The $\aleph_2$-Souslin problem

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Conventions

Throughout, $\kappa$ denotes a regular uncountable cardinal, and $\lambda$ denotes an uncountable cardinal.
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For a set of ordinals \( C \), write:

- \( \text{acc}(C) := \{ \alpha < \sup(C) \mid \sup(C \cap \alpha) = \alpha > 0 \} \); 
- \( \text{nacc}(C) := C \setminus \text{acc}(C) \).
Trees

Definition

- A tree is a poset \((T, \triangleleft)\) in which \(x_\downarrow := \{y \in T \mid y \triangleleft x\}\) is well-ordered for all \(x \in T\);
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- **A tree** is a poset $(T, \triangleleft)$ in which $x_\downarrow := \{ y \in T \mid y \triangleleft x \}$ is well-ordered for all $x \in T$;
- The **height** of $x \in T$ is $ht(x) := \text{otp}(x_\downarrow, \triangleleft)$;
- The **height** of $T$ is $\sup\{ht(x) \mid x \in T\}$;
- $T_\delta = \{x \in T \mid ht(x) = \delta\}$ is the $\delta^{th}$-level of $T$. 
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- \(T_\delta = \{x \in T \mid ht(x) = \delta\}\) is the \(\delta^{th}\)-level of \(T\).
- \((T, \triangleleft)\) is \(\chi\)-complete if any \(\triangleleft\)-increasing sequence of length \(< \chi\) admits a bound.
**Definition**

A $\kappa$-tree is a tree $(T, \triangleleft)$ of height $\kappa$ whose levels are of size $< \kappa$. 

- **Aronszajn**: if it has no chains of size $\kappa$;
- **Souslin**: if it has no chains or antichains of size $\kappa$;
- **Special**: if there exists a function $f : T \to T$ such that:
  1. $f(t) \triangleleft t$ for all non-minimal nodes $t$ in $T$;
  2. for all $t \in T$, $f^{-1}\{t\}$ is the union of $< \kappa$-many antichains.
\( \kappa \)-trees

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What are we doing here?

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First, let us recall some equiconsistency results.
Equiconsistency results

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A cardinal $\kappa$ is **Mahlo** if $\{\alpha < \kappa \mid \text{cf}(\alpha) = \alpha\}$ is stationary in $\kappa$. 
Equiconsistency results

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Fact
The following are equiconsistent:

- There exists a Mahlo cardinal;
- There are no special $\aleph_2$-Aronszajn trees;
- $\Box_{\omega_1}$ fails;
- Every stationary subset of $E_{\omega_2}$ reflects;
- FRP($\omega_2$) holds.
Equiconsistency results

Definition
\( \kappa \) is **weakly compact** if it is inaccessible and \( \neg \exists \kappa \)-Aronszajn trees.

Recall (Hanf, 1964)
If \( \kappa \) is weakly compact, then \( \{ \alpha < \kappa \mid \alpha \text{ is Mahlo} \} \) is stationary.
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The following are equiconsistent:
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- \( \Box (\omega_2) \) fails;
- Every stationary subset of \( E_\omega^{\omega_2} \) reflects;
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- Every pair of stationary subsets of \( E_{\omega_2} \) reflect simultaneously;
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- \( \square(\omega_2) \) fails;
- Every pair of stationary subsets of \( E_\omega^{\omega_2} \) reflect simultaneously;
- Every stationary subset of \( [\omega_2]^\omega \) reflects.
Equiconsistency results

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κ is weakly compact if it is inaccessible and ¬∃κ-Aronszajn trees.

Fact
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- There exists a weakly compact cardinal;
- There are no ℵ_2-Aronszajn trees;
- □(ω_2) fails;
- Every pair of stationary subsets of E_ω^ω_2 reflect simultaneously;
- Every stationary subset of [ω_2]ω reflects;
- For some regular cardinal κ ≥ ω_2, κ-cc × κ-cc = κ-cc.
The $\aleph_2$-Souslin problem
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In the 1970’s, Jensen proved that the existence of an $\aleph_1$-Souslin tree is independent of GCH.

