Diamond on successors of singulars

ESI workshop on large cardinals
and descriptive set theory

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Diamond on successor cardinals

Definition (Jensen, ‘72). For an infinite cardinal, λ, and a stationary set $S \subseteq \lambda^+$, $\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for all $A \subseteq \lambda^+$.

Theorem (Jensen, '72). In Gödel's constructible universe, $\diamondsuit(S)$ holds for every stationary $S \subseteq \lambda^+$ and every infinite cardinal, λ.

Notation and conventions

Let $E^{\lambda^+}_{\kappa} := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \kappa\}$,
and $E^{\lambda^+}_{\neq\kappa} := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \kappa\}$.

We shall say that $S \subseteq \lambda^+$ *reflects* iff the following set is stationary:

$$\mathrm{Tr}(S) := \{\gamma < \lambda^+ \mid \mathrm{cf}(\gamma) > \omega, S \cap \gamma \text{ is stationary}\}.$$

Diamond vs. GCH

Observation. For $S \subseteq \lambda^+$, $\diamondsuit(S) \Rightarrow \diamondsuit(\lambda^+) \Rightarrow 2^\lambda = \lambda^+$.

Theorem (Jensen, '74). CH $\not\Rightarrow \diamondsuit(\aleph_1)$.

Theorem (Gregory, '76). GCH $\Rightarrow \diamondsuit(\aleph_2)$. Moreover:

GCH entails $\diamondsuit(S)$ for every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$.

Theorem (Shelah, '78). For uncountable cardinal λ:

GCH entails $\diamondsuit(S)$ for every stationary $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$.

Successors of regulars

Theorem (Shelah, ‘80). For every regular uncountable cardinal, λ:

$$\mathsf{GCH} + \neg\diamondsuit(E^{\lambda^+}_{\mathsf{cf}(\lambda)}) \text{ is consistent.}$$

Thus, the possible behaviors of diamond at successors of regulars, in the presence of GCH, are well-understood.

Successors of singulars

Theorem (Shelah, ‘84). For every singular cardinal, λ, for some non-reflecting stationary set $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$:

$$\mathrm{GCH} + \neg\diamondsuit(S) \text{ is consistent.}$$

and in the other direction:

Theorem (Shelah, ‘84). For every singular cardinal, λ: (GCH and $\square^*_\lambda$) entails $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Questions

For 25-30 years, the following questions remained open:

Question 1. Could GCH be replaced with "$2^\lambda = \lambda^+$" in the above combinatorial theorems?

Question 2. To what extent can $\square^*_\lambda$ be weakened?

Question 3. Can $\square^*_\lambda$ be completely eliminated?
put differently, can GCH hold while $\diamondsuit(S)$ fails for a set $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects?

Status

Question 1 has recently been answered in the affirmative(!)

Theorem (Shelah, 2007). For uncountable cardinal, λ:

$2^\lambda = \lambda^+$ entails $\diamondsuit(S)$ for every stationary $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$.

Theorem (Zeman, 2008). For a singular cardinal, λ: ($2^\lambda = \lambda^+$ and $\square^*_\lambda$) entails $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Thus, this talk will be focused on Questions 2 and 3. In particular, we shall assume throughout that λ denotes a singular cardinal.

Reducing weak square

Weak Square

Definition (Jensen '72). $\square^*_\lambda$ asserts the existence of a sequence $\overrightarrow{\mathcal{P}} = \langle \mathcal{P}_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

1. $\mathcal{P}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{P}_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2. for every limit $\gamma < \lambda^+$, there exists a club $C_\gamma \subseteq \gamma$ satisfying:

$$C_\gamma \cap \alpha \in \mathcal{P}_\alpha \text{ for all } \alpha \in C_\gamma.$$

Remark. By Jensen, $\square^*_\lambda$ is equivalent to the existence of a special Aronszajn tree of height λ^+.

Approachability Property

Definition (Foreman-Magidor. implicit in Shelah '78). AP_λ asserts the existence of a seq. $\overrightarrow{\mathcal{P}} = \langle \mathcal{P}_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

1. $\mathcal{P}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{P}_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2. for club many $\gamma < \lambda^+$, there exists an unbounded $A_\gamma \subseteq \gamma$ satisfying:

$$A_\gamma \cap \alpha \in \mathcal{P}_\alpha \text{ for all } \alpha \in A_\gamma.$$

Stationary Approachability Property

Definition. SAP_λ asserts that for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects, there exists a seq. $\overrightarrow{\mathcal{P}_S} = \langle \mathcal{P}_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

1. $\mathcal{P}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{P}_\alpha| = \lambda$ for all $\alpha < \lambda^+$;

2. for stationarily many $\gamma \in \mathrm{Tr}(S)$, there exists a stationary $S_\gamma \subseteq S \cap \gamma$ satisfying:

$$S_\gamma \cap \alpha \in \bigcup \{\mathcal{P}(X) \mid X \in \mathcal{P}_\alpha\} \text{ for all } \alpha \in S_\gamma.$$

Answering question 2

Trivial Fact. $\square^*_\lambda \Longrightarrow \mathsf{SAP}_\lambda$.

