The $\aleph_2$-Souslin problem

Casa Matemática Oaxaca

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Conventions

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For a set of ordinals C, write:

- $\text{acc}(C) := \{\alpha < sup(C) \mid \sup(C \cap \alpha) = \alpha > 0\}$;
- $\text{nacc}(C) := C \setminus \text{acc}(C)$.

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- $T_\delta = \{x \in T \mid ht(x) = \delta\}$ is the δ^{th}-level of T.
- $(T, \lhd)$ is $\chi-$complete if any $\lhd$-increasing sequence of length $< \chi$ admits a bound.

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  - $f(t) \lhd t$ for all non-minimal nodes t in T;
  - for all $t \in T$, $f^{-1}\{t\}$ is the union of $<\kappa$-many antichains.

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The following are equiconsistent:

- There exists a Mahlo cardinal;
- There are no special $\aleph_2$-Aronszajn trees;
- $\square_{\omega_1}$ fails;
- Every stationary subset of $E^{\omega_2}_{\omega}$ reflects;
- $\mathrm{FRP}(\omega_2)$ holds.

Equiconsistency results

Definition

κ is weakly compact if it is inaccessible and $\neg\exists\kappa$-Aronszajn trees.

Recall (Hanf, 1964)

If κ is weakly compact, then $\{\alpha < \kappa \mid \alpha \text{ is Mahlo}\}$ is stationary.

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- $\square(\omega_2)$ fails;
- Every pair of stationary subsets of $E^{\omega_2}_{\omega}$ reflect simultaneously;
- Every stationary subset of $[\omega_2]^{\omega}$ reflects;
- For some regular cardinal $\kappa \geq \omega_2$, κ-cc$\times\kappa$-cc$=\kappa$-cc.

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If GCH *holds, and there exist no $\aleph_2$-Souslin trees, then $\aleph_2$ is a Mahlo cardinal in L.*

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Given the above-mentioned equiconsistency results, the general belief is that Gregory's lower bound should be increased from Mahlo to a weakly compact. Also, add to it the following:

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Theorem (Jensen, 1972)

If $V = L$, then for every regular uncountable cardinal κ, TFAE:

- *κ is not weakly compact;*
- *There exists a κ-Aronszajn tree;*
- *There exists a κ-Souslin tree.*

The $\aleph_2$-Souslin problem

From the Kanamori-Magidor 1978 survey article (p. 261):

The consistency problem for SH_κ when $\kappa > \omega_1$ seems to be much more difficult, especially if we want to retain the GCH. To bring matters into focus, we make some remarks which recall and amplify §21. First of all, Jensen[1972] had actually established that in L, weak compactness for κ is equivalent to SH_κ, for regular κ. We are interested in SH_κ for small κ, and the Mitchell-Silver model cited in §21 certainly satisfied SH_{ω_2}, as there were not even any ω_2-Aronszajn trees in that model. However, $2^\omega = \omega_2$ held in that model, and in fact a classical result of Specker[1951] as cited in §5 necessitates something like this: if $2^\omega = \omega_1$, then there is an ω_2-Aronszajn tree. No such result seems available for ω_2-Souslin trees, so the focal problem in this area is to get SH_{ω_2} and the GCH to hold.

This problem has been extensively investigated by Gregory[1976] who established in particular that: If $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $E^\omega_{\omega_2}$ hold, then SH_{ω_2} is false, i.e. there is an ω_2-Souslin tree. Hence, if we want SH_{ω_2} and the GCH to hold, we need to guarantee the failure of $E^\omega_{\omega_2}$. As pointed out in §21, this necessitates at least the consistency strength of the existence of a Mahlo cardinal, and very likely, of a weakly compact cardinal.

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Whether GCH entails the existence of an $\aleph_2$-Souslin tree remains open, however, the trees we get here are of a particular kind:

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Theorem (Todorcevic, 1981)

After Lévy-collapsing a weakly compact cardinal to $\aleph_2$ over a model of GCH*:* GCH *holds, and every $\aleph_2$-Aronszajn tree contains an $\aleph_1$-Aronszajn subtree.*

Stating the results

For almost two years now, Ari Brodsky and myself been studying a parameterized proxy principle, denoted $P(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma, \mathcal{E})$, and its effect on the existence of different types of κ-Souslin trees.

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Remark

A club-regressive κ-tree contains no ν-Aronszajn subtrees nor ν-Cantor subrtrees for every regular cardinal $\nu < \kappa$.