Open problem
Does GCH entail the existence of an $\aleph_2$-Souslin tree?
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Theorem (Gregory, 1976)
If GCH holds, and there exist no $\aleph_2$-Souslin trees, then $\aleph_2$ is a Mahlo cardinal in $L$. 
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Given the above-mentioned equiconsistency results, the general belief is that Gregory’s lower bound should be increased from Mahlo to a weakly compact. Also, add to it the following:
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Theorem (Jensen, 1972)
If $V = L$, then for every regular uncountable cardinal $\kappa$, TFAE:

- $\kappa$ is not weakly compact;
- There exists a $\kappa$-Aronszajn tree;
- There exists a $\kappa$-Souslin tree.
The $\aleph_2$-Souslin problem

From the Kanamori-Magidor 1978 survey article (p. 261):

The consistency problem for $\text{SH}_\kappa$ when $\kappa > \omega_1$ seems to be much more difficult, especially if we want to retain the GCH. To bring matters into focus, we make some remarks which recall and amplify §21. First of all, Jensen[1972] had actually established that in L, weak compactness for $\kappa$ is equivalent to $\text{SH}_\kappa$, for regular $\kappa$. We are interested in $\text{SH}_\kappa$ for small $\kappa$, and the Mitchell-Silver model cited in §21 certainly satisfied $\text{SH}_{\omega_2}$, as there were not even any $\omega_2$-Aronszajn trees in that model. However, $2^\omega = \omega_2$ held in that model, and in fact a classical result of Specker[1951] as cited in §5 necessitates something like this: if $2^\omega = \omega_1$, then there is an $\omega_2$-Aronszajn tree. No such result seems available for $\omega_2$-Souslin trees, so the focal problem in this area is to get $\text{SH}_{\omega_2}$ and the GCH to hold.

This problem has been extensively investigated by Gregory[1976] who established in particular that: if $2^\omega = \omega_1$, $2^\omega = \omega_2$, and $\text{E}^\omega_{\omega_2}$ hold, then $\text{SH}_{\omega_2}$ is false, i.e. there is an $\omega_2$-Souslin tree. Hence, if we want $\text{SH}_{\omega_2}$ and the GCH to hold, we need to guarantee the failure of $\text{E}^\omega_{\omega_2}$. As pointed out in §21, this necessitates at least the consistency strength of the existence of a Mahlo cardinal, and very likely, of a weakly compact cardinal.
The $\aleph_2$-Souslin problem

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Theorem (Gregory, 1976)
*If GCH holds, and there exist no $\aleph_2$-Souslin trees, then $\aleph_2$ is a Mahlo cardinal in $L$.**
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Does GCH entail the existence of an $\aleph_2$-Souslin tree?

Theorem (Gregory, 1976)
If GCH holds, and there exist no $\aleph_2$-Souslin trees, then $\aleph_2$ is a Mahlo cardinal in $L$.

Theorem (2016)
If GCH holds, and there exist no $\aleph_2$-Souslin trees, then $\aleph_2$ is a weakly compact cardinal in $L$. 
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Whether GCH entails the existence of an $\aleph_2$-Souslin tree remains open, however, the trees we get here are of a particular kind:

**Theorem (2016)**

If GCH holds and $\aleph_2$ is not weakly compact in $L$, then there exists an $\aleph_2$-Souslin tree with no $\aleph_1$-Aronszajn subtrees.
Whether GCH entails the existence of an $\aleph_2$-Souslin tree remains open, however, the trees we get here are of a particular kind:

**Theorem (2016)**

If GCH holds and $\aleph_2$ is not weakly compact in $L$, then there exists an $\aleph_2$-Souslin tree with no $\aleph_1$-Aronszajn subtrees.

