The extent of the failure of Ramsey's theorem at successor cardinals

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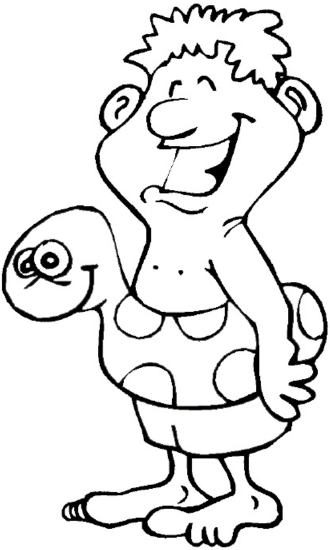
Introduction



Ramsey's theorem

The square-bracket relation

Let $\lambda \to [\lambda]^2_\kappa$ denote the assertion:
For every function $f : [\lambda]^2 \to \kappa$, there exists a subset $H \subseteq \lambda$ of size λ such that $f``[H]^2 \neq \kappa$.

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Theorem (Ramsey, 1929)

$\omega \to [\omega]^2_2$ *holds.*

I.e., if we partition the set of (unordered) pairs of natural numbers into two sets A_0, A_1, then there exists an infinite set H and an index $i < 2$, for which the square satisfies $[H]^2 \subseteq A_i$.

Ramsey's theorem (Cont.)

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Theorem (Sierpiński, 1933)

$\omega_1 \not\to [\omega_1]^2_2$.

I.e., there exists a partition $[\omega_1]^2 = A_0 \uplus A_1$, such that for every uncountable $H \subseteq \omega_1$, we have $[H]^2 \cap A_i \neq \emptyset$ for both $i < 2$.

Generalizing Sierpiński

Theorem (Sierpiński, 1933)

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Sierpiński theorem handles partitions of the form $[\omega_1]^2 = A_0 \uplus A_1$.
How about partitions of the form $[\omega_1]^2 = \biguplus_{i<\omega_1} A_i$?

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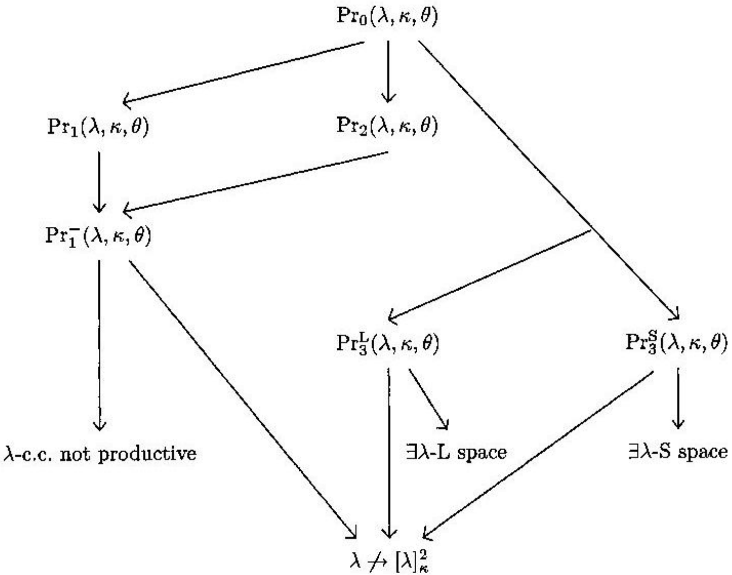
- A function witnessing the failure of the square bracket relation is considered as a strong coloring.

Shelah's study of strong colorings



The rectangular square-bracket relation

Negative square-bracket relation

Let $\lambda \not\to [\lambda]^2_\kappa$ denote the assertion:
There exists a function $f : [\lambda]^2 \to \kappa$, such that for every subset $X \subseteq \lambda$ of size λ, we have $f``[X]^2 = \kappa$.

Negative rectangular square-bracket relation

Let $\lambda \not\to [\lambda; \lambda]^2_\kappa$ denote the assertion:
There exists a function $f : [\lambda]^2 \to \kappa$, such that for every subsets X, Y of λ, each of size λ, we have $f[X \circledast Y] = \kappa$.

The rectangular square-bracket relation (Cont.)

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Main result: comparing squares with rectangles

Theorem

TFAE for all cardinals λ, κ:

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The above theorem was the missing link to the following corollary.

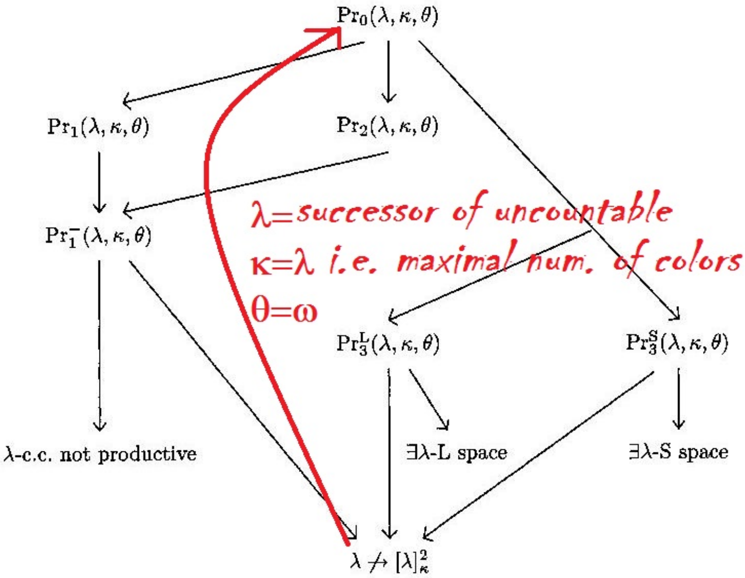
Corollary (Eisworth+Shelah+R.)