Theorem. Suppose SAP_λ holds.
Then $2^\lambda = \lambda^+$ entails $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Theorem. It is relatively consistent with the existence of a supercompact that $\mathsf{SAP}_{\aleph_\omega}$ holds, while $\square^*_{\aleph_\omega}$ fails.

Moreover, $\mathsf{SAP}_{\aleph_\omega}$ is consistent with $\mathsf{Refl}^*([\aleph_{\omega+1}]^\omega)$.

by-product: a tree from a small forcing

In one of our failed attempts to construct a model of $\mathrm{SAP}_\lambda + \neg\square^*_\lambda$, we ended-up proving the following counterintuitive fact.

Theorem. It is relatively consistent with the existence of two supercompact cardinals that there exists a cofinality-preserving forcing of size $\aleph_3$ that introduces a special Aronszajn tree of height $\aleph_{\omega_1+1}$.

A possible rant on our solution to Q. 2

"*I do not know what SAP is, and I don't like new definitions. I know that weak square implies AP, and implies a better scale, so why don't you try to reduce the weak square hypothesis from the Shelah-Zeman theorem to these well-studied principles?*"

Fortunately, we have a satisfactory response..

In the absence of SAP

Answering question 3

Theorem (Gitik-R.). It is relatively consistent with the existence of a supercompact cardinal that the GCH holds, while $\diamondsuit(S)$ **fails** for some $S \subseteq E^{\aleph_{\omega+1}}_{\omega}$ that reflects.

Note that $\square^*_{\aleph_\omega}$ necessarily fails in our model, hence, the large cardinal hypothesis.

More on question 3

To justify the notion of SAP_λ, we also prove:

Theorem (Gitik-R.). Starting with a supercompact cardinal, we can force to get:
(1) a strong limit $\lambda > \mathrm{cf}(\lambda) = \omega$ with $2^\lambda = \lambda^+$;
(2) $\diamondsuit(S)$ **fails** for some $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects;
in conjunction with any of the following:

- $\mathsf{AP}_\lambda + \mathsf{Refl}(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$;
- a very good scale for λ;
- $\exists\kappa < \lambda$ supercompact;
- Martin's Maximum (so $S \subseteq E^{\lambda^+}_\omega$ is $(\omega_1 + 1)$-fat.)

Revisiting the weak square

Forcing axioms vs. Square, I

Magidor, extending Todorcevic, proved that PFA entails the failure of $\square_{\kappa,\omega_1}$ for all $\kappa > \omega$.

He also proved the following:

Theorem (Magidor).
(1) PFA is consistent with $\square^*_\kappa$ for all κ;
(2) MM entails that $\square^*_\kappa$ fails for all $\kappa > \mathrm{cf}(\kappa) = \omega$.

It is natural to ask whether MM can be reduced to PFA^+, in this context.

▶ It turns out that diamond helps..

Forcing axioms vs. Square, II

Theorem. Suppose:
(1) λ is a singular strong limit;
(2) $2^\lambda = \lambda^+$;
(3) $\square^*_\lambda$ holds;
(4) every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects.
then $\diamondsuit^*(\lambda^+)$ holds.

Remark. Replacing $\square^*_\lambda$ with SAP_λ in (3), does not yield the conclusion! In fact, this is the approach eventually taken to establish that SAP_λ is strictly weaker than $\square^*_\lambda$.

Corollary. Assume PFA^+.
If $\lambda > \mathrm{cf}(\lambda) = \omega$ is a strong limit, then $\square^*_\lambda$ fails.

A quick proof

Corollary. Assume PFA$^+$.
If $\lambda > \mathrm{cf}(\lambda) = \omega$ is a strong limit, then $\square^*_\lambda$ fails.

Proof. Suppose not. Force with Add$(\lambda^+, \lambda^{++})$. Then $\diamondsuit^*(\lambda^+)$ fails, while $\square^*_\lambda$ and PFA$^+$ are preserved, and λ remains a strong limit.

It follows from the previous theorem that $\diamondsuit^*(\lambda^+)$ holds. A contradiction. ■

Remark. After our lecture, J. Krueger informed us of another, already known, proof of the above corollary.

Summary: Coherence vs. Guessing

Let Refl_λ denote the assertion that every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects. Then, for λ singular, we have:

1. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \square^*_\lambda \Rightarrow \diamondsuit^*(\lambda^+)$;
2. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \mathsf{SAP}_\lambda \not\Rightarrow \diamondsuit^*(\lambda^+)$;
3. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \mathsf{SAP}_\lambda \Rightarrow \diamondsuit(S)$ for every stat. $S \subseteq \lambda^+$;
4. $\mathsf{GCH} + \mathsf{Refl}_\lambda + \mathsf{AP}_\lambda \not\Rightarrow \diamondsuit(S)$ for every stat. $S \subseteq \lambda^+$.

Remark. here, the non-implication symbol, $\not\Rightarrow$, is a slang for a consistency result modulo the existence of a supercompact cardinal.

Open problems

Open problems

Let λ denote a singular cardinal.

Question I. Does $2^\lambda = \lambda^+$ entail $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$?

equivalently:

Question II. Does $2^\lambda = \lambda^+$ entail the existence of a stationary $\mathcal{S} \subseteq [\lambda^+]^{<\lambda}$ on which $X \mapsto \sup(X)$ is an injective map from $\mathcal{S}$ to $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$?

Thank you!