**Theorem (Todorcevic, 1981)**

*After Lévy-collapsing a weakly compact cardinal to $\aleph_2$ over a model of GCH: GCH holds, and every $\aleph_2$-Aronszajn tree contains an $\aleph_1$-Aronszajn subtree.*
Stating the results

For almost two years now, Ari Brodsky and myself have been studying a parameterized proxy principle, denoted $P(\kappa, \mu, R, \theta, S, \nu, \sigma, E)$, and its effect on the existence of different types of $\kappa$-Souslin trees.
Stating the results

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**Theorem (Brodsky-Rinot, 2015)**

\( ♠^-(κ) + ♦(κ) \) entails a club-regressive \( κ \)-Souslin tree.
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Simpler instances of the principle were isolated, yielding the following simple statements:

**Theorem (Brodsky-Rinot, 2015)**

\[ \Box^-(\kappa) + \Diamond(\kappa) \text{ entails a club-regressive } \kappa\text{-Souslin tree.} \]

**Remark**

A club-regressive \( \kappa \)-tree contains no \( \nu \)-Aronszajn subtrees nor \( \nu \)-Cantor subtrees for every regular cardinal \( \nu < \kappa \).
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\[ \Box^- (\kappa) + \Diamond (\kappa) \text{ entails a club-regressive } \kappa\text{-Souslin tree.} \]

**Theorem (Brodsky-Rinot, 2015)**

\[ \Box'(E_{\geq \chi}^\kappa) + \Diamond (\kappa) \text{ entails a } \chi\text{-complete } \kappa\text{-Souslin tree, provided that } \kappa \text{ is } \chi\text{-inaccessible.} \]
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**Theorem (Brodsky-Rinot, 2015)**

\[ \square^{\neg} (\kappa) \land \lozenge (\kappa) \text{ entails a club-regressive } \kappa\text{-Souslin tree.} \]

**Theorem (Brodsky-Rinot, 2015)**

\[ \square' (E^{\kappa}_{\geq \chi}) \land \lozenge (\kappa) \text{ entails a } \chi\text{-complete } \kappa\text{-Souslin tree, provided that } \kappa \text{ is } \chi\text{-inaccessible.} \]

**Remark**

The classic way to obtain \( \chi\)-completeness is to move from \( \lozenge (\kappa) \) to \( \lozenge (E^{\kappa}_{\geq \chi}) \). Unfortunately, \( \lozenge (\kappa) \) is consistent with the failure of \( \lozenge (E^{\kappa}_{\geq \chi}) \):

**Theorem (Shelah, 1980)**

\[ \text{GCH} + \lozenge (\omega_2) + \neg \lozenge (E^{\omega_2}_{\omega_1}) \text{ is consistent.} \]
Stating the results

Simpler instances of the principle were isolated, yielding the following simple statements:

Theorem (Brodsky-Rinot, 2015)
$\blacklozenge^-(\kappa) + \lozenge(\kappa)$ entails a club-regressive $\kappa$-Souslin tree.

Theorem (Brodsky-Rinot, 2015)
$\blacklozenge'(E^{\kappa}_{\geq \chi}) + \lozenge(\kappa)$ entails a $\chi$-complete $\kappa$-Souslin tree, provided that $\kappa$ is $\chi$-inaccessible.

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\[ \Box^-(\kappa) + \Diamond(\kappa) \text{ entails a club-regressive } \kappa \text{-Souslin tree.} \]

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Remark
All previous \( \Diamond \)-based constructions of \( \kappa \)-Souslin trees involved sealing antichains at levels \( \alpha \in S \) for some stationary \( S \) that does not reflect.
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Simpler instances of the principle were isolated, yielding the following simple statements:

Theorem (Brodsky-Rinot, 2015)
\(\bigstar^- (\kappa) + \lozenge (\kappa)\) entails a club-regressive \(\kappa\)-Souslin tree.

Theorem (Brodsky-Rinot, 2015)
\(\bigstar' (E_{\kappa}^{\kappa}) + \lozenge (\kappa)\) entails a \(\chi\)-complete \(\kappa\)-Souslin tree, provided that \(\kappa\) is \(\chi\)-inaccessible.

