Tutorial on Strong colorings and their applications Part III

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Coloring groups

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Of course there is! Identify the underlying set of the group with $\aleph_1$, and let c be a witness to $\aleph_1 \not\rightarrow [\aleph_1]^2_{\aleph_1}$. Then $\langle G, c \rangle$, let alone $\langle G, *, c \rangle$, has no proper uncountable substructures.

Corrected question

Is it the case that for every group $\langle G, * \rangle$ of size $\aleph_1$, there exists a unary operation c, such that all proper substructures of $\langle G, *, c \rangle$ are countable?

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Unary operations are 1-dimensional coloring, and reminds us of Hindman's algebraic pigeonhole principle:

Hindman's theorem

Suppose $\langle G, + \rangle$ is an Abelian group. Then $G \to [\aleph_0]^{\mathrm{FS}}_2$ holds. That is, for every coloring $c : G \to \{0, 1\}$, there exists a set A of size $\aleph_0$ such that $\mathrm{FS}(A)$ is monochromatic for c.

$FS(A) := \{a_0 + \cdots + a_k \mid k < \omega, a_0, \ldots, a_k$ pairwise distinct elements of $A\}$.

Well-behaved groups

Definition

Let us say that a group $\langle G, *\rangle$ is well-behaved if there exists a map $s : G \to [G]^{<\omega}$ such that:

1. s is countable-to-one;
2. for all $x, y \in G$: $s(x) \triangle s(y) \subseteq s(x * y) \subseteq s(x) \cup s(y)$.

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Example (2)

Define $s : \mathbb{R} \rightarrow [\mathbb{R}]^{<\omega}$, as follows.
Let B be a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. For each real r, let $s(r)$ be the smallest subset $x \subseteq B$ such that $r \in \mathrm{Span}_{\mathbb{Q}}(x)$.

All Abelian groups are well-behaved

Let $\langle G, +\rangle$ be an arbitrary Abelian group.
By a standard fact from group theory, there exists a countable-to-one map $\mathbf{supp} : G \to [G]^{<\omega}$ such that for all $x, y \in G$:

$$\mathbf{supp}(x) \triangle \mathbf{supp}(y) \subseteq \mathbf{supp}(x + y) \subseteq \mathbf{supp}(x) \cup \mathbf{supp}(y).$$

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We now define a transformation $\mathbf{t} : G \to [G]^{<\omega}$, as follows.
Let $\preceq$ be some well-ordering of G.
Let $c : \omega \to [\omega]^{<\omega}$ be Cohen-generic over all definable dense sets.

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Theorem (Fernández-Bretón-Rinot, 2017)

For every uncountable $A \subseteq G$, there exists $B \subseteq G$ with $|B| = |A|$ such that $\mathbf{t}``\mathrm{FS}(A) \supseteq [B]^{<\omega}$.

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Corollary 1

Suppose $\langle G,+\rangle$ is an Abelian group. Then $G\nrightarrow[\aleph_1]^{\mathrm{FS}}_{\aleph_0}$.

Simply let $c(x):=|\mathbf{t}(x)|$.

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Compare with:

Hindman's theorem

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Corollary 2

The following is independent of ZFC (modulo large cardinals). Suppose $\langle G, +\rangle$ is an Abelian group. Then $G \not\rightarrow [\aleph_1]^{\mathrm{FS}}_{\aleph_1}$.

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Corollary 3

Suppose $\langle G, +\rangle$ is an Abelian group of size $\aleph_1$. Then $G \nrightarrow [\aleph_1]^{\mathrm{FS}}_{\aleph_1}$.

Fix an injective ennumeration $G = \{g_\alpha \mid \alpha < \omega_1\}$. Take d witnessing $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$, and let $c(x) := d(\alpha_1, \alpha_2)$ whenever $\mathbf{t}(x) = \{g_{\alpha_1}, g_{\alpha_2}, \ldots\}$.

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Let c witness $G \nrightarrow [\aleph_1]^{\mathrm{FS}}_{\aleph_1}$. Then c is unary, and identifying the range of c with G, all proper substructures of $\langle G, *, c\rangle$ are countable.

All Abelian groups are well-behaved (cont.)

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For every Abelian group $\langle G, +\rangle$, there is a transformation $\mathbf{t} : G \to [G]^{<\omega}$ satisfying that for every uncountable $A \subseteq G$, there exists $B \subseteq G$ with $|B| = |A|$ such that $\mathbf{t}``\mathrm{FS}(A) \supseteq [B]^{<\omega}$.

