists an $(\aleph_0, \lambda^+)$-graph of size λ^+.

Indeed, the example graph is of the form $G(\overrightarrow{C})$, for $G = E^{\lambda^+}_{\lambda}$, and a certain $\boxed{\clubsuit}_\lambda$-like sequence $\overrightarrow{C}$ that exists in the generic extension.

Simple way to derive graphs - application #3

Theorem (Shelah, 1990)

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P. Komjáth, BSL VII(4), page 540, 2001.

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Theorem ([Rin2])

If $\square_\lambda + \mathsf{CH}_\lambda$ holds, λ singular, then there exists $\overrightarrow{C}$ and G such that $G(\overrightarrow{C})$ is an $(\aleph_0, \lambda^+)$-chromatic graph of size λ^+.

Towards a solution

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By Hajnal's proposition, if $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \lambda$, while $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) = \aleph_0$, then $\mathcal{G}$ and $\mathcal{H}$ are $(\aleph_0, \lambda)$-chromatic.

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A careful look at Hajnal's 1985 construction hints on a partial converse:

Suppose that $\mathcal{G} = (\theta, E)$ is an almost countably chromatic graph. Define $\mathcal{G}^*$, by letting:

- $V(\mathcal{G}^*) := \{c \in {}^{<\theta}\omega \mid c \text{ is } E\text{-chromatic}\}$;
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Then the half graph $\{(\alpha, c) \mid \mathrm{dom}(c) > \alpha\}$ of $\mathcal{G} \times \mathcal{G}^*$ is countably chromatic, as witnessed by the coloring $(\alpha, c) \mapsto c(\alpha)$.

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2. what about the diagonal $\{(\alpha, c) \mid \mathrm{dom}(c) = \alpha\}$?
3. why should $\mathcal{G}^*$ have an high chromatic number?
4. how does this relate to a workshop in Forcing and Large Cardinals?

   Indeed, large cardinals were seen to (outright) resolve one side of the problem, but if you are going to work in L for the other side, where would the forcing come from?

Towards a solution (cont.)

Question 2

What about the diagonal $\{(\alpha, c) \mid \mathrm{dom}(c) = \alpha\}$?

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Short Answer

We don't care.

Towards a solution (cont.)

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Short Answer

We don't care.

Long Answer

This can be found already in Hajnal's original construction.
Split θ into two equipotent sets T_0 and T_1. Then, identify $V(\mathcal{G})$ with T_0, and restrict the vertices set of the corresponding $\mathcal{G}^*$ to the functions h only with $\text{dom}(h) \in T_1$.
This way, the diagonal turns void.

Towards a solution (cont.)

Question 3

Why should $\mathcal{G}^*$ have an high chromatic number?

Answer

Recall that $\mathcal{G}_{\mathcal{T}}$ for a Souslin tree $\mathcal{T}$ is uncountably chromatic, since $\mathcal{T}$ admits a branch in some σ-distributive forcing extension.
Now, suppose that θ is regular, and that $\mathcal{G} = (\theta, E)$ is more than just $(\aleph_0, \theta)$-chromatic — it can be made countably chromatic in some $(< \theta)$-distributive forcing extension. Then the chromatic number of the corresponding $\mathcal{G}^*$ would have to be $\geq \theta$.

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But, is this at all possible?

Theorem ([Rin4])

In L, for every infinite λ, there exists a sequence $\overrightarrow{C}$, a set $G \subseteq \lambda^+$, and a λ-distributive notion of forcing $\mathbb{P}$ of size λ^+, such that:

- $L \models \mathrm{Chr}(G(\overrightarrow{C})) = \lambda^+$;
- $L^{\mathbb{P}} \models \mathrm{Chr}(G(\overrightarrow{C})) = \aleph_0$.

Towards a solution (cont.)

Question 4

When will you be using forcing?!

Towards a solution (cont.)

Question 4

When will you be using forcing?!

Answer

I just did!

We saw that if $V = L$, then there exists a graph $\mathcal{G}$ such that:

- $\mathcal{G}$ has size and chromatic number λ^+;
- $\mathcal{G}^*$ has size and chromatic number λ^+ (by a forcing argument!);
- the half graph of $\mathcal{G} \times \mathcal{G}^*$ is countably chromatic;
- wlog, it has no diagonal.