Remark
All previous \(\lozenge\)-based constructions of \(\kappa\)-Souslin trees involved sealing antichains at levels \(\alpha \in S\) for some stationary \(S\) that does not reflect.

In contrast, Lambie-Hanson proved that \(\bigstar^- (\aleph_{\omega + 1}) + \lozenge (\aleph_{\omega + 1})\) is consistent with the reflection of all stationary subsets of \(\aleph_{\omega + 1}\).
Stating the results

Theorem (Brodsky-Rinot, 2015)
\[\Box^-(\kappa) + \lozenge(\kappa) \text{ entails a club-regressive } \kappa\text{-Souslin tree.}\]

Theorem (Brodsky-Rinot, 2015)
\[\Box'(E_{\geq \chi}^\kappa) + \lozenge(\kappa) \text{ entails a } \chi\text{-complete } \kappa\text{-Souslin tree, provided that } \kappa \text{ is } \chi\text{-inaccessible.}\]

Theorem (2016)
\[\Box(\lambda^+) + \text{GCH } \textit{entails } \Box^-(\lambda^+);\]
Stating the results

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\[ \Box(\lambda^+) + \text{GCH } \text{ entails } \Box^-(\lambda^+); \]

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Theorem (2016)
\(\square(\lambda^+) + \text{GCH}\) entails \(\Box^-(\lambda^+)\);
\(\square(\lambda^+) + \text{GCH}\) entails \(\Box'(E_{\text{cf}(\lambda)}^{\lambda^+})\).

Corollary
\(\square(\lambda^+) + \text{GCH}\) entails a club-regressive \(\lambda^+\)-Souslin tree;
Stating the results

Theorem (Brodsky-Rinot, 2015)
\[ \blacklozenge^-(\kappa) + \Diamond(\kappa) \text{ entails a club-regressive } \kappa\text{-Souslin tree.} \]

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Corollary
\[ \square(\lambda^+) + \text{GCH entails a club-regressive } \lambda^+\text{-Souslin tree;} \]
\[ \square(\lambda^+) + \text{GCH entails a cf}(\lambda)\text{-complete } \lambda^+\text{-Souslin tree.} \]
Elements of the proofs
A $C$-sequence is a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

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A **C-sequence** is a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

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A coherent C-sequence is a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- For every limit $\alpha < \kappa$, $C_\alpha$ is a club in $\alpha$;
- if $\bar{\alpha} \in \text{acc}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
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Easiest way? Take a club $D$ in $\kappa$, and put:

$$C_\alpha := \begin{cases} D \cap \alpha, & \text{if } \sup(D \cap \alpha) = \alpha; \\ \alpha \setminus \sup(D \cap \alpha), & \text{if } \sup(D \cap \alpha) < \alpha. \end{cases}$$
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**Definition (Todorcevic, 1987)**

$\square(\kappa)$ asserts the existence of a coherent $C$-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for every club $D \subseteq \kappa$, there exists some $\alpha \in \text{acc}(D)$ satisfying $C_\alpha \neq D \cap \alpha$. 
Square principles (cont.)

Definition (Brodsky-Rinot, 2015)

For a stationary $S \subseteq \kappa$, $\square^-(S)$ asserts the existence of a coherent $C$-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for every cofinal $A \subseteq \kappa$, there exists some limit $\alpha \in S$ satisfying $\sup(\text{nacc}(C_\alpha) \cap A) = \alpha$. 
Square principles (cont.)

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For a stationary $S \subseteq \kappa$, $\square^-(S)$ asserts the existence of a coherent $C$-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for every cofinal $A \subseteq \kappa$, there exists some limit $\alpha \in S$ satisfying $\sup(\text{nacc}(C_\alpha) \cap A) = \alpha$.

