The extent of the failure of Ramsey’s theorem at successor cardinals

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Introduction
Ramsey’s theorem

The square-bracket relation
Let $\lambda \to [\lambda]^2_\kappa$ denote the assertion:
For every function $f : [\lambda]^2 \to \kappa$, there exists a subset $H \subseteq \lambda$ of size $\lambda$ such that $f''[H]^2 \neq \kappa$. 
Ramsey’s theorem

The square-bracket relation

Let $\lambda \rightarrow [\lambda]^2_\kappa$ denote the assertion:
For every function $f : [\lambda]^2 \rightarrow \kappa$, there exists a subset $H \subseteq \lambda$ of size $\lambda$ such that $f"[H]^2 \neq \kappa$.

Theorem (Ramsey, 1929)

$\omega \rightarrow [\omega]^2_2$ holds.

I.e., if we partition the set of (unordered) pairs of natural numbers into two sets $A_0, A_1$, then there exists an infinite set $H$ and an index $i < 2$, for which the square satisfies $[H]^2 \subseteq A_i$. 
Ramsey’s theorem (Cont.)

Theorem (Ramsey, 1929)
\[ \omega \rightarrow [\omega]^2_2. \]

Ramsey’s theorem is very pleasing. Unfortunately, it does not generalize to higher cardinals.
Ramsey’s theorem (Cont.)

Theorem (Ramsey, 1929)
\[ \omega \to [\omega]^2_2. \]

Ramsey’s theorem is very pleasing. Unfortunately, it does not generalize to higher cardinals.

Theorem (Sierpiński, 1933)
\[ \omega_1 \not\to [\omega_1]^2_2. \]

I.e., there exists a partition \([\omega_1]^2 = A_0 \uplus A_1\), such that for every uncountable \(H \subseteq \omega_1\), we have \([H]^2 \cap A_i \neq \emptyset\) for both \(i < 2\).
Generalizing Sierpiński

Theorem (Sierpiński, 1933)

\[ \omega_1 \not\rightarrow [\omega_1]^2_2. \]

Sierpiński theorem handles partitions of the form \([\omega_1]^2 = A_0 \cup A_1\). How about partitions of the form \([\omega_1]^2 = \bigcup_{i < \omega_1} A_i\)?
Generalizing Sierpiński

Theorem (Sierpiński, 1933)
\[ \omega_1 \not\rightarrow [\omega_1]^2. \]

Sierpiński theorem handles partitions of the form \([\omega_1]^2 = A_0 \uplus A_1\).
How about partitions of the form \([\omega_1]^2 = \biguplus_{i<\omega_1} A_i\)?

Theorem (Erdős-Hajnal-Rado, 1965)

*CH entails* \(\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}\).
Generalizing Sierpiński

Theorem (Erdös-Hajnal-Rado, 1965)

$\text{CH} \text{ entails } \omega_1 \notightarrow [\omega_1]^{\omega_1}_\omega$.

Theorem (Todorčević, 1987)

$\omega_1 \notightarrow [\omega_1]^{\omega_1}_\omega$ holds in ZFC.
Generalizing Sierpiński

Theorem (Sierpiński, 1933)
\[ \omega_1 \nleftrightarrow [\omega_1]^2_2. \]

Theorem (Todorčević, 1987)
\[ \omega_1 \nleftrightarrow [\omega_1]^2_{\omega_1}. \]

- A function witnessing the failure of the square bracket relation is considered as a strong coloring.
Shelah’s study of strong colorings

Pr₀(λ, κ, θ) → Pr₁(λ, κ, θ) → Pr⁻₁(λ, κ, θ) → λ-c.c. not productive

Pr₂(λ, κ, θ) → Pr₃⁻(λ, κ, θ) → λ ↯ [λ]²ₖ

Pr₃⁺(λ, κ, θ) → ∃λ-L space

∃λ-S space
**The rectangular square-bracket relation**

**Negative square-bracket relation**

Let $\lambda \not\rightarrow [\lambda]_\kappa^2$ denote the assertion:
There exists a function $f : [\lambda]^2 \rightarrow \kappa$, such that for every subset $X \subseteq \lambda$ of size $\lambda$, we have $f\ "[X]^2 = \kappa$.

