Tutorial on
Strong colorings and their applications
Part I

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Colorings

Notation

$[X]^{\mu} := \{A \subseteq X \mid \mathrm{otp}(A) = \mu\}$.

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- We think of Y as *the set of possible colors*, and sometime say that c is a Y-coloring.
- If $X \subseteq$ ordinals, then we identify $[X]^2$ with the "half-graph" $X \circledast X$, where $X \circledast Y := \{(\alpha, \beta) \in X \times Y \mid \alpha < \beta\}$.

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Example (3)

Any undirected graph $\mathcal{G} = \langle V, E\rangle$ induces a coloring $c_{\mathcal{G}} : [V]^2 \to \{0, 1\}$ by letting $c_{\mathcal{G}}(\{x, y\}) := 1$ iff $\{x, y\} \in E$.

Special type of sets (w.r.t. a given coloring)

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Example (2)

If $\langle P, <\rangle$ is a poset and $\mathcal{G}$ is its corresponding comparability graph, then any monochromatic set $A \subseteq P$ for $c_{\mathcal{G}}$, is either a chain or an antichain.

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Example (3)

If $\langle G, +\rangle$ is an Abelian group, and $A \subseteq G$ is omnichromatic for $+ : [G]^2 \to G$, then A is in particular a generating set, i.e., $\langle A\rangle = G$.

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- rainbow for c, if $c \restriction [A]^\mu$ is injective.

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Example (4)

If $\langle G, + \rangle$ is an Abelian group, then every triangle is rainbow for $+$:

$$|\{x, y, z\}| = 3 \quad \Longrightarrow \quad |\{x + y, x + z, y + z\}| = 3.$$

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Example (5)

If $\langle X, d \rangle$ is an ultrametric space, then d admits no rainbow triangles.

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$\mathsf{MA}_{\aleph_1}$ *entails that the product of any two ccc posets is ccc.*

Theorem (Shelah, 1980)

There exists an uncountable group all of whose proper subgroups are countable.

Coloring objects of size $\aleph_0$

Pigeonhole Principle

$\aleph_0 \rightarrow [\aleph_0]^1_2$. *That is, every 1-dimensional coloring* $c : [\aleph_0]^1 \rightarrow \{0,1\}$ *admits a monochromatic set of size* $\aleph_0$.

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Theorem (Ramsey, 1930)

$\aleph_0 \to [\aleph_0]^2_2$. *That is, every 2-dimensional coloring* $c : [\aleph_0]^2 \to \{0, 1\}$ *admits a monochromatic set of size* $\aleph_0$.

In particular, every infinite poset contains an infinite chain, or an infinite antichain.

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Theorem (Erdős-Rado, 1952 [special case])

$\aleph_0 \not\rightarrow [\aleph_0]^\omega_2$. *That is,* $\exists$ *an* ω*-dimensional coloring* $c : [\aleph_0]^\omega \rightarrow \{0,1\}$ *that does not admit a monochromatic set of size* $\aleph_0$.

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Proof.

Fix a nontrivial ultrafilter $\mathcal{U}$ over ω, and derive a coloring c as follows. Given an infinite set A, with increasing enumeration $\{a_n \mid n < \omega\}$, set $I_A := [a_0, a_1) \uplus [a_2, a_3) \uplus [a_4, a_5) \uplus \ldots$, and let $c(A) := 1$ iff $I_A \in \mathcal{U}$. □

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Algebraic Pigeonhole Principle (Hindman, 1974)

$\mathbb{N} \to [\aleph_0]^{\mathrm{FS}}_2$. *That is, every 1-dimensional coloring* $c : \mathbb{N} \to \{0, 1\}$ *admits a set* A *of size* $\aleph_0$ *which is monochromatic with respect to finite sums: there is* $i \in \{0, 1\}$ *such that for every finitely many elements of* A, $a_1 < \ldots < a_k$, $c(a_1 + \cdots + a_k) = i$.

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Topological-strong coloring (Baumgartner, 1986)

$\mathbb{Q} \nrightarrow [\text{top}\,\mathbb{Q}]^2_{\aleph_0}$. *That is,* $\exists$ *a 2-dimensional coloring* $c : [\mathbb{Q}]^2 \to \aleph_0$ *such that any topological copy of* $\mathbb{Q}$ *is omnichromatic.*

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Proof.

Fix a partition of $\aleph_1$ into pairwise disjoint stationary sets, $\langle S_\gamma \mid \gamma < \aleph_1 \rangle$, and derive a coloring $c : [\aleph_1]^\omega \to \aleph_1$ as follows.

