A relative of the approachability ideal, diamond and non-saturation

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Diamond on successor cardinals

Definition (Jensen, ‘72). For a cardinal λ, and a stationary set $S \subseteq \lambda^+$, $\diamondsuit(S)$ asserts the existence of a collection $\{A_\alpha \mid \alpha \in S\}$ such that $\{\alpha \in S \mid A \cap \alpha = A_\alpha\}$ is stationary for all $A \subseteq \lambda^+$.

Observation. $\diamondsuit(S) \Rightarrow \diamondsuit(\lambda^+) \Rightarrow 2^\lambda = \lambda^+$.

Questions. 1. Does $2^\lambda = \lambda^+$ imply $\diamondsuit(\lambda^+)$?
2. What about $\diamondsuit(S)$ for a particular S?

History of the problem, I

Let $E^{\lambda^+}_{\kappa} := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \kappa\}$,
and $E^{\lambda^+}_{\neq\kappa} := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) \neq \kappa\}$.

Theorem (Jensen, '74). $2^{\aleph_0} = \aleph_1 \not\Rightarrow \diamondsuit(\aleph_1)$.

Theorem (Gregory, '76). $2^{\aleph_1} = \aleph_2 \Rightarrow \diamondsuit(\aleph_2)$ *provided* that CH holds.

More specifically, CH + $2^{\aleph_1} = \aleph_2$ entails:

$$\diamondsuit(S) \text{ for every stationary } S \subseteq E^{\aleph_2}_{\aleph_0}.$$

History of the problem, II

Theorem (Shelah, '78). Assume GCH. Then for every uncountable cardinal λ:

$$\diamondsuit(S) \text{ for every stationary } S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}.$$

Since then, a chain of results of Shelah recently culminated in:

Theorem (Shelah, 2008). If $2^\lambda = \lambda^+$, then:

$$\diamondsuit(S) \text{ for every stationary } S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}.$$

In particular, for every uncountable cardinal λ:

$$2^\lambda = \lambda^+ \iff \diamondsuit(\lambda^+).$$

Refining the question, I

Refined Question. Suppose $2^\lambda = \lambda^+$ for an uncountable cardinal, λ;

For which $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$, must $\diamondsuit(S)$ hold?

Theorem (Shelah, '80). For every regular uncountable cardinal, λ:

$$\mathsf{GCH} + \neg\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)}) \text{ is consistent.}$$

Theorem (Shelah, '84). For every singular cardinal, λ, for some non-reflecting stationary set $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$:

$$\mathsf{GCH} + \neg\diamondsuit(S) \text{ is consistent.}$$

Refining the question, II

We shall say that $S \subseteq \lambda^+$ *reflects (stationarily often)* iff the following set is stationary:

$$\mathrm{Tr}(S) := \{\gamma < \lambda^+ \mid \mathrm{cf}(\gamma) > \omega, S \cap