**Negative rectangular square-bracket relation**

Let $\lambda \not\rightarrow [\lambda; \lambda]_\kappa^2$ denote the assertion:
There exists a function $f : [\lambda]^2 \rightarrow \kappa$, such that for every subsets $X, Y$ of $\lambda$, each of size $\lambda$, we have $f[X \otimes Y] = \kappa$. 
The rectangular square-bracket relation (Cont.)

Theorem (Erdös-Hajnal-Rado, 1965)

$CH$ entails $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$.

Theorem (Todorčević, 1987)

$\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$ holds in ZFC.
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Theorem (Erdős-Hajnal-Rado, 1965)

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Theorem (Todorčević, 1987)

$\omega_1 \not\rightarrow [\omega_1]^{2}_{\omega_1}$ holds in ZFC.

Theorem (Moore, 2006)

$\omega_1 \not\rightarrow [\omega_1; \omega_1]^{2}_{\omega_1}$ holds in ZFC.
Theorem

TFAE for all cardinals $\lambda, \kappa$:

- $\lambda^+ \nleftrightarrow [\lambda^+]^2_\kappa$
- $\lambda^+ \nleftrightarrow [\lambda^+; \lambda^+]^2_\kappa$
Main result: comparing squares with rectangles

Theorem
TFAE for all cardinals $\lambda, \kappa$:
- $\lambda^+ \not\rightarrow [\lambda^+]^2_\kappa$
- $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_\kappa$

The above theorem was the missing link to the following corollary.

Corollary (Eisworth+Shelah+R.)
TFAE for every uncountable cardinal $\lambda$:
- $\lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+}$
- $\Pr_0(\lambda^+, \lambda^+, \omega)$

For the definition of $\Pr_0$, see appendix.
Surprise, Surprise!!

\[ \lambda = \text{successor of uncountable} \]
\[ \kappa = \lambda \text{ i.e. maximal num. of colors} \]
\[ \theta = \omega \]

\[ \lambda \text{-c.c. not productive} \]

\[ \lambda \not\in [\lambda]^2_\kappa \]

\[ \exists \lambda \text{-L space} \]

\[ \exists \lambda \text{-S space} \]

\[ \Pr_0(\lambda, \kappa, \theta) \]
\[ \Pr_2(\lambda, \kappa, \theta) \]
\[ \Pr_3(\lambda, \kappa, \theta) \]
\[ \Pr_3^L(\lambda, \kappa, \theta) \]
\[ \Pr_3^S(\lambda, \kappa, \theta) \]
Main result in two parts

Theorem

TFAE for all cardinals $\lambda, \kappa$:

- $\lambda^+ \not
\rightarrow [\lambda^+]_{\kappa}^2$
- $\lambda^+ \not
\rightarrow [\lambda^+; \lambda^+]_{\kappa}^2$

The theorem will follow from the following two ZFC results:

1. if $\lambda = \text{cf}(\lambda)$, then $\lambda^+ \not
\rightarrow [\lambda^+; \lambda^+]_{\lambda^+}^2$ holds;
Main result in two parts

Theorem

*TFAE for all cardinals* $\lambda, \kappa$:

- $\lambda^+ \not\leftrightarrow [\lambda^+]_\kappa^2$
- $\lambda^+ \not\leftrightarrow [\lambda^+; \lambda^+]_\kappa^2$

The theorem will follow from the following two ZFC results:

1. if $\lambda = \text{cf}(\lambda)$, then $\lambda^+ \not\leftrightarrow [\lambda^+]_{\lambda^+}^2$ holds;

2. if $\lambda > \text{cf}(\lambda)$, then there exists a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ such that for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \otimes Y] \supseteq Z \otimes Z$. 
Successors of regulars
Successors of regulars — in ZFC

Let $\lambda$ denote a regular cardinal. Then:

1. (Todorčević, 1987) $\lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+}$ [Partitioning pairs of countable ordinals]

Corollary (Shelah+Moore) $\lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+}$ holds for every regular cardinal $\lambda$. 

Remark In a recent joint work with Todorčević, we found a uniform proof of the above 3 + 4 + 5.
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2. (Shelah, 1987) $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_\lambda$, if $\lambda > 2^{\aleph_0}$ [Sh:280]

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4. (Shelah, 1996) \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\lambda^+} \), if \( \lambda = \aleph_1 \) [Sh:572]

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5. (Moore, 2006) $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$, if $\lambda = \aleph_0$ [A solution to the L space problem]
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Corollary (Shelah+Moore)

$\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$ holds for every regular cardinal $\lambda$. 
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Corollary (Shelah+Moore)

$\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^{2}_{\lambda^+}$ holds for every regular cardinal $\lambda$.