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The classic way to obtain χ-completeness is to move from $\diamondsuit(\kappa)$ to $\diamondsuit(E^\kappa_{\geq\chi})$. Unfortunately, $\diamondsuit(\kappa)$ is consistent with the failure of $\diamondsuit(E^\kappa_{\geq\chi})$:

Theorem (Shelah, 1980)

$\mathsf{GCH} + \diamondsuit(\omega_2) + \neg\diamondsuit(E^{\omega_2}_{\omega_1})$ *is consistent.*

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All previous $\diamondsuit$-based constructions of κ-Souslin trees involved sealing antichains at levels $\alpha \in S$ for some stationary S that does not reflect.

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In contrast, Lambie-Hanson proved that $\boxtimes^-(\aleph_{\omega+1}) + \diamondsuit(\aleph_{\omega+1})$ is consistent with the reflection of all stationary subsets of $\aleph_{\omega+1}$.

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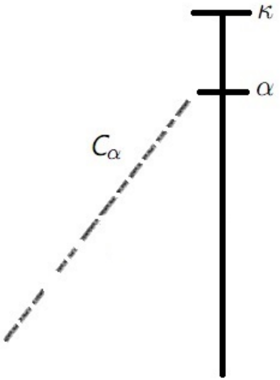
$\square(\lambda^+) + \mathsf{GCH}$ entails a club-regressive λ^+-Souslin tree;
$\square(\lambda^+) + \mathsf{GCH}$ entails a $\mathrm{cf}(\lambda)$-complete λ^+-Souslin tree.

Elements of the proofs

C-sequences

A <u>C-sequence</u> is a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- For every limit $\alpha < \kappa$, C_α is a club in α.



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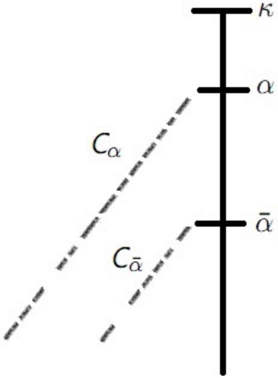

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- For every limit $\alpha < \kappa$, C_α is a club in α;
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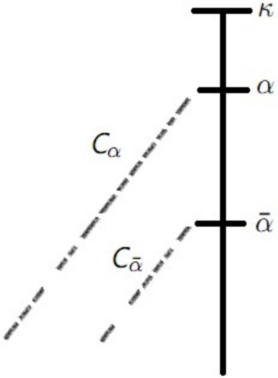

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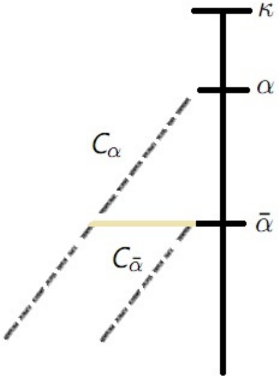

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Easiest way? Take a club D in κ, and put:

$$C_\alpha := \begin{cases} D \cap \alpha, & \text{if } \sup(D \cap \alpha) = \alpha; \\ \alpha \setminus \sup(D \cap \alpha), & \text{if } \sup(D \cap \alpha) < \alpha. \end{cases}$$

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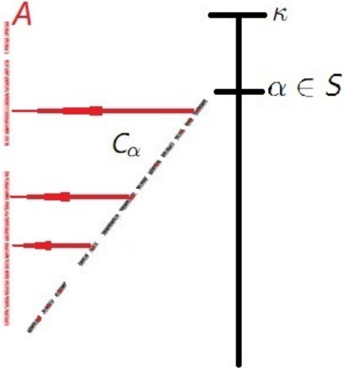
Definition (Todorcevic, 1987)

$\square(\kappa)$ asserts the existence of a coherent C-sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that for every club $D \subseteq \kappa$, there exists some $\alpha \in \mathrm{acc}(D)$ satisfying $C_\alpha \neq D \cap \alpha$.

Square principles (cont.)

Definition (Brodsky-Rinot, 2015)

For a stationary $S \subseteq \kappa$, $\boxtimes^-(S)$ asserts the existence of a coherent C-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for every cofinal $A \subseteq \kappa$, there exists some limit $\alpha \in S$ satisfying $\sup(\mathrm{nacc}(C_\alpha) \cap A) = \alpha$.



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Observation: $\boxtimes^-(\kappa) \implies \square(\kappa)$

Given a club $D \subseteq \kappa$, put $A := \mathrm{acc}(D)$.
Pick a limit $\alpha < \kappa$ such that $\sup(\mathrm{nacc}(C_\alpha) \cap A) = \alpha$. In particular, $\alpha \in \mathrm{acc}(D)$, and $\sup(\mathrm{nacc}(C_\alpha) \cap \mathrm{acc}(D)) = \alpha$ so that $C_\alpha \neq D \cap \alpha$.

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Remark

The standard way to force $\square(\kappa)$ is via the poset of all coherent C-sequences of successor length $< \kappa$ (ordered by end-extension).

Square principles (cont.)

Definition (Brodsky-Rinot, 2015)

For a stationary $S \subseteq \kappa$, $\boxtimes^-(S)$ asserts the existence of a coherent C-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that for every cofinal $A \subseteq \kappa$, there exists some limit $\alpha \in S$ satisfying $\sup(\mathrm{nacc}(C_\alpha) \cap A) = \alpha$.

Observation: $\boxtimes^-(\kappa) \implies \square(\kappa)$

Given a club $D \subseteq \kappa$, put $A := \mathrm{acc}(D)$.
Pick a limit $\alpha < \kappa$ such that $\sup(\mathrm{nacc}(C_\alpha) \cap A) = \alpha$. In particular, $\alpha \in \mathrm{acc}(D)$, and $\sup(\mathrm{nacc}(C_\alpha) \cap \mathrm{acc}(D)) = \alpha$ so that $C_\alpha \neq D \cap \alpha$.

Remark

The standard way to force $\square(\kappa)$ is via the poset of all coherent C-sequences of successor length $< \kappa$ (ordered by end-extension). The generic for this poset is in fact a $\boxtimes^-(\kappa)$-sequence!

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Question

Does $\square(\kappa) \implies \boxtimes^-(\kappa)$?
($V = L$ entails an affirmative answer)

$J[\kappa]$: A new normal ideal over κ

$S \in \mathcal{P}(\kappa)$ is in $J[\kappa]$ iff there exists a club $C \subseteq \kappa$ and a sequence of functions $\langle f_i : \kappa \to \kappa \mid i < \kappa \rangle$ satisfying the following. For every $\alpha \in S \cap C$, every regressive function $f : \alpha \to \alpha$, and every cofinal subset $B \subseteq \alpha$, there exists some $i < \alpha$ such that

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If $\diamondsuit(\kappa)$ holds and $S \in J[\kappa]$ is stationary, then $\square(\kappa)$ entails $\boxtimes^-(S)$.

Corollary

For all $\lambda \geq \beth_\omega$ satisfying $2^\lambda = \lambda^+$:
$\square(\lambda^+)$ entails the existence of a club-regressive λ^+-Souslin tree.

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Assuming GCH, for every infinite cardinals $\theta < \lambda$ with $\mathrm{cf}(\theta) = \theta$ and $\mathrm{cf}(\theta) \neq \mathrm{cf}(\lambda)$, $J[\lambda^+]$ contains a stationary subset of $E^{\lambda^+}_\theta$.

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Corollary

Assuming GCH, for every infinite cardinals $\theta < \lambda$ with $\mathrm{cf}(\theta) = \theta$ and $\mathrm{cf}(\theta) \neq \mathrm{cf}(\lambda)$, $\boxtimes^-(E^{\lambda^+}_\theta)$ holds.

A slightly weaker principle

$\boxtimes'(S)$ is obtained from $\boxtimes^{-}(S)$ by replacing the coherence requirement with coherence modulo bounded.

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$\square(\lambda^+) + \mathsf{GCH}$ *entails* $\boxtimes'(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ *for every uncountable cardinal* λ*, and hence the existence of a* $\mathrm{cf}(\lambda)$*-complete* λ^+*-Souslin tree.*

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Corollary

$\square(\lambda^+) + \mathrm{GCH}$ entails $\boxtimes'(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ for every uncountable cardinal λ, and hence the existence of a $\mathrm{cf}(\lambda)$-complete λ^+-Souslin tree.

Proof.

Pick a regular cardinal $\theta < \lambda$ with $\theta \neq \mathrm{cf}(\lambda)$. Then $J[\lambda^+]$ contains a stationary subset S of $E^{\lambda^+}_\theta$. So, $\boxtimes^-(E^{\lambda^+}_\theta)$ holds, let alone $\boxtimes^-(\lambda^+)$ and $\boxtimes'(\lambda^+)$. By GCH and a theorem of Gregory/Shelah, $\diamondsuit(\lambda^+)$ holds. Consequently, $\boxtimes'(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds.

Altogether, there exists a $\mathrm{cf}(\lambda)$-complete λ^+-Souslin tree. $\square$

The λ^+-Souslin problem for λ singular

Open problem

Suppose that λ is a singular cardinal.
Does $\mathsf{GCH} + \square^*_\lambda$ entail the existence of a λ^+-Souslin tree?

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However, we have identified the following obstruction:

Theorem (Brodsky-Rinot, 2016)

Suppose that λ is a strongly inaccessible cardinal, and $\mathbb{P}$ is a λ^+-cc notion of forcing of size $\leq 2^\lambda = \lambda^+$ that makes λ into a singular cardinal. Then $\mathbb{P}$ introduces a λ^+-Souslin tree.
(Moreover, $V^{\mathbb{P}} \models \boxtimes^*(\lambda^+) + \diamondsuit(\lambda^+)$.)