Thus, we are left with only one problem - taming the other half.

Main Result

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Theorem ([Rin3])

Suppose that $\boxtimes_\lambda$ holds.
Then there exist graphs $\mathcal{G}, \mathcal{H}$ of size λ^+ such that:

- $\mathrm{Chr}(\mathcal{G}) = \mathrm{Chr}(\mathcal{H}) = \lambda^+$;
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Theorem ([Rin3])

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Bigger products ([Rin3])

Suppose that $\aleph_0 \leq \kappa \leq \lambda$, $n < \omega$, and $\boxed{\diamondsuit}_\lambda$ holds.
Then there exist graphs $\mathcal{G}_0, \ldots, \mathcal{G}_{n+1}$ of size λ^+ such that:

- $\mathrm{Chr}(\prod_{i \in I} \mathcal{G}_i) = \lambda^+$ whenever $\emptyset \neq I \subsetneqq \{0, \ldots, n+1\}$;
- $\mathrm{Chr}(\prod_{i \in I} \mathcal{G}_i) = \kappa$ for $I = \{0, \ldots, n+1\}$.

Consequences of the main theorem

In the constructible universe:

- the Infinite Weak Hedetniemi Conjecture fails, since $\varphi(\kappa)$ would have to be larger than λ for every cardinal λ.
- GCH holds, and there are graphs $\mathcal{G}, \mathcal{H}$ with $\mathsf{Chr}(\mathcal{G}) = \mathsf{Chr}(\mathcal{H}) \geq \aleph_\omega$ and $\mathsf{Chr}(\mathcal{G} \times \mathcal{H}) < \aleph_\omega$.
- GCH holds and there exist graphs of size and chromatic number $\aleph_3$ whose tensor product is countably chromatic.
- for every positive integer n, there are graphs $\mathcal{G}, \mathcal{H}$ such that

$$\mathsf{Chr}(\mathcal{G} \times \mathcal{H}) = \underbrace{\log \cdots \log}_{n \text{ times}}(\min\{\mathsf{Chr}(\mathcal{G}), \mathsf{Chr}(\mathcal{H})\}).$$

Outline of the proof of the main theorem

Utilize $\boxtimes_\lambda$ to construct a $\square_\lambda$-sequence $\overrightarrow{C}$, pairwise disjoint nonreflecting stationary sets G_0, G_1, and notions of forcing $\mathbb{P}_0, \mathbb{P}_1$, such that for $\mathcal{G}_0 := G_0(\overrightarrow{C}), \mathcal{G}_1 := G_1(\overrightarrow{C})$, we have:

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1.	$\mathbb{P}_0$ is λ-distributive of size λ^+	$\mathbb{P}_1$ is λ-distributive of size λ^+
2.	$V^{\mathbb{P}_0} \models \mathrm{Chr}(\mathcal{G}_0) = \aleph_0$	$V^{\mathbb{P}_1} \models \mathrm{Chr}(\mathcal{G}_0) = \lambda^+$
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In particular, $\mathcal{G}_0$ and $\mathcal{G}_1$ are pairwise non-embeddable.

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For $i < 2$, define $\mathcal{H}_i$ as follows:

- $V(\mathcal{H}_i) := \{h \in V(\mathcal{G}^*_{1-1}) \mid \mathrm{dom}(h) \in V(\mathcal{G}_i)\}$;
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Note that $\mathrm{Chr}(\mathcal{H}_i) \leq \mathrm{Chr}(\mathcal{G}_i)$ by lifting of colorings.

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Utilize $\boxed{\Diamond}_\lambda$ to construct a $\square_\lambda$-sequence $\vec{C}$, pairwise disjoint nonreflecting stationary sets G_0, G_1, and notions of forcing $\mathbb{P}_0, \mathbb{P}_1$, such that for $\mathcal{G}_0 := G_0(\vec{C}), \mathcal{G}_1 := G_1(\vec{C})$, we have:

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Note that $\mathrm{Chr}(\mathcal{H}_i) \leq \mathrm{Chr}(\mathcal{G}_i)$ by lifting of colorings.
As $\mathcal{H}_0$ is a subgraph of $\mathcal{G}^*_1$, and the half-graph of $\mathcal{G}_0 \times \mathcal{G}^*_0$ is countably chromatic, so is the half-graph of $\mathcal{G}_0 \times \mathcal{H}_1$. Then, by lifting, the half-graph of $\mathcal{H}_0 \times \mathcal{H}_1$ is countably chromatic.