Corollary 4

For every regular uncountable cardinal κ, the following are equivalent:

- κ is not Jónsson;
- $G \nrightarrow [\kappa]^{\mathrm{FS}}_{\kappa}$ for every Abelian group $\langle G, +\rangle$ of cardinality κ.

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Theorem (Fernández-Bretón-Rinot, 2017)

For every Abelian group $\langle G, +\rangle$, there is a transformation $\mathbf{t} : G \rightarrow [G]^{<\omega}$ satisfying that for every uncountable $A \subseteq G$, there exists $B \subseteq G$ with $|B| = |A|$ such that $\mathbf{t}``\mathrm{FS}(A) \supseteq [B]^{<\omega}$.

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For every regular uncountable cardinal κ, the following are equivalent:

- There exists an algebra of cardinality κ with no proper subalgebra of cardinality κ;
- $G \nrightarrow [\kappa]^{\mathrm{FS}}_{\kappa}$ for every Abelian group $\langle G, +\rangle$ of cardinality κ.

Sums of length 2

Question

What happens if we restrict our attention to sums of length 2? i.e., to $\mathrm{FS}_2(A) := \{a_1 + a_2 \mid a_1 \neq a_2 \text{ from } A\}$.

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Proposition

If $G \nrightarrow [\lambda]^{\mathrm{FS}_2}_{\theta}$ for an Abelian group $\langle G, +\rangle$ of size κ, then $\kappa \nrightarrow [\lambda]^2_{\theta}$.

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Proposition

If $G \nrightarrow [\lambda]^{\mathrm{FS}_2}_\theta$ for an Abelian group $\langle G, +\rangle$ of size κ, then $\kappa \nrightarrow [\lambda]^2_\theta$.

Proof.

Let c witness $G \nrightarrow [\lambda]^{\mathrm{FS}_2}_\theta$. Fix injective enumeration $\{g_\alpha \mid \alpha < \kappa\}$ of G. Define $d : [\kappa]^2 \to \theta$ by stipulating $d(\alpha, \beta) := c(g_\alpha + g_\beta)$. □

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Theorem (Hindman-Leader-Strauss, 2015)

$\mathbb{R} \not\rightarrow [2^{\aleph_0}]_2^{\mathrm{FS}_2}$. *That is, there exists a coloring $c : \mathbb{R} \rightarrow \{0, 1\}$ such that for any $X \subseteq \mathbb{R}$ with $|X| = |\mathbb{R}|$, $\mathrm{FS}_2(X)$ is omnichromatic.*

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$\mathbb{R} \not\rightarrow [2^{\aleph_0}]^{\mathrm{FS}_2}_2$. *That is, there exists a coloring* $c : \mathbb{R} \rightarrow \{0, 1\}$ *such that for any* $X \subseteq \mathbb{R}$ *with* $|X| = |\mathbb{R}|$, $\mathrm{FS}_2(X)$ *is omnichromatic.*

Theorem (Komjáth and independently D. Soukup-Weiss, 2016)

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Note

It follows from a theorem of Shelah (1988) that the KSW result is optimal in the sense that $\mathbb{R} \nrightarrow [\aleph_1]^{FS_2}_3$ is not provable in ZFC.

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Recall (Galvin-Shelah, 1973)

$2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$.

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The proof contrasts three different characters of the real line: it is a vector space, a linearly ordered set, and a metric space.

The proof of the special case that $|\mathbb{R}|$ is a regular cardinal

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E.g., consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$, and let $B \subseteq [0,1)$ be the corresponding Hamel basis. For each $x \in \mathbb{R}$, there is a unique $v_x \in \mathbb{Q}^B$ such that $x = \sum_{b \in B} v_x(b) b$, so we may let $s(x) := \{b \in B \mid v_x(b) \neq 0\}$.

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Namely, for every $n \in \mathbb{N}$, every sequence of injections $\langle f_r : n \to E \mid r \in \mathbb{R} \rangle$ with pairwise disjoint images, and any subset I of n, there exist $r < s$ such that $f_r(i) < f_s(i)$ iff $i \in I$.

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- For each $y \neq z$ from $[0,1)$, let $\Delta(y,z) := \max\{m \in \mathbb{N} \mid |y-z| \leq 2^{-m}\}$.

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▶ Define $c : \mathbb{R} \to \mathbb{N}$ as follows. Let $x \in \mathbb{R}$ with $|s(x)| > 1$ be arbitrary.

(For all $x \in \mathbb{R}$ with $|s(x)| \leq 1$, we simply let $c(x) := 0$.)