Remark

In a recent joint work with Todorčević, we found a uniform proof of the above 3 + 4 + 5.
Successors of singulars

The Anatomy of a Latte

- Coffee
- Bullshit
Successor of singulars — in ZFC

**Theorem (Shelah, 1990’s)**

\[ \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\text{cf}(\lambda)} \] holds for every singular cardinal \( \lambda \).

**Theorem (Eisworth, 2010)**

If \( \lambda \) is a singular cardinal of countable cofinality, then \( E^{\lambda^+} \) carries a club-guessing matrix of a very strong form.

**Still Open**

Whether \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2 \) hold for all singular \( \lambda \), in ZFC.
Successor of singulars — in ZFC

Theorem (Shelah, 1990’s)
\[
\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^{2}_{\text{cf}(\lambda)}
\] holds for every singular cardinal \( \lambda \).

Theorem (Shelah, 1990’s)
If \( \lambda \) is a singular cardinal of uncountable cofinality, then \( E^{\lambda^+}_{\text{cf}(\lambda)} \) carries a club-guessing sequence of a very strong form.

Theorem (Eisworth, 2010)
If \( \lambda \) is a singular cardinal of countable cofinality, then \( E^{\lambda^+}_{\omega_1} \) carries a club-guessing matrix of a very strong form.
Successor of singulars — in ZFC

Theorem (Shelah, 1990’s)
\( \lambda^+ \not
\rightarrow [\lambda^+;\lambda^+]_\text{cf}(\lambda)^2 \) holds for every singular cardinal \( \lambda \).

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If \( \lambda \) is a singular cardinal of uncountable cofinality, then \( E^{\lambda^+}_{\text{cf}(\lambda)} \) carries a club-guessing sequence of a very strong form.

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If \( \lambda \) is a singular cardinal of countable cofinality, then \( E^{\lambda^+}_{\omega_1} \) carries a club-guessing matrix of a very strong form.

Still Open
Whether \( \lambda^+ \not\rightarrow [\lambda^+]_\lambda^+ \) hold for all singular \( \lambda \), in ZFC.
Transforming Rectangles into Squares — in ZFC

Main technical result
For every singular cardinal $\lambda$, there exists a function $rts : [\lambda^+]^2 \rightarrow [\lambda^+]^2$ such that for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \odot Y] \supseteq Z \odot Z$.

Remark: our proof builds heavily on previous arguments of Shelah, Todorčević, and most notably — Eisworth.
Transforming Rectangles into Squares — in ZFC

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The definition of $rts$

- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle$ that incorporates a club-guessing sequence/matrix.
Transforming Rectangles into Squares — in ZFC

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For every singular cardinal $\lambda$, there exists a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ such that for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \odot Y] \supseteq Z \odot Z$.

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- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle$ that incorporates a club-guessing sequence/matrix.
- Adapt Shelah’s proof of $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_{\text{cf}(\lambda)}$, to get a function $f : [\lambda^+]^2 \to <\omega \text{cf}(\lambda) \times <\omega \text{cf}(\lambda)$ with strong properties.
Transforming Rectangles into Squares — in ZFC

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For every singular cardinal $\lambda$, there exists a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ such that for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \diamond Y] \supseteq Z \diamond Z$.

Remark: our proof builds heavily on previous arguments of Shelah, Todorčević, and most notably — Eisworth.

The definition of $rts$

- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle$ that incorporates a club-guessing sequence/matrix.
- Adapt Shelah’s proof of $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]_{\text{cf}(\lambda)}^2$, to get a function $f : [\lambda^+]^2 \to \omega \times \omega$ with strong properties.
- Given $\alpha < \beta < \lambda^+$, consider $(\sigma, \eta) = f(\alpha, \beta)$;
Transforming Rectangles into Squares — in ZFC

Main technical result
For every singular cardinal \( \lambda \), there exists a function \( rts : [\lambda^+]^2 \to [\lambda^+]^2 \) such that for every cofinal subsets \( X, Y \) of \( \lambda^+ \), there exists a cofinal subset \( Z \subseteq \lambda^+ \) such that \( rts[X \odot Y] \supseteq Z \odot Z \).

Remark: our proof builds heavily on previous arguments of Shelah, Todorčević, and most notably — Eisworth.