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Given an uncountable $B \subseteq \aleph_1$, the set $\{\delta < \aleph_1 \mid \sup(B \cap \delta) = \delta\}$ is a club. □

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Given an uncountable $B \subseteq \aleph_1$, the set $\{\delta < \aleph_1 \mid \sup(B \cap \delta) = \delta\}$ is a club.
So for each color $\gamma < \aleph_1$, we may find $\delta \in S_\gamma$ such that $\sup(B \cap \delta) = \delta$.
Pick $A \in [B]^\omega$ with $\sup(A) = \delta$; then $c(A) = \gamma$. □

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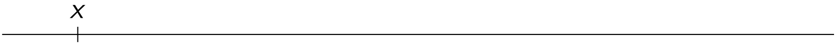

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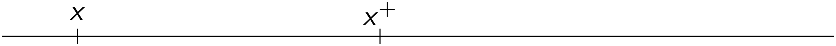

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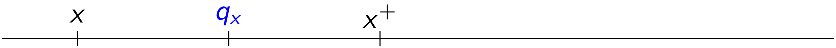

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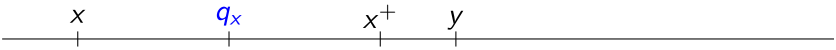

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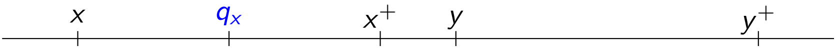

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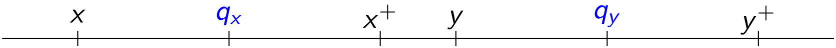

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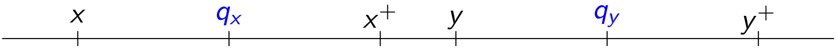

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The argument for the non-existence of an uncountable antichain is similar.

Sierpiński's poset (cont.)

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An uncountable poset with no uncountable chains or antichains.

Theorem (Sierpiński, 1933)

A Sierpiński poset exists. In particular, $\aleph_1 \nrightarrow [\aleph_1]^2_2$.

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Exercise

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In particular, $t_\alpha{}^\frown\langle\gamma\rangle \subseteq t_\beta$, so that $c(\alpha, \beta) = t_\beta(\alpha) = \gamma$. □

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A: Yes, it does! This is due to Todorcevic (1987).

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The more pairwise orthogonal orderings there are, the more colors we get!

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Moore (2006) demonstrated that the class of uncountable linear orderings can consistently have a finite basis. This shows that Sierpiński's argument cannot be strengthened to yield $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0}$.

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An application in the theory of large cardinals

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If $\kappa > \mu$ are infinite regular cardinals, then $\kappa \nrightarrow [\kappa]^{\mu}_{\kappa}$. I.e., there exists a coloring $c : [\kappa]^{\mu} \rightarrow \kappa$ such that any cofinal $A \subseteq \kappa$ is omnichromatic.

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Proof.

Write $\delta := \mathrm{crit}(j)$. Let λ be the first cardinal $> \delta$ such that $j(\lambda) = \lambda$.
That is, $\lambda = \sup\{\delta, j(\delta), j(j(\delta), j(j(j(\delta))), \ldots\}$.

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Write $\delta := \mathrm{crit}(j)$. Let λ be the first cardinal $> \delta$ such that $j(\lambda) = \lambda$.
That is, $\lambda = \sup\{\delta, j(\delta), j(j(\delta), j(j(j(\delta))), \ldots\}$.
Fix a strong coloring $c : [\lambda^+]^{\omega} \to \lambda^+$ using $\kappa = \lambda^+$ and $\mu = \omega$ above.
Towards a contradiction, suppose that $j``\lambda^+ \in M$.

□

An application in the theory of large cardinals

The partition-into-stationary-sets argument

If $\kappa > \mu$ are infinite regular cardinals, then $\kappa \nrightarrow [\kappa]^{\mu}_{\kappa}$. I.e., there exists a coloring $c : [\kappa]^{\mu} \rightarrow \kappa$ such that any cofinal $A \subseteq \kappa$ is omnichromatic.

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Pick $y \in [j``\lambda^+]^{\omega}$ such that $d(y) = \delta$. Let $x \in [\lambda^+]^{\omega}$ be s.t. $j``x = y$.
By $j(\omega) = \omega$, we have $j(x) = y$. Then $j(c(x)) = j(c)(j(x)) = d(y) = \delta$, contradicting the fact that $\delta \notin \mathrm{Im}(j)$. □

An application in graph theory

A graph

A graph is a structure $\mathcal{G} = \langle V, E \rangle$, where $E \subseteq [V]^2$.
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The chromatic number of $\mathcal{G} = \langle V, E \rangle$, denoted $\mathrm{Chr}(\mathcal{G})$, is the least cardinal κ for which V may be covered by κ many E-independent sets.
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If $\mathcal{G}$ is a graph, k is a some positive integer, and all finite subgraphs of $\mathcal{G}$ have chromatic number $\leq k$, then $\mathrm{Chr}(\mathcal{G}) \leq k$.