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- For each $y \neq z$ from $[0,1)$, let $\Delta(y,z) := \max\{m \in \mathbb{N} \mid |y - z| \leq 2^{-m}\}$.
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The proof of the special case that $|\mathbb{R}|$ is a regular cardinal

Theorem (Fernández-Bretón-Rinot, 2017)

$\mathbb{R} \nrightarrow [2^{\aleph_0}]^{FS_2}_{\aleph_0}$.

▶ Fix a countable-to-one map $s : \mathbb{R} \rightarrow [0,1)^{<\omega}$ satisfying for all x, y:

$$s(x) \triangle s(y) \subseteq s(x+y) \subseteq s(x) \cup s(y).$$

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Continuation of the proof

To see that $c : \mathbb{R} \to \mathbb{N}$ works, fix an arbitrary $A \subseteq \mathbb{R}$ of size $2^{\aleph_0}$.
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Unbalanced partition relations

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Theorem (Dushnik-Erdős-Miller, 1941)

For every infinite cardinal κ, $\kappa \to (\kappa, \omega)^2$. That is, for every 2-dimensional coloring $c : [\kappa]^2 \to \{0, 1\}$, at least one of the following holds:

1. There exists $A \in [\kappa]^\kappa$ which is 0-monochromatic;
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Application

Every function from some infinite cardinal κ to the ordinals is either constant or strictly increasing on some cofinal subset of κ.

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If there exists a κ^+-Souslin tree, then $\kappa^+ \nrightarrow (\kappa^+, \lambda + 2)^2$.

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One can push the Raghavan-Todorcevic argument a little further, to show that the same hypothesis entails the existence of a coloring that witnesses $\kappa^+ \nrightarrow (\kappa^+, \lambda+2)^2$ and $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$ simultaneously.

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Fix a prolific κ^+-Souslin tree $\mathcal{T}$:

- $\mathcal{T} \subseteq {}^{<\kappa^+}\kappa^+$ and $|\mathcal{T}| = \kappa^+$;
- $(\mathcal{T}, \subseteq)$ has no antichains of size κ^+;
- for every $t \in \mathcal{T}$ and $\gamma \in \mathrm{dom}(t)$, $t \restriction \gamma \in \mathcal{T}$ and $t ^\frown \langle \gamma \rangle \in \mathcal{T}$.

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Theorem (Raghavan-Todorcevic, 2015)

1. *Every $A \subseteq \kappa^+$ of order-type κ^+ is omnichromatic for c;*
2. *No $B \subseteq \kappa^+$ of order-type $\lambda + 2$ is 1-monochromatic.*

Scales

Recall

Definition

- For $f, g \in {}^{\omega}\omega$, let $f \leq^{m} g$ iff $f(n) \leq g(n)$ for all $n \in [m, \omega)$;
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Definition

Say that $\mathcal{F} \subseteq {}^{\omega}\omega$ is bounded iff there exists some $g \in {}^{\omega}\omega$ such that $f <^* g$ for all $f \in \mathcal{F}$. Otherwise, $\mathcal{F}$ is said to be unbounded.

Definition

The smallest size of an unbounded subfamily $\mathcal{F} \subseteq {}^{\omega}\omega$ is denoted by $\mathfrak{b}$.

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Exercise

- *$\mathfrak{b}$ is regular and uncountable;*
- *There exists a $\mathfrak{b}$-scale. That is, a $<^*$-increasing sequence $\langle f_\alpha \mid \alpha < \mathfrak{b}\rangle$ s.t. $\{f_\alpha \mid \alpha < \mathfrak{b}\}$ is unbounded, and each $f_\alpha : \omega \to \omega$ is increasing.*

A strong coloring from a $\mathfrak{b}$-scale

Fix a $\mathfrak{b}$-scale, $\vec{f} = \langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$. Derive a coloring $c : [\mathfrak{b}]^2 \rightarrow \{0, 1\}$ by letting for all $\alpha < \beta < \mathfrak{b}$: $c(\alpha, \beta) := 1$ iff $f_\alpha \leq^0 f_\beta$.

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Lemma (Todorcevic, 1988)

Suppose $A \subseteq \mathfrak{b}$ is cofinal. Then there are $\alpha < \beta$ from A with $c(\alpha, \beta) = 1$.

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Proof.