Observation: $\square^-(\kappa) \implies \square(\kappa)$
Given a club $D \subseteq \kappa$, put $A := \text{acc}(D)$.
Pick a limit $\alpha < \kappa$ such that $\sup(\text{nacc}(C_\alpha) \cap A) = \alpha$. In particular, $\alpha \in \text{acc}(D)$, and $\sup(\text{nacc}(C_\alpha) \cap \text{acc}(D)) = \alpha$ so that $C_\alpha \neq D \cap \alpha$. 
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Given a club $D \subseteq \kappa$, put $A := \text{acc}(D)$. Pick a limit $\alpha < \kappa$ such that $\sup(\text{nacc}(C_{\alpha}) \cap A) = \alpha$. In particular, $\alpha \in \text{acc}(D)$, and $\sup(\text{nacc}(C_{\alpha}) \cap \text{acc}(D))) = \alpha$ so that $C_{\alpha} \neq D \cap \alpha$.

Remark
The standard way to force $\square(\kappa)$ is via the poset of all coherent $C$-sequences of successor length $< \kappa$ (ordered by end-extension).
Square principles (cont.)

Definition (Brodsky-Rinot, 2015)

For a stationary $S \subseteq \kappa$, $\square^-(S)$ asserts the existence of a coherent $C$-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for every cofinal $A \subseteq \kappa$, there exists some limit $\alpha \in S$ satisfying $\sup(\text{nacc}(C_\alpha) \cap A) = \alpha$.

Observation: $\square^-(\kappa) \implies \square(\kappa)$

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Remark

The standard way to force $\square(\kappa)$ is via the poset of all coherent $C$-sequences of successor length $< \kappa$ (ordered by end-extension). The generic for this poset is in fact a $\square^-(\kappa)$-sequence!
Square principles (cont.)

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Question

Does $\square(\kappa) \implies \square^-(\kappa)$?

$(V = L$ entails an affirmative answer)
$J[\kappa]$: A new normal ideal over $\kappa$

$S \in \mathcal{P}(\kappa)$ is in $J[\kappa]$ iff there exists a club $C \subseteq \kappa$ and a sequence of functions $\langle f_i : \kappa \to \kappa \mid i < \kappa \rangle$ satisfying the following. For every $\alpha \in S \cap C$, every regressive function $f : \alpha \to \alpha$, and every cofinal subset $B \subseteq \alpha$, there exists some $i < \alpha$ such that

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Theorem

If $\Diamond(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\Box(\kappa)$ entails $\blacklozenge \neg(S)$.
\( J[\kappa] \): A new normal ideal over \( \kappa \)

\( S \in \mathcal{P}(\kappa) \) is in \( J[\kappa] \) iff there exists a club \( C \subseteq \kappa \) and a sequence of functions \( \langle f_i : \kappa \to \kappa \mid i < \kappa \rangle \) satisfying the following. For every \( \alpha \in S \cap C \), every regressive function \( f : \alpha \to \alpha \), and every cofinal subset \( B \subseteq \alpha \), there exists some \( i < \alpha \) such that

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**Theorem**

If \( \Diamond(\kappa) \) holds and \( S \in J[\kappa] \) is stationary, then \( \square(\kappa) \) entails \( \boxminus(S) \).

A comparison with the nonstationary ideal

1. \( J[\omega_1] = NS[\omega_1] \);
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$S \in \mathcal{P}(\kappa)$ is in $J[\kappa]$ iff there exists a club $C \subseteq \kappa$ and a sequence of functions $\langle f_i : \kappa \to \kappa \mid i < \kappa \rangle$ satisfying the following. For every $\alpha \in S \cap C$, every regressive function $f : \alpha \to \alpha$, and every cofinal subset $B \subseteq \alpha$, there exists some $i < \alpha$ such that

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Theorem

If $\Diamond(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\Box(\kappa)$ entails $\Diamond^{-}(S)$.

A comparison with the nonstationary ideal

1. $J[\omega_1] = NS[\omega_1]$;
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$J[\kappa]$: A new normal ideal over $\kappa$

$S \in \mathcal{P}(\kappa)$ is in $J[\kappa]$ iff there exists a club $C \subseteq \kappa$ and a sequence of functions $\langle f_i : \kappa \to \kappa \mid i < \kappa \rangle$ satisfying the following. For every $\alpha \in S \cap C$, every regressive function $f : \alpha \to \alpha$, and every cofinal subset $B \subseteq \alpha$, there exists some $i < \alpha$ such that

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**Theorem**

If $\lozenge(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\square(\kappa)$ entails $\lozenge^-(S)$.