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Utilize $\boxed{\diamondsuit}_\lambda$ to construct a $\square_\lambda$-sequence $\vec{C}$, pairwise disjoint nonreflecting stationary sets G_0, G_1, and notions of forcing $\mathbb{P}_0, \mathbb{P}_1$, such that for $\mathcal{G}_0 := G_0(\vec{C}), \mathcal{G}_1 := G_1(\vec{C})$, we have:

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Likewise, the half-graph of $\mathcal{H}_1 \times \mathcal{H}_0$ is countably chromatic.
As $G_0 \cap G_1 = \emptyset$, the diagonal is vacuous.
Altogether, $\mathrm{Chr}(\mathcal{H}_0 \times \mathcal{H}_1) = \aleph_0$.

Outline of the proof of the main theorem

Utilize $\boxed{\diamondsuit}_\lambda$ to construct a $\square_\lambda$-sequence $\vec{C}$, pairwise disjoint nonreflecting stationary sets G_0, G_1, and notions of forcing $\mathbb{P}_0, \mathbb{P}_1$, such that for $\mathcal{G}_0 := G_0(\vec{C}), \mathcal{G}_1 := G_1(\vec{C})$, we have:

1.	$\mathbb{P}_0$ is λ-distributive of size λ^+	$\mathbb{P}_1$ is λ-distributive of size λ^+
2.	$V^{\mathbb{P}_0} \models \mathrm{Chr}(\mathcal{G}_0) = \aleph_0$	$V^{\mathbb{P}_1} \models \vec{C} \restriction G_0$ is Ostaszewski
3.	$V^{\mathbb{P}_0} \models \vec{C} \restriction G_1$ is Ostaszewski	$V^{\mathbb{P}_1} \models \mathrm{Chr}(\mathcal{G}_1) = \aleph_0$

In particular, $\mathcal{G}_0$ and $\mathcal{G}_1$ are pairwise non-embeddable.

For $i < 2$, define $\mathcal{H}_i$ as follows:

- $V(\mathcal{H}_i) := \{h \in V(\mathcal{G}^*_{1-1}) \mid \mathrm{dom}(h) \in V(\mathcal{G}_i)\}$;
- $E(\mathcal{H}_i) := \{\{h, h'\} \in E(\mathcal{G}^*_{1-i}) \mid \{\mathrm{dom}(h), \mathrm{dom}(h')\} \in E(\mathcal{G}_i)\}$.

The half-graph of $\mathcal{H}_0 \times \mathcal{H}_1$ is countably chromatic.
Likewise, the half-graph of $\mathcal{H}_1 \times \mathcal{H}_0$ is countably chromatic.
As $G_0 \cap G_1 = \emptyset$, the diagonal is vacuous.
Altogether, $\mathrm{Chr}(\mathcal{H}_0 \times \mathcal{H}_1) = \aleph_0$.
By the third column of the above table, $\mathrm{Chr}(\mathcal{H}_0) = \lambda^+$.
By the second column of the above table, $\mathrm{Chr}(\mathcal{H}_1) = \lambda^+$. $\square$

Further investigation

1. In the proof we just discussed, the analysis of the constructed graphs was done by passing to forcing extensions of the universe. Is there a forcing-free proof?
2. We saw that the existence of class many strongly compact cardinals entails the infinite weak Hedetniemi conjecture.
    2.1 Does the weak conjecture follow from a bounded amount of large cardinals?
    2.2 Is it consistent that the weak conjecture hold and the witnessing φ satisfies $\varphi(\kappa) \leq \kappa^{+\mu}$ for some fixed cardinal μ? How about $\varphi(\kappa) \leq \beth_\mu(\kappa)$?
3. What is the consistency strength of the statement: "there exist infinite cardinals κ, ψ, such that if $\mathcal{G}, \mathcal{H}$ are graphs with $\min\{\mathrm{Chr}(\mathcal{G}), \mathrm{Chr}(\mathcal{H})\} \geq \psi$, then $\mathrm{Chr}(\mathcal{G} \times \mathcal{H}) \geq \kappa$" ? How about $\kappa = \aleph_1$?