Since ${}^{<\omega}\omega$ is countable, let us fix a large enough $\delta < \mathfrak{b}$ such that

$$\{f_\alpha \restriction n \mid n < \omega, \alpha \in A\} = \{f_\alpha \restriction n \mid n < \omega, \alpha \in A \cap \delta\}.$$

□

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Suppose $A \subseteq \mathfrak{b}$ is cofinal. Then there are $\alpha < \beta$ from A with $c(\alpha, \beta) = 1$.

Proof.

Since ${}^{<\omega}\omega$ is countable, let us fix a large enough $\delta < \mathfrak{b}$ such that for every $t \in {}^{<\omega}\omega$: if there is $\alpha \in A$ with $t \subseteq f_\alpha$, then there is such α in $A \cap \delta$.
Fix $m < \omega$ such that $A_m := \{\beta \in A \mid \delta < \beta \ \&\ f_\delta \leq^m f_\beta\}$ is cofinal in $\mathfrak{b}$.

□

A strong coloring from a $\mathfrak{b}$-scale

Fix a $\mathfrak{b}$-scale, $\vec{f} = \langle f_\alpha \mid \alpha < \mathfrak{b}\rangle$. Derive a coloring $c : [\mathfrak{b}]^2 \rightarrow \{0, 1\}$ by letting for all $\alpha < \beta < \mathfrak{b}$: $c(\alpha, \beta) := 1$ iff $f_\alpha \leq^0 f_\beta$.

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Then $\{f_\beta \mid \beta \in A_n\}$ is an unbounded family.

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Then $\{f_\beta \mid \beta \in A_n\}$ is an unbounded family. In particular, there are infinitely many $n < \omega$ such that $\sup\{f_\beta(n) \mid \beta \in A_m\} = \omega$.

□

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Fix $m < \omega$ such that $A_m := \{\beta \in A \mid \delta < \beta \ \&\ f_\delta \leq^m f_\beta\}$ is cofinal in $\mathfrak{b}$.
Let $n < \omega$ be the least to satisfy $\sup\{f_\beta(n) \mid \beta \in A_m\} = \omega$.

$\square$

A strong coloring from a $\mathfrak{b}$-scale

Fix a $\mathfrak{b}$-scale, $\vec{f} = \langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$. Derive a coloring $c : [\mathfrak{b}]^2 \to \{0,1\}$ by letting for all $\alpha < \beta < \mathfrak{b}$: $c(\alpha, \beta) := 1$ iff $f_\alpha \leq^0 f_\beta$.

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Let $n < \omega$ be the least to satisfy $\sup\{f_\beta(n) \mid \beta \in A_m\} = \omega$.
By minimality of n, $\{f_\beta \restriction n \mid \beta \in A_m\}$ is finite.

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Let $n < \omega$ be the least to satisfy $\sup\{f_\beta(n) \mid \beta \in A_m\} = \omega$.
By minimality of n, $\{f_\beta \restriction n \mid \beta \in A_m\}$ is finite. Therefore, there exists some $t : n \rightarrow \omega$ such that $\sup\{f_\beta(n) \mid \beta \in A_m \;\&\; t \subseteq f_\beta\} = \omega$.

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Pick $\beta \in A_m$ with $t \subseteq f_\beta$ such that $f_\beta(n) > f_\alpha(k+m)$. Altogether:

- If $i < n$, then $f_\alpha(i) = t(i) = f_\beta(i)$;

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Pick $\beta \in A_m$ with $t \subseteq f_\beta$ such that $f_\beta(n) > f_\alpha(k+m)$. Altogether:

- If $i < n$, then $f_\alpha(i) = t(i) = f_\beta(i)$;
- If $n \leq i \leq k+m$, then $f_\alpha(i) \leq f_\alpha(k+m) < f_\beta(n) \leq f_\beta(i)$;

□

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- If $i < n$, then $f_\alpha(i) = t(i) = f_\beta(i)$;
- If $n \leq i \leq k+m$, then $f_\alpha(i) \leq f_\alpha(k+m) < f_\beta(n) \leq f_\beta(i)$;
- If $i > k+m$, then by $f_\alpha \leq^k f_\delta$ and $f_\delta \leq^m f_\beta$, we have $f_\alpha(i) \leq f_\beta(i)$. □

A strong coloring from a $\mathfrak{b}$-scale (cont.)

Lemma (Todorcevic, 1986)

Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ *is a family of pairwise disjoint finite* 0*-mono. sets. Then there exist* $\alpha \neq \beta$ *such that* $a_\alpha \cup a_\beta$ *is* 0*-monochromatic.*

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Proof.