A comparison with the nonstationary ideal

1. $J[\omega_1] = NS[\omega_1]$;
2. It is consistent that $J[\omega_2] = NS[\omega_2]$;
3. If $\kappa$ is inaccessible, then $J[\kappa] = NS[\kappa]$;
\( J[\kappa] \): A new normal ideal over \( \kappa \)

\( S \in \mathcal{P}(\kappa) \) is in \( J[\kappa] \) iff there exists a club \( C \subseteq \kappa \) and a sequence of functions \( \langle f_i : \kappa \to \kappa \mid i < \kappa \rangle \) satisfying the following. For every \( \alpha \in S \cap C \), every regressive function \( f : \alpha \to \alpha \), and every cofinal subset \( B \subseteq \alpha \), there exists some \( i < \alpha \) such that

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Theorem

*If \( \Diamond(\kappa) \) holds and \( S \in J[\kappa] \) is stationary, then \( \Box(\kappa) \) entails \( \Box^{-}(S) \).*

A comparison with the nonstationary ideal

1. \( J[\omega_1] = NS[\omega_1] \);
2. It is consistent that \( J[\omega_2] = NS[\omega_2] \);
3. If \( \kappa \) is inaccessible, then \( J[\kappa] = NS[\kappa] \);
4. If \( \lambda \geq \beth_\omega \), then \( J[\lambda^+] \neq NS[\lambda^+] \).
$J[\kappa]$: A new normal ideal over $\kappa$

$S \in \mathcal{P}(\kappa)$ is in $J[\kappa]$ iff there exists a club $C \subseteq \kappa$ and a sequence of functions $\langle f_i : \kappa \rightarrow \kappa \mid i < \kappa \rangle$ satisfying the following. For every $\alpha \in S \cap C$, every regressive function $f : \alpha \rightarrow \alpha$, and every cofinal subset $B \subseteq \alpha$, there exists some $i < \alpha$ such that

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Theorem

If $\diamondsuit(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\square(\kappa)$ entails $\ominus^-(S)$.

Corollary

For all $\lambda \geq \beth_\omega$ satisfying $2^\lambda = \lambda^+$:

$\square(\lambda^+)$ entails the existence of a club-regressive $\lambda^+$-Souslin tree.
$J[\kappa]$: A new normal ideal over $\kappa$

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Theorem

If $\diamondsuit(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\square(\kappa)$ entails $\boxminus(S)$.

Theorem

Assuming GCH, for every infinite cardinals $\theta < \lambda$ with $\text{cf}(\theta) = \theta$ and $\text{cf}(\theta) \neq \text{cf}(\lambda)$, $J[\lambda^+]$ contains a stationary subset of $E^{\lambda^+}_\theta$. 

$J[\kappa]$: A new normal ideal over $\kappa$

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**Theorem**

If $\diamondsuit(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\square(\kappa)$ entails $\boxminus^{-}(S)$.

**Theorem**

Assuming GCH, for every infinite cardinals $\theta < \lambda$ with $\text{cf}(\theta) = \theta$ and $\text{cf}(\theta) \neq \text{cf}(\lambda)$, $J[\lambda^+]$ contains a stationary subset of $E_{\theta}^{\lambda^+}$.

**Corollary**

Assuming GCH, for every infinite cardinals $\theta < \lambda$ with $\text{cf}(\theta) = \theta$ and $\text{cf}(\theta) \neq \text{cf}(\lambda)$, $\boxminus^{-}(E_{\theta}^{\lambda^+})$ holds.
A slightly weaker principle

\( \Box'(S) \) is obtained from \( \Box^-(S) \) by replacing the coherence requirement with coherence modulo bounded.
A slightly weaker principle

\( \mathcal{F}'(S) \) is obtained from \( \mathcal{F}^-(S) \) by replacing the coherence requirement with coherence modulo bounded.