The definition of \( rts \)

- Fix a matrix of local clubs \( \langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix.
- Adapt Shelah’s proof of \( \lambda^+ \not\rightarrow [\lambda^+]^2 \text{cf}(\lambda) \), to get a function \( f : [\lambda^+]^2 \to <\omega \text{cf}(\lambda) \times <\omega \text{cf}(\lambda) \) with strong properties.
- Given \( \alpha < \beta < \lambda^+ \), consider \((\sigma, \eta) = f(\alpha, \beta)\);
- Let \( \beta_0 := \beta \), and \( \beta_{n+1} := \min(C^{\sigma(n)}_{\beta_n} \setminus \alpha) \) for all \( n \in \text{dom}(\sigma) \);
Transforming Rectangles into Squares (Cont.)

The definition of \( rts \)

- Fix a matrix of local clubs \( \langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix;
- Fix a function \( f : [\lambda^+]^2 \rightarrow <\omega \text{ cf}(\lambda) \times <\omega \text{ cf}(\lambda) \) with strong coloring properties;
- Given \( \alpha < \beta < \lambda^+ \), consider \( (\sigma, \eta) = f(\alpha, \beta) \);
- Let \( \beta_0 := \beta \), and \( \beta_{n+1} := \min(\langle C^\sigma_{\beta_n} \rangle \setminus \alpha) \) for all \( n \in \text{dom}(\sigma) \);
The definition of $rts$

- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle$ that incorporates a club-guessing sequence/matrix;
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- Let $\beta_0 := \beta$, and $\beta_{n+1} := \min(C^{\sigma(n)}_{\beta_n} \setminus \alpha)$ for all $n \in \text{dom}(\sigma)$;
- Let $\gamma := \max\{\sup(C^{\sigma(n)}_{\beta_n} \cap \alpha) \mid n \in \text{dom}(\sigma)\}$.
The definition of \textit{rts}

\begin{itemize}
\item Fix a matrix of local clubs \( \langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix;
\item Fix a function \( f : [\lambda^+]^2 \rightarrow \omega \times \omega \) with strong coloring properties;
\item Given \( \alpha < \beta < \lambda^+ \), consider \( (\sigma, \eta) = f(\alpha, \beta) \);
\item Let \( \beta_0 := \beta \), and \( \beta_{n+1} := \min(C^{\sigma(n)}_{\beta_n} \setminus \alpha) \) for all \( n \in \text{dom}(\sigma) \);
\item Let \( \gamma := \max\{\sup(C^{\sigma(n)}_{\beta_n} \cap \alpha) \mid n \in \text{dom}(\sigma)\} \);
\item Let \( \alpha_0 := \alpha \), and \( \alpha_{m+1} := \min(C^{\eta(m)}_{\alpha_m} \setminus \gamma + 1) \) for \( m \in \text{dom}(\eta) \).
\end{itemize}
The definition of \( rts \)

- Fix a matrix of local clubs \( \langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix;
- Fix a function \( f : [\lambda^+]^2 \rightarrow <\omega \text{ cf}(\lambda) \times <\omega \text{ cf}(\lambda) \) with strong coloring properties;
- Given \( \alpha < \beta < \lambda^+ \), consider \( (\sigma, \eta) = f(\alpha, \beta) \);
- Let \( \beta_0 := \beta \), and \( \beta_{n+1} := \min(C_{\beta_n}^{\sigma(n)} \setminus \alpha) \) for all \( n \in \text{dom}(\sigma) \);
- Let \( \gamma := \max\{\sup(C_{\beta_n}^{\sigma(n)} \cap \alpha) \mid n \in \text{dom}(\sigma)\} \);
- Let \( \alpha_0 := \alpha \), and \( \alpha_{m+1} := \min(C_{\alpha_m}^{\eta(m)} \setminus \gamma + 1) \) for \( m \in \text{dom}(\eta) \);
- Put \( rts(\alpha, \beta) := (\alpha_{\text{dom}(\eta)}, \beta_{\text{dom}(\sigma)}) \).
The definition of $rts$

