Tutorial on Strong colorings and their applications Part II

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For each $\beta < \omega_1$, fix an injection $t_\beta : \beta \to x_\beta$.
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Proposition

For every $n < \omega$, and every uncountable subfamily $\mathcal{A} \subseteq [\omega_1]^{<\omega}$ of pairwise disjoint sets, there exist $a, b \in \mathcal{A}$ such that:

- $\max(a) < \min(b)$;
- $c[a \times b]$ has size $|a \times b|$;
- $c[a \times b]$ is disjoint from n.

The strongest coloring

Now, force to add a Cohen real $r : \omega \to \omega$.
Every unctble family in $V[r]$ contains an unctble subfamily in V, so that every condition $p : n \to \omega$ forces that $d := r \circ c$ is the wildest coloring.

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For each $\beta < \omega_1$, define $f_\beta : \omega_1 \to 2$ by stipulating:

$$f_\beta(\alpha) := \begin{cases} 1, & \text{if } \alpha < \beta \text{ and } d(\alpha, \beta) = 1; \\ 1, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

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This is an L space

$\{f_\beta \mid \beta < \omega_1\}$ is a regular, Hausdorff subspace of 2^{ω_1} which is hereditarily Lindelöf but not hereditarily separable.

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Adding a Cohen real adds a strong L space (Roitman, 1979)

$\{f_\beta \mid \beta < \omega_1\}$ is a regular, Hausdorff subspace of 2^{ω_1} which is hereditarily Lindelöf in all finite powers but not separable.

By Kunen (1977), $MA_{\aleph_1}$ entails that there are no strong L spaces.

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Let $f^*_\beta(\alpha) := f_\alpha(\beta)$. That is, define $f^*_\beta : \omega_1 \to 2$ by stipulating:

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This is a strong S space

$\{f^*_\beta \mid \beta < \omega_1\}$ is a regular, Hausdorff subspace of 2^{ω_1} which is hereditarily separable in all finite powers but not Lindelöf.

By Zenor (1980), there is a strong S space iff there is a strong L space.

Stretching exercises

Getting more colors

Definition

A subset $T \subseteq \mathbb{N}$ is said to be thick if for every positive integer m, there exists a positive integer k such that $\{k+1, \ldots, k+m\} \subseteq T$.

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Proof.

Let $\{p_i \mid i < \omega\}$ be the increasing enumeration of all prime numbers.
Define $\psi : \mathbb{N} \rightarrow \mathbb{N}$ by stipulating that $\psi(n)$ be the least $i < \omega$ such that p_i does not divide n.

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So $\psi \circ c$ witnesses $\kappa \nrightarrow [\kappa]^2_{\aleph_0}$. □

Getting more colors

Proposition 2

Suppose $c : [\kappa]^2 \rightarrow \mathbb{R}$ satisfies that for every cofinal $B \subseteq \kappa$, $c``[B]^2$ is dense. Then $\kappa \nrightarrow [\kappa]^2_{\aleph_0}$ holds.

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Getting more colors (cont.)

Proposition 3

Suppose κ is a regular uncountable and $c : [\kappa]^2 \rightarrow \kappa$ satisfies that for every cofinal $B \subseteq \kappa$, $c``[B]^2$ covers a club.
Then $\kappa \nrightarrow [\kappa]^2_\kappa$ holds.

Getting more colors (cont.)

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Fix a partition of κ into pairwise disjoint stationary sets, $\langle S_\gamma \mid \gamma < \kappa \rangle$.

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For every club $D \subseteq \kappa$, we have $D \cap S_\gamma \neq \emptyset$ for all $\gamma < \kappa$, so that $\psi``D = \kappa$.

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Consequently, $d := \psi \circ c$ witnesses $\kappa \nrightarrow [\kappa]^2_\kappa$. □

Getting more colors (cont.)

Proposition 3*

Suppose κ is regular uncountable, $S \subseteq \kappa$ is stationary, and $c : [\kappa]^2 \rightarrow \kappa$ satisfies that for every cofinal $B \subseteq \kappa$, $c``[B]^2$ covers a club relative to S.
Then $\kappa \nrightarrow [\kappa]^2_\kappa$ holds.

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The following are equivalent:

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To see that d witnesses $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$, fix a cofinal $B \subseteq \lambda^+$ and some color $\gamma < \lambda^+$. Wlog, $\min(B) > \gamma$.

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3d stretching

Stretching functions could get more and more exotic.