**Theorem**

For \( \kappa \geq \omega_2 \), \( \mathcal{F}'(\kappa) + \diamond (\kappa) \) entails \( \mathcal{F}'(S) \) for all stationary \( S \subseteq \kappa \).
A slightly weaker principle

\( \Box'(S) \) is obtained from \( \Box^-(S) \) by replacing the coherence requirement with coherence modulo bounded.

**Theorem**

For \( \kappa \geq \omega_2 \), \( \Box'(\kappa) + \Diamond(\kappa) \) entails \( \Box'(S) \) for all stationary \( S \subseteq \kappa \).

**Corollary**

\( \square(\lambda^+) + \text{GCH} \) entails \( \Box'(E_{\text{cf}(\lambda)}^{\lambda^+}) \) for every uncountable cardinal \( \lambda \), and hence the existence of a \( \text{cf}(\lambda) \)-complete \( \lambda^+ \)-Souslin tree.
A slightly weaker principle

$\mathbb{P}^\prime(S)$ is obtained from $\mathbb{P}^{-}(S)$ by replacing the coherence requirement with coherence modulo bounded.

**Theorem**

For $\kappa \geq \omega_2$, $\mathbb{P}^\prime(\kappa) + \diamond (\kappa)$ entails $\mathbb{P}^\prime(S)$ for all stationary $S \subseteq \kappa$.

**Corollary**

$\square(\lambda^+) + \text{GCH}$ entails $\mathbb{P}^\prime(E_{\text{cf}(\lambda)}^{\lambda^+})$ for every uncountable cardinal $\lambda$, and hence the existence of a $\text{cf}(\lambda)$-complete $\lambda^+$-Souslin tree.

**Proof.**

Pick a regular cardinal $\theta < \lambda$ with $\theta \neq \text{cf}(\lambda)$. Then $J[\lambda^+]$ contains a stationary subset $S$ of $E_{\theta}^{\lambda^+}$. So, $\mathbb{P}^{-}(E_{\theta}^{\lambda^+})$ holds, let alone $\mathbb{P}^{-}(\lambda^+)$ and $\mathbb{P}^\prime(\lambda^+)$. By GCH and a theorem of Gregory/Shelah, $\diamond (\lambda^+)$ holds. Consequently, $\mathbb{P}^\prime(E_{\text{cf}(\lambda)}^{\lambda^+})$ holds. Altogether, there exists a $\text{cf}(\lambda)$-complete $\lambda^+$-Souslin tree. \qed
The $\lambda^+$-Souslin problem for $\lambda$ singular

Open problem

Suppose that $\lambda$ is a singular cardinal. Does GCH + $\Box^*_\lambda$ entail the existence of a $\lambda^+$-Souslin tree?
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Open problem

Suppose that $\lambda$ is a singular cardinal.
Does $\text{GCH} + \Box^*_\lambda$ entail the existence of a $\lambda^+$-Souslin tree?

Solutions to problems concerning the combinatorics of successor of singulars often goes through Prikry/Magidor/Radin forcing.
However, we have identified the following obstruction:
The $\lambda^+$-Souslin problem for $\lambda$ singular

Open problem
Suppose that $\lambda$ is a singular cardinal. Does GCH $+ \square^*_\lambda$ entail the existence of a $\lambda^+$-Souslin tree?

Solutions to problems concerning the combinatorics of successor of singulars often goes through Prikry/Magidor/Radin forcing. However, we have identified the following obstruction:

Theorem (Brodsky-Rinot, 2016)
Suppose that $\lambda$ is a strongly inaccessible cardinal, and $\mathbb{P}$ is a $\lambda^+$-cc notion of forcing of size $\leq 2^\lambda = \lambda^+$ that makes $\lambda$ into a singular cardinal. Then $\mathbb{P}$ introduces a $\lambda^+$-Souslin tree. (Moreover, $V^{\mathbb{P}} \models \square^*(\lambda^+) + \diamond(\lambda^+).$)