- Fix a matrix of local clubs $\langle C_{\alpha}^i \mid \alpha < \lambda^+, i < cf(\lambda) \rangle$ that incorporates a club-guessing sequence/matrix;
- Fix a function $f : [\lambda^+]^2 \rightarrow \omega \times \omega$ with strong coloring properties;
- Given $\alpha < \beta < \lambda^+$, consider $(\sigma, \eta) = f(\alpha, \beta)$;
- Let $\beta_0 := \beta$, and $\beta_{n+1} := \min(C_{\beta_n}^{\sigma(n)} \setminus \alpha)$ for all $n \in \text{dom}(\sigma)$;
- Let $\gamma := \max\{\sup(C_{\beta_n}^{\sigma(n)} \cap \alpha) \mid n \in \text{dom}(\sigma)\}$;
- Let $\alpha_0 := \alpha$, and $\alpha_{m+1} := \min(C_{\alpha_m}^{\eta(m)} \setminus \gamma + 1)$ for $m \in \text{dom}(\eta)$;
- Put $rts(\alpha, \beta) := (\alpha_{\text{dom}(\eta)}, \beta_{\text{dom}(\sigma)})$.

The definition of $rts$ is quite natural in this context, and so the main point is to verify that the definition does the job.
Why does \textit{rts} work

- For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type $p$ in the language of the matrix-based walks, let $X_p(\delta) := \{ \alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p \}$;
Why does \textit{rts} work

For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type $p$ in the language of the matrix-based walks, let

$$X_p(\delta) := \{ \alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p \};$$

Denote $S_p^X := \{ \delta < \lambda^+ \mid \sup(X_p(\delta)) = \sup(X) \};$
Why does \textit{rts} work

- For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type $p$ in the language of the matrix-based walks, let $X_p(\delta) := \{ \alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p \}$;
- Denote $S_p^X := \{ \delta < \lambda^+ \mid \sup(X_p(\delta)) = \sup(X) \}$;
- Use the fact that the chosen matrix incorporates club guessing to argue that for every cofinal subsets of $\lambda^+$, $X$ and $Y$, there exists a type $p$, for which $S_p^X \cap S_p^Y$ is stationary;
Why does $rts$ work

For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type $p$ in the language of the matrix-based walks, let $X_p(\delta) := \{ \alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p \}$;

Denote $S^X_p := \{ \delta < \lambda^+ \mid \sup(X_p(\delta)) = \sup(X) \}$;

Use the fact that the chosen matrix incorporates club guessing to argue that for every cofinal subsets of $\lambda^+$, $X$ and $Y$, there exists a type $p$, for which $S^X_p \cap S^Y_p$ is stationary;

Use the fact that $f$ oscillates quite nicely on rectangles $X \ast Y$, so that it can produce sequences $(\sigma, \eta)$ with successful guidelines on which columns of the matrix to advise throughout the walks, and at which step of the walks to stop. This insures that the type $p$ gets realized quite frequently;
Why does $rts$ work

- For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type $p$ in the language of the matrix-based walks, let $X_p(\delta) := \{\alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p\}$;
- Denote $S^X_p := \{\delta < \lambda^+ \mid \sup(X_p(\delta)) = \sup(X)\}$;
- Use the fact that the chosen matrix incorporates club guessing to argue that for every cofinal subsets of $\lambda^+$, $X$ and $Y$, there exists a type $p$, for which $S^X_p \cap S^Y_p$ is stationary;
- Use the fact that $f$ oscillates quite nicely on rectangles $X \otimes Y$, so that it can produce sequences $(\sigma, \eta)$ with successful guidelines on which columns of the matrix to advise throughout the walks, and at which step of the walks to stop. This insures that the type $p$ gets realized quite frequently;
- Conclude that $rts[X \otimes Y] \supseteq [S^X_p \cap S^Y_p \cap C]^2$ for the club $C$ of ordinals of the form $M \cap \lambda^+$, for elementary submodels $M \prec H_\chi$ of size $\lambda$, that contains all relevant objects.
Thank you!

The slides of this talk may be found at the following address: http://papers.assafrinot.com/?talk=cms2011
Appendix

Definition (Shelah)

Pr$_0(\lambda, \lambda, \omega)$ asserts the existence of a function $f : [\lambda]^2 \to \lambda$ satisfying the following.
For every $n < \omega$, every $g : n \times n \to \lambda$, and every collection $A \subseteq [\lambda]^n$ of mutually disjoint sets, of size $\lambda$, there exists some $x, y \in A$ with $\max(x) < \min(y)$ such that

$$f(x(i), y(j)) = g(i, j) \text{ for all } i, j < n.$$