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Theorem (Foreman-Laver, 1988)

Relative to a large cardinal hypothesis, it is consistent that every graph of size and chromatic number $\aleph_2$ contains a subgraph of size and chromatic number $\aleph_1$.

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Terminology

Say that a is 0-monochromatic iff $c``[a]^2 \subseteq \{0\}$.

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Pick $\alpha < \beta$ both from A_i such that $c(\alpha, \beta) = 1$.
Then $\{\alpha, \beta\} \in E$, contradicting the fact that A_i is E-independent. $\square$

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by $a, b \in L_n$, we have $a \cup b \in \mathbb{P}_\delta$, so that $c(\alpha, \beta) \in c``[a \cup b]^2 \subseteq \{0\}$.
Thus, $c``[A_n]^2 \subseteq \{0\}$.

An application in graph theory (cont.)

Lemma 0

Suppose that for every uncountable $\mathcal{A} \subseteq [\mathfrak{c}]^{<\omega}$ consisting of pairwise disjoint 0-monochromatic sets, there exist $a \neq b$ in $\mathcal{A}$ s.t. $a \cup b$ is 0-mono. Then $\mathrm{Chr}(\mathcal{G} \restriction \delta) \leq \aleph_0$ for all $\delta < \mathfrak{c}$.

Proof.

Fix $\delta < \mathfrak{c}$. Let $\mathbb{P}_\delta$ consists of all $a \in [\delta]^{<\omega}$ such that $c``[a]^2 \subseteq \{0\}$.
By the hypothesis of the lemma and the Δ-system lemma, $(\mathbb{P}_\delta, \supseteq)$ is *ccc*.
Since MA implies that every *ccc* poset of size $< \mathfrak{c}$ is σ-linked, we have that $\mathbb{P}_\delta = \bigcup_{n<\omega} L_n$, where every two elements of L_n admit a bound in $\mathbb{P}_\delta$.
For each $n < \omega$, put $A_n := \bigcup L_n$. So $\delta = \bigcup \mathbb{P}_\delta = \bigcup \bigcup_{n<\omega} L_n = \bigcup_{n<\omega} A_n$.
For all $\alpha, \beta \in A_n$, there exist $a, b \in L_n$ such that $\alpha \in a$ and $\beta \in b$;
by $a, b \in L_n$, we have $a \cup b \in \mathbb{P}_\delta$, so that $c(\alpha, \beta) \in c``[a \cup b]^2 \subseteq \{0\}$.
Thus, $c``[A_n]^2 \subseteq \{0\}$.
So A_n is E-independent for all $n < \omega$, testifying that $\mathrm{Chr}(\mathcal{G} \restriction \delta) \leq \aleph_0$. $\square$

A strong coloring from a $\mathfrak{b}$-scale

Definition

- For $f, g \in {}^{\omega}\omega$, let $f \leq^m g$ iff $f(n) \leq g(n)$ for all $n \in [m, \omega)$;

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A strong coloring from a $\mathfrak{b}$-scale (cont.)

Fix a $\mathfrak{b}$-scale, $\vec{f} = \langle f_\alpha \mid \alpha < \mathfrak{b} \rangle$.
Derive a coloring $c : [\mathfrak{b}]^2 \to \{0, 1\}$ by letting for all $\alpha < \beta < \mathfrak{b}$:

$$c(\alpha, \beta) := 1 \text{ iff } f_\alpha \leq^0 f_\beta.$$

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1. *For every cofinal $A \subseteq \mathfrak{b}$, there are $\alpha < \beta$ in A with $c(\alpha, \beta) = 1$;*
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The proof is illuminating, and will be presented on Thursday.

The special case $\mathfrak{b} = \aleph_1$

Oscillation of a $\mathfrak{b}$-scale

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Corollary (Todorcevic, 1986)

If $\mathfrak{b} = \aleph_1$, then there exists a *ccc* poset which is not Knaster.

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- By Clause (2), $(\mathbb{P}, \supseteq)$ is *ccc*;

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- By Clause (2), $(\mathbb{P}, \supseteq)$ is *ccc*;
- By Clause (1), $(\mathbb{P}, \supseteq)$ is not Knaster. □