By iterative applications of the pigeonhole principle and the Dushnik-Miller theorem, we may thin out to ensure the existence of $k, m < \omega$ and that:

-
-
-
-

□

A strong coloring from a $\mathfrak{b}$-scale (cont.)

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Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ is a family of pairwise disjoint finite 0-mono. sets. Then there exist $\alpha \neq \beta$ such that $a_\alpha \cup a_\beta$ is 0-monochromatic.

Proof.

By iterative applications of the pigeonhole principle and the Dushnik-Miller theorem, we may thin out to ensure the existence of $k, m < \omega$ and that:

- For all $\alpha < \omega_1$, $|a_\alpha| = k$;
-
-
-

□

A strong coloring from a $\mathfrak{b}$-scale (cont.)

Lemma (Todorcevic, 1986)

Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ is a family of pairwise disjoint finite 0-mono. sets. Then there exist $\alpha \neq \beta$ such that $a_\alpha \cup a_\beta$ is 0-monochromatic.

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By iterative applications of the pigeonhole principle and the Dushnik-Miller theorem, we may thin out to ensure the existence of $k, m < \omega$ and that:

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-
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- For all $\alpha < \omega_1$, $\{\alpha_i \mid i < k\}$ is the increasing enumeration of a_α;
- For all $\alpha < \omega_1$ and $i \neq j$, exists $m' < m$ such that $f_{\alpha_i}(m') > f_{\alpha_j}(m')$;
-
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A strong coloring from a $\mathfrak{b}$-scale (cont.)

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- For all $\alpha < \omega_1$, $\{\alpha_i \mid i < k\}$ is the increasing enumeration of a_α;
- For all $\alpha < \omega_1$ and $i \neq j$, exists $m' < m$ such that $f_{\alpha_i}(m') > f_{\alpha_j}(m')$;
  - for $j < i$, we use the fact that $f_{\alpha_j} \leq^* f_{\alpha_i}$;
  - for $i < j$, we use the fact that a_α is 0-monochromatic, so $f_{\alpha_i} \not\leq^0 f_{\alpha_j}$.

□

A strong coloring from a $\mathfrak{b}$-scale (cont.)

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Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ *is a family of pairwise disjoint finite* 0*-mono. sets. Then there exist* $\alpha \neq \beta$ *such that* $a_\alpha \cup a_\beta$ *is* 0*-monochromatic.*

Proof.

By iterative applications of the pigeonhole principle and the Dushnik-Miller theorem, we may thin out to ensure the existence of $k, m < \omega$ and that:

- For all $\alpha < \omega_1$, $\{\alpha_i \mid i < k\}$ is the increasing enumeration of a_α;
- For all $\alpha < \omega_1$ and $i \neq j$, exists $m' < m$ such that $f_{\alpha_i}(m') > f_{\alpha_j}(m')$;
- The map $\alpha \mapsto \langle f_{\alpha_i} \restriction m \mid i < k \rangle$ is constant;
-

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A strong coloring from a $\mathfrak{b}$-scale (cont.)

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- The map $\alpha \mapsto \langle f_{\alpha_i} \restriction m \mid i < k\rangle$ is constant;
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A strong coloring from a $\mathfrak{b}$-scale (cont.)

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Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ is a family of pairwise disjoint finite 0-mono. sets. Then there exist $\alpha \neq \beta$ such that $a_\alpha \cup a_\beta$ is 0-monochromatic.

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By iterative applications of the pigeonhole principle and the Dushnik-Miller theorem, we may thin out to ensure the existence of $k, m < \omega$ and that:

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- The map $\alpha \mapsto \langle f_{\alpha_i} \restriction m \mid i < k\rangle$ is constant;
- The map $\alpha \mapsto \alpha_i$ is strictly increasing for all $i < k$.

As before, fix large enough $\delta < \omega_1$ s.t. $\{\langle f_{\alpha_i} \restriction n \mid i < k\rangle \mid n < \omega, \alpha < \omega_1\}$ is equal to $\{\langle f_{\alpha_i} \restriction n \mid i < k\rangle \mid n < \omega, \alpha < \delta\}$.

□

A strong coloring from a $\mathfrak{b}$-scale (cont.)

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Let $\beta := \delta + 1$. Fix a large enough $n < \omega$ such that $f_{\delta_i} <^n f_{\beta_i}$ for all $i < k$.

$\square$

A strong coloring from a $\mathfrak{b}$-scale (cont.)