TFAE for every uncountable cardinal λ:

- $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$
- $\mathrm{Pr}_0(\lambda^+, \lambda^+, \omega)$

For the definition of Pr_0, see appendix.

Surprise, Surprise!!



Main result in two parts

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The theorem will follow from the following two ZFC results:

1. if $\lambda = \mathrm{cf}(\lambda)$, then $\lambda^+ \not\to [\lambda^+; \lambda^+]^2_{\lambda^+}$ holds;

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2. if $\lambda > \mathrm{cf}(\lambda)$, then there exists a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ such that for every cofinal subsets X, Y of λ^+, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \circledast Y] \supseteq Z \circledast Z$.

Successors of regulars

Successors of regulars — in ZFC

Let λ denote a regular cardinal. Then:

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Corollary (Shelah+Moore)

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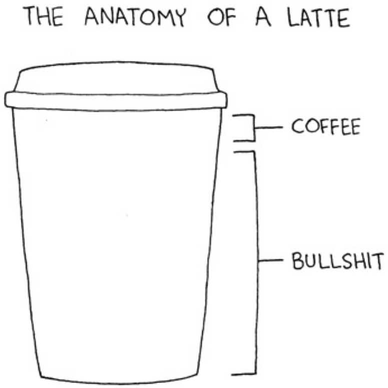
Remark

In a recent joint work with Todorčević, we found a uniform proof of the above $3+4+5$.

Successors of singulars



Successor of singulars — in ZFC

Theorem (Shelah, 1990's)

$\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\mathsf{cf}(\lambda)}$ *holds for every singular cardinal* λ.

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Theorem (Shelah, 1990's)

If λ *is a singular cardinal of uncountable cofinality, then* $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *carries a club-guessing sequence of a very strong form.*

Theorem (Eisworth, 2010)

If λ *is a singular cardinal of countable cofinality, then* $E^{\lambda^+}_{\omega_1}$ *carries a club-guessing <u>matrix</u> of a very strong form.*

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Still Open

Whether $\lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+}$ hold for all singular λ, in ZFC.

Transforming Rectangles into Squares — in ZFC

Main technical result

For every singular cardinal λ, there exists a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ such that for every cofinal subsets X, Y of λ^+, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \circledast Y] \supseteq Z \circledast Z$.

Remark: our proof builds heavily on previous arguments of Shelah, Todorčević, and most notably — Eisworth.

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- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \mathrm{cf}(\lambda)\rangle$ that incorporates a club-guessing sequence/matrix.

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- Adapt Shelah's proof of $\lambda^+ \not\to [\lambda^+; \lambda^+]^2_{\mathrm{cf}(\lambda)}$, to get a function $f : [\lambda^+]^2 \to {}^{<\omega}\mathrm{cf}(\lambda) \times {}^{<\omega}\mathrm{cf}(\lambda)$ with strong properties.

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- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \mathrm{cf}(\lambda)\rangle$ that incorporates a club-guessing sequence/matrix;
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The definition of *rts* is quite natural in this context, and so the main point is to verify that the definition does the job.

Why does *rts* work

- For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type p in the language of the matrix-based walks, let $X_p(\delta) := \{\alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p\}$;

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- Use the fact that f oscillates quite nicely on rectangles $X \circledast Y$, so that it can produce sequences (σ, η) with successful guidelines on which columns of the matrix to advise throughout the walks, and at which step of the walks to stop. This insures that the type p gets realized quite frequently;

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- Conclude that $rts[X \circledast Y] \supseteq [S_p^X \cap S_p^Y \cap C]^2$ for the club C of ordinals of the form $M \cap \lambda^+$, for elementary submodels $M \prec H_\chi$ of size λ, that contains all relevant objects.

Thank you!

The slides of this talk may be found at the following address:
http://papers.assafrinot.com/?talk=cms2011

Appendix

Definition (Shelah)

$\mathrm{Pr}_0(\lambda, \lambda, \omega)$ asserts the existence of a function $f : [\lambda]^2 \to \lambda$ satisfying the following.
For every $n < \omega$, every $g : n \times n \to \lambda$, and every collection $\mathcal{A} \subseteq [\lambda]^n$ of mutually disjoint sets, of size λ,
there exists some $x, y \in A$ with $\max(x) < \min(y)$ such that

$$f(x(i), y(j)) = g(i, j) \text{ for all } i, j < n.$$