In Moore (2006), an L space is derived from a coloring; this coloring is the outcome of an unorthodox stretching — motivated by Kronecker's theorem on diophantine approximations — of some map $\mathrm{osc} : [\aleph_1]^2 \to \mathbb{N}$.

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3. Find a suitable sequence of real numbers $\langle r_\alpha\mid\alpha<\aleph_1\rangle$.

Derive $c:[\aleph_1]^2\rightarrow\aleph_0$ by letting for all $\alpha<\beta<\omega_1$:

$$c(\alpha,\beta):=\lfloor g(f(r_\alpha\cdot\mathrm{osc}(\alpha,\beta)+r_\beta)\rfloor.$$

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Does $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ hold for every singular cardinal λ?

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Theorem (Shelah/Todorcevic, 1980's)

Suppose λ is a singular cardinal, and $\kappa \nrightarrow [\kappa]^2_\kappa$ for a tail of regular $\kappa < \lambda$. Then $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$.

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Then $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$.

Theorem (Eisworth, 2013)

Suppose λ is a singular cardinal, and $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ for all $\theta < \lambda$.
Then $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$.

It is not always possible to get more colors

Note

After adding weakly compact many Cohen reals:

1. $2^{\aleph_0} \not\rightarrow [2^{\aleph_0}]^2_{\aleph_0}$ *holds;*
2. $2^{\aleph_0} \not\rightarrow [2^{\aleph_0}]^2_{\aleph_1}$ *fails.*

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Conversely, by Rinot (2014), if the cardinal $\mu := \mathrm{cf}(2^{\aleph_0})$ is not weakly compact in L, then $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_1}$ holds.

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The following is open:

Problem 2

Suppose κ is a regular uncountable cardinal. Are the following equivalent?

- $\kappa \nrightarrow [\kappa]^2_2$;
- $\kappa \nrightarrow [\kappa]^2_{\aleph_0}$.

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Is there any reason to prefer a coloring that natively produces a pallet of colors over another coloring that produces the same, but through some clever stretching?

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A possible answer

Suppose that c is some strong coloring in our universe V, and that V' is some cofinality-preserving forcing extension.

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Suppose that c is some strong coloring in our universe V, and that V' is some cofinality-preserving forcing extension. If c is derived from a Souslin tree or a $\mathfrak{b}$-scale, then it is quite possible that in V', these objects have lost their defining feature.

To stretch or not to stretch, that is the question

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The same issue arise with regards to the stretching component.

Inner partitioning into stationary sets

Definition (Woodin)

A regular uncountable cardinal κ is ω-strongly measurable in HOD if there exists $\theta < \kappa$ such that:

1. $(2^\theta)^{\mathsf{HOD}} < \kappa$;
2. There is no partition $\langle S_\gamma \mid \gamma < \theta\rangle$ of $\kappa \cap \mathrm{cof}(\omega)$ into stationary sets such that $\langle S_\gamma \mid \gamma < \theta\rangle \in \mathsf{HOD}$.

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HOD Dichotomy theorem (Woodin, 2010)

Suppose that δ is an extendible cardinal. Then one of the following hold.

1. *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD.
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Alternative 2 should be understood as an abstract generalization of "$0^\sharp$ exists".

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Inner partitioning into stationary sets also plays a role in the Friedman-Magidor paper (2009) on the number of normal measures.

Inner partitioning into stationary sets from a coloring

Observation

Suppose $W \subseteq V$ is an inner model of ZFC, and c is some coloring in W.

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Suppose $W \subseteq V$ is an inner model of ZFC, and c is some coloring in W. In V, suppose that c witnesses $\kappa \nrightarrow [\kappa]^2_\theta$ for some regular uncountable cardinal κ, and some $\theta < \kappa$.

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For each $\alpha < \kappa$, let $S^\alpha_\gamma := \{\beta \mid \alpha < \beta < \kappa, c(\alpha, \beta) = \gamma\}$ for all $\gamma < \theta$.

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Put $D := \bigcap_{\alpha<\theta} D_\alpha$. Pick $\alpha < \beta$ both from $S \cap D$ such that $c(\alpha, \beta) = \gamma^*$.

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Put $D := \bigcap_{\alpha<\theta} D_\alpha$. Pick $\alpha < \beta$ both from $S \cap D$ such that $c(\alpha, \beta) = \gamma^*$.
Then $\beta \in D \cap S^\alpha_{\gamma^*} \subseteq D_\alpha \cap S^\alpha_{\gamma_\alpha}$, contradicting the choice of D_α. □

Never say never

In Dobrinen (2008), it is proved that if $W \subseteq V$ is an inner model of ZFC with the same ordinals, then the existence of an ω-sequence in $V \setminus W$ together with the existence of mildly-strong colorings in W imply that for a tail of regular cardinals κ, for all cardinals $\lambda > \kappa$, $\mathcal{P}_\kappa(\lambda) \setminus W$ is stationary.

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The above is an application of a strong coloring (indeed, to analyzing outer models) that do not depend on whether the coloring remains strong in the outer model.

Playing with a Luzin set

Luzin set

An uncountable set of reals whose intersection with any nowhere dense set is countable.

Note. CH $\implies$ there exists a Luzin set $\implies$ $\mathfrak{b} = \aleph_1$.

Playing with a Luzin set

Suppose that $\{x_\beta \mid \beta < \omega_1\}$ is the injective enumeration of a Luzin subset of ${}^\omega\omega$. Derive $\langle f_n : \omega_1 \to \omega \mid n < \omega\rangle$ by stipulating:

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Since B is uncountable, $X_B := \{x_\beta \mid \beta \in B\}$ is somewhere dense.
Pick $t \in {}^{<\omega}\omega$ such that X_B is dense in the basic open set $[t] := \{f \in {}^\omega\omega \mid t \subseteq f\}$. Then for every $s \in {}^{<\omega}\omega$ extending t and every $m < \omega$, there exists $\beta \in B$ such that $x_\beta \in [s^\frown\langle m \rangle]$. $\square$

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Claim 2

c witnesses $\aleph_1 \nrightarrow [\aleph_0; \aleph_1]^2_{\aleph_0}$.
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Pick a large enough $n \in t``A$ such that $f_n \restriction B'$ is onto.

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Pick a large enough $n \in t``A$ such that $f_n \restriction B'$ is onto.
So, for $\alpha \in A$ such that $t(\alpha) = n$, we have $\{c(\alpha, \beta) \mid \beta \in B'\} = \aleph_0$. □

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And, if we wish, we can apply some stretching:

Theorem (Todorcevic, 1987 [extending Erdős-Hajnal-Milner, 1966])

If there exists a Luzin set, then $\aleph_1 \nrightarrow [\aleph_0; \aleph_1]^2_{\aleph_1}$.

Rectangles vs. squares

Theorem (Todorcevic, 1987)

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$2^{\aleph_0} \not\rightarrow [2^{\aleph_0}]^2_{\aleph_0}$.

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The following is still open.

Problem 3

Does $2^{\aleph_0} \nrightarrow [2^{\aleph_0}; 2^{\aleph_0}]^2_{\aleph_0}$ follow from ZFC?

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Rectangular strong colorings are universal

Theorem (Erdős-Hajnal, 1978)

Suppose that $c : [\aleph_1]^2 \rightarrow \theta$ *witnesses* $\aleph_1 \nrightarrow [\aleph_1;\aleph_1]^2_\theta$.
That is, for every $A, B \in [\aleph_1]^{\omega_1}$, $c[A \circledast B] = \theta$.
Then c *contains a copy of any countable 2-dimensional* θ*-coloring.*

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Then c *contains a copy of any countable 2-dimensional* θ*-coloring.*

Reminder

$c : [\aleph_1]^2 \to \theta$ is said to contain a copy of $d : [\aleph_0]^2 \to \theta$ if there exists an increasing sequence $\{\alpha_n\}_{n=0}^{\infty} \subseteq \aleph_1$ such that

$$c(\alpha_n, \alpha_m) = d(n, m) \text{ for all } n, m.$$

Rectangular strong colorings are universal

Theorem (Erdős-Hajnal, 1978)

Suppose that $c : [\aleph_1]^2 \rightarrow \aleph_0$ *witnesses* $\aleph_1 \nrightarrow [\aleph_1; \aleph_1]^2_{\aleph_0}$.
That is, for every $A, B \in [\aleph_1]^{\omega_1}$, $c[A \circledast B] = \aleph_0$.
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Shelah's approach is surprising, so let us look at it.

A strong coloring with no rainbow triangles

Shelah's idea: use a ready-made coloring, and only construct its domain.

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The Cantor space

Consider the Cantor space $\langle 2^\omega, d\rangle$, where $d : [2^\omega]^2 \to \mathbb{Q}$ is derived from the product topology. That is, $d(f,g) := \frac{1}{2^{\Delta(f,g)}}$, where

$$\Delta(f,g) := \min\{m < \omega \mid f(m) \neq g(m)\}.$$

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Theorem (Shelah, 1975)

CH *entails* $X \subseteq 2^\omega$ *of size* $\aleph_1$ *such that* $c \restriction [X]^2$ *witnesses* $\aleph_1 \not\to [\aleph_1]^2_{\aleph_0}$.

Transforming rectangles into squares

Recall Problem 3

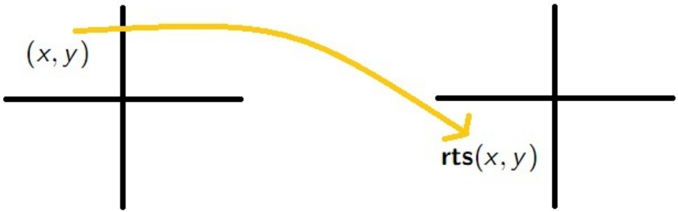
Does $2^{\aleph_0} \nrightarrow [2^{\aleph_0}; 2^{\aleph_0}]^2_2$ follow from ZFC?

Fact

For every **successor** *cardinal κ, and every θ, the following are equivalent:*

- $\kappa \nrightarrow [\kappa]^2_\theta$;
- $\kappa \nrightarrow [\kappa; \kappa]^2_\theta$.

Transforming rectangles into squares



Theorem

For every infinite cardinal λ*, there exists a function* $\mathbf{rts} : [\lambda^+]^2 \rightarrow [\lambda^+]^2$ *with the following property;*

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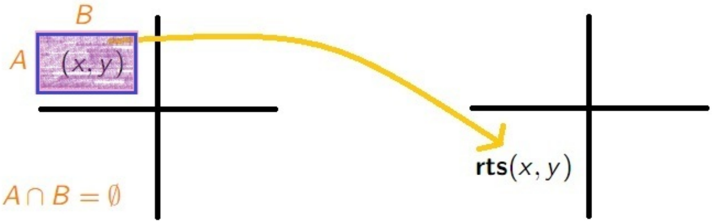

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For every infinite cardinal λ, there exists a function $\mathbf{rts} : [\lambda^+]^2 \to [\lambda^+]^2$ *with the following property; For all subsets* $A, B \subseteq \lambda^+$ *of size* λ^+,

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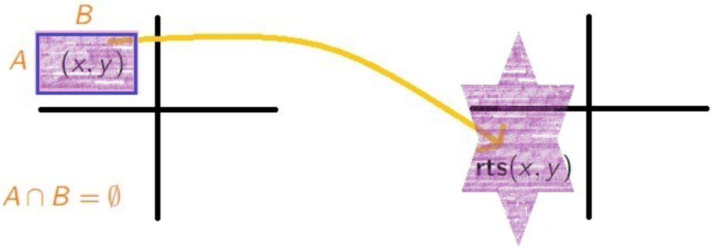

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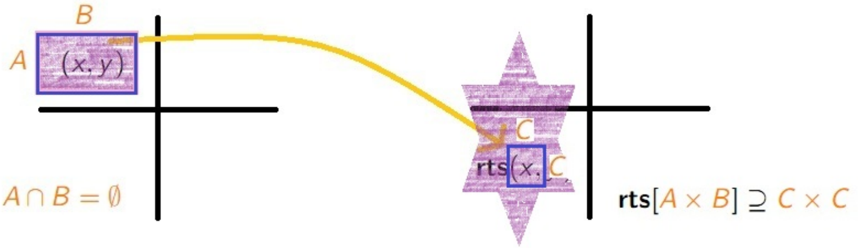

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$$\mathbf{rts}[A \circledast B] \supseteq C \circledast C.$$

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- (Shelah, 1990) for λ regular $> 2^{\aleph_0}$;
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- (Todorcevic-Rinot, 2013) uniform proof for all regular λ.

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Corollary

For every successor cardinal κ, and every θ, the following are equivalent:

- $\kappa \nrightarrow [\kappa]^2_\theta$;
- $\kappa \nrightarrow [\kappa;\kappa]^2_\theta$.

Proof.

If c witnesses $\kappa \nrightarrow [\kappa]^2_\theta$, then $c \circ$ **rts** witnesses $\kappa \nrightarrow [\kappa;\kappa]^2_\theta$. □

Colorings triples

Theorem (Shore, 1974)

If $\aleph_1 \nrightarrow [\aleph_0; \aleph_1]^2_{\aleph_1}$ *and there exists a Kurepa tree, then* $\aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_1}$.

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$$\delta(\alpha,\beta,\gamma) := \{\Delta(b_\alpha, b_\beta), \Delta(b_\alpha, b_\gamma), \Delta(b_\beta, b_\gamma)\},$$

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