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Let $\beta := \delta + 1$. Fix a large enough $n < \omega$ such that $f_{\delta_i} <^n f_{\beta_i}$ for all $i < k$. Pick $\alpha < \delta$ such that $\langle f_{\alpha_i} \restriction (n+1) \mid i < k\rangle = \langle f_{\beta_i} \restriction (n+1) \mid i < k\rangle$. □

A strong coloring from a $\mathfrak{b}$-scale (cont.)

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Find $\alpha < \delta < \beta$ and $n < \omega$ such that $f_{\delta_i} <^n f_{\beta_i}$ for all $i < k$, and $\langle f_{\alpha_i} \restriction (n+1) \mid i < k\rangle = \langle f_{\beta_i} \restriction (n+1) \mid i < k\rangle$.

□

A strong coloring from a $\mathfrak{b}$-scale (cont.)

Lemma (Todorcevic, 1986)

Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ is a family of pairwise disjoint finite 0-mono. sets. Then there exist $\alpha \neq \beta$ such that $a_\alpha \cup a_\beta$ is 0-monochromatic.

Proof.

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A strong coloring from a $\mathfrak{b}$-scale (cont.)

Lemma (Todorcevic, 1986)

Suppose $\{a_\alpha \mid \alpha < \omega_1\}$ *is a family of pairwise disjoint finite* 0*-mono. sets. Then there exist* $\alpha \neq \beta$ *such that* $a_\alpha \cup a_\beta$ *is* 0*-monochromatic.*

Proof.

By iterative applications of the pigeonhole principle and the Dushnik-Miller theorem, we may thin out to ensure the existence of $k, m < \omega$ and that:

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- ▶ If $i \neq j$, then exists $m' < m$ such that $f_{\alpha_i}(m') > f_{\alpha_j}(m') = f_{\delta_j}(m')$.

Altogether, $c(\alpha_i, \delta_j) = 0$ for all $i, j < k$, so that $a_\alpha \cup a_\delta$ is 0-mono. □

A stronger coloring from a $\mathfrak{b}$-scale

Definition (Shelah, 1988)

$\mathrm{Pr}_1(\kappa,\theta)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ as follows. For every $\mathcal{A} \subseteq [\kappa]^{<\omega}$ of size κ, consisting of pairwise disjoint sets, and every $\gamma < \theta$, there are $a, b \in \mathcal{A}$ with $\max(a) < \min(b)$ such that $c[a \times b] = \{\gamma\}$.

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An argument we gave yesterday readily verifies that adding a Cohen real introduces a witness to $\mathrm{Pr}_1(\aleph_1, \aleph_0)$.

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Theorem (Galvin, 1980)

If CH *holds, then so does* $\mathrm{Pr}_1(\aleph_1, \aleph_1)$.

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Then $(\mathbb{Q}_0, \supseteq)$ and $(\mathbb{Q}_1, \supseteq)$ are κ-cc.

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Then $(\mathbb{Q}_0, \supseteq)$ and $(\mathbb{Q}_1, \supseteq)$ are κ-cc.
In contrast, $\{(\{\alpha\}, \{\alpha\}) \mid \alpha < \kappa\}$ is an antichain in their product.

pcf scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a *pcf* scale.
That is, a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$, along with an increasing sequence of regular cardinals $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ converging to λ, such that:

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Corollary (Shelah, 1990's)

Suppose λ is a singular cardinal. Then $\lambda^+ \nrightarrow [\lambda^+]^2_{\mathrm{cf}(\lambda)}$ holds.

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Proof sketch.

Fix a *pcf* scale $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^+ \rangle$, and derive $\Gamma : [\lambda^+]^2 \rightarrow \mathrm{cf}(\lambda)$ as follows. For all $\alpha < \beta < \lambda^+$, let

$$\Gamma(\alpha, \beta) := \sup\{i < \mathrm{cf}(\lambda) \mid f_\alpha(i) \geq f_\beta(i)\}.$$

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Suppose $A \subseteq \lambda^+$ is cofinal. As $\vec{f}$ is a scale, we have $\sup_{\beta \in A} f_\beta(i) = \lambda_i$ for a tail of $i < \mathrm{cf}(\lambda)$.

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Corollary (Shelah, 1990's)

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The very same coloring actually establishes more:

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Theorem (Rinot, 2012, building on Eisworth)

Suppose λ is a singular cardinal. For every θ, $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ iff $\mathrm{Pr}_1(\lambda^+, \theta)$.

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Are the following two equivalent?

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Suppose κ is a regular cardinal $\geq \aleph_2$. Are the following two equivalent?

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By Rinot (2014), if κ is a regular cardinal $\geq \aleph_2$ and the product of any two κ-*cc* posets is κ-*cc*, then κ is a weakly compact cardinal in L.

Walks on ordinals

Fix a regular uncountable cardinal κ along with $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ such that each C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$.

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For all $\alpha < \beta < \kappa$, define $\mathrm{Tr}(\alpha, \beta) \in {}^\omega\kappa$, by recursion on $n < \omega$:

$$\mathrm{Tr}(\alpha,\beta)(n) := \begin{cases} \beta & n = 0 \\ & \\ & \end{cases}$$

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Fix a regular uncountable cardinal κ along with $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ such that each C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$.

Definition (Todorcevic, 1987)

For all $\alpha < \beta < \kappa$, define $\mathrm{Tr}(\alpha, \beta) \in {}^\omega\kappa$, by recursion on $n < \omega$:

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Examples of educated choices of $\vec{C}$

- $\vec{C}$ that avoids a stationary set;
- $\vec{C}$ that nicely-swallows a club-guessing sequence;
- a $\square(\kappa)$-sequence;
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Fact (Todorcevic, 1987 / Shelah, 1987)

If $\vec{C}$ avoids a stationary set S, then for every cofinal $A \subseteq \kappa$, the set $\bigcup\{\mathrm{Im}(\mathrm{Tr}(\alpha,\beta)) \mid \alpha < \beta \text{ both from } A\}$ covers a club relative to S.

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Fact (Shelah, 1990's)

If $\vec{C}$ nicely-swallows a club-guessing sequence, then for every cofinal $A \subseteq \kappa$, the set $\bigcup\{\mathrm{Im}(\mathrm{Tr}(\alpha,\beta)) \mid \alpha < \beta$ both from $A\}$ is positive with respect to some normal ideal.

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Fact (Rinot, 2014)

If $\vec{C}$ is a $\square(\kappa)$-sequence, then for a suitable $\psi : \kappa \rightarrow \kappa$, for every cofinal $A \subseteq \kappa$, the set $\psi``\bigcup\{\mathrm{Im}(\mathrm{Tr}(\alpha, \beta)) \mid \alpha < \beta \text{ both from } A\}$ is equal to κ.

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- $\vec{C}$ that avoids a stationary set;
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The better the $\vec{C}$ is, the easier it is to derive a strong coloring.

Exercise

If $\vec{C}$ is a $\clubsuit_\lambda$-sequence, and $\psi : \lambda \to \lambda$ is a surjection with all preimages of size λ, then $\psi \circ \rho_1$ witnesses $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$.

Definition

$\rho_1(\alpha, \beta) := \max\{\mathrm{otp}(C_{\mathrm{Tr}(\alpha,\beta)(n)} \cap \alpha) \mid n < \omega, \mathrm{Tr}(\alpha, \beta)(n) \neq \alpha\}$.

Oscillation functions

Goal: picking one point from the walk

Suppose we walk along a sequence $\vec{C}$ satisfying that for every cofinal $A \subseteq \kappa$, the set $\bigcup\{\mathrm{Im}(\mathrm{Tr}(\alpha,\beta)) \mid \alpha < \beta$ both from $A\}$ is "large".

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Oscillation of the upper trace

The first example of such an oscillation map was introduced by Todorcevic (1987).

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Oscillation of the lower trace

In his solution of the L space problem, Moore (2006) introduced an oscillation map $\mathrm{osc} : [\omega_1]^2 \to \omega$, witnessing $\omega_1 \nrightarrow [\omega_1;\omega_1]^2_\omega$ (and more).

Oscillation functions

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Oscillation functions

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Oscillation functions

Goal: picking one point from the walk

Suppose we walk along a sequence $\vec{C}$ satisfying that for every cofinal $A \subseteq \kappa$, the set $\bigcup\{\mathrm{Im}(\mathrm{Tr}(\alpha, \beta)) \mid \alpha < \beta \text{ both from } A\}$ is "large".
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Oscillation of a *pcf* scale

Suppose we walk below λ^+, where λ is a singular cardinal. Let $\Gamma : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ be the oscillation of *pcf* scales that we've met before:

$$\Gamma(\alpha, \beta) := \sup\{i < \mathrm{cf}(\lambda) \mid f_\alpha(i) \geq f_\beta(i)\}.$$

Eisworth (2009) derives an oscillation $o : [\lambda^+]^2 \to \omega$, as follows.
For $\alpha < \beta$, let $o(\alpha, \beta)$ be the least n s.t. $\Gamma(\alpha, \beta) \neq \Gamma(\alpha, \mathrm{Tr}(\alpha, \beta)(n))$.

Oscillation functions

Goal: picking one point from the walk

Suppose we walk along a sequence $\vec{C}$ satisfying that for every cofinal $A \subseteq \kappa$, the set $\bigcup\{\mathrm{Im}(\mathrm{Tr}(\alpha,\beta)) \mid \alpha < \beta$ both from $A\}$ is "large".
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Another oscillation of a *pcf* scale

In Rinot (2012), we walk below λ^+, where λ is singular of cofinality ω, and $\vec{C}$ is not a sequence, but a matrix $\langle C_{\alpha,n} \mid \alpha < \lambda^+, n < \omega\rangle$.

Oscillation functions

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Another oscillation of a *pcf* scale

In Rinot (2012), we walk below λ^+, where λ is singular of cofinality ω, and $\vec{C}$ is not a sequence, but a matrix $\langle C_{\alpha,n} \mid \alpha < \lambda^+, n < \omega\rangle$.
We use a bijection $\omega \leftrightarrow {}^{<\omega}\omega$ to identify Γ with a map $\Gamma : [\lambda^+]^2 \to {}^{<\omega}\omega$.
Then we walk from β down to α, letting $\sigma := \Gamma(\alpha,\beta)$, $\beta_0 := \beta$, and $\beta_{n+1} := \min(C_{\beta_n,\sigma(n)} \setminus \alpha)$ for all n. We then stop at $\beta_{\ell(\sigma)}$.

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The oscillation oracle $\mathrm{P}\ell_6(\kappa,\theta,\chi)$

$\exists$function c such that for every club $E \subseteq \kappa$, every regressive map $\varphi : E \to \theta$, and every sequence $\langle (u_\alpha, v_\alpha, \sigma_\alpha) \mid \alpha \in E\rangle$, satisfying

1.
2.
3.

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there exist $\alpha < \beta$ in E such that
for all $\eta \in u_\alpha$ and $\sigma \in v_\beta$: $c(\eta^\frown\sigma) = (\eta,\sigma)$.

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Oscillation functions

Theorem (Rinot, 2014)

$\mathrm{P}\ell_6(\lambda^+, \omega, \lambda)$ holds for every infinite regular cardinal λ.

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We have used $\mathrm{P}\ell_6$ in proving that if κ, χ are regular cardinals with $\kappa > \chi^+$, and either $\square(\kappa)$ holds or $E^\kappa_{\geq\chi}$ admits a nonreflecting stationary set, then $\mathrm{Pr}_1(\kappa, \kappa, \chi)$ holds.

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We have used $\mathrm{P}\ell_6$ in proving that if κ, χ are regular cardinals with $\kappa > \chi^+$, and either $\square(\kappa)$ holds or $E^\kappa_{\geq\chi}$ admits a nonreflecting stationary set, then $\mathrm{Pr}_1(\kappa, \kappa, \chi)$ holds.

Definition (Shelah, 1988)

$\mathrm{Pr}_1(\kappa, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \rightarrow \theta$ as follows. For every $\mathcal{A} \subseteq [\kappa]^{<\chi}$ of size κ, consisting of pairwise disjoint sets, and every $\gamma < \theta$, there are $a, b \in \mathcal{A}$ with $\sup(a) < \min(b)$ such that $c[a \times b] = \{\gamma\}$.

Oscillation functions

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The proof of $\mathrm{P}\ell_6(\lambda^+,\omega,\lambda)$ takes advantage of oscillation theory of a simple weight function for C-sequences. Plus, at some point, it splits into two:

- if $\lambda^{<\lambda}>\lambda$, then we use a particular club guessing at $E^{\lambda^+}_{\lambda}$;
- if $\lambda^{<\lambda}=\lambda$, then we use a higher analogue of Todorcevic's strong coloring derived from the complete binary tree ${}^{<\omega}2$.

Concluding slides

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- an inner partition into stationary sets;
- an oracle for picking a point from the upper trace of a walk;
- strong counterexamples to higher analogs of Hindman's theorem.

Major open problems

Successors of singulars

Suppose λ is a singular cardinal. Does $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ hold?

Inaccessibles

Suppose κ is an inaccessible cardinal that is not weakly compact.
Do all of the following hold?

- $\kappa \nrightarrow [\kappa]^2_\omega$
- $\kappa \nrightarrow [\kappa;\kappa]^2_2$
- $\mathrm{Pr}_1(\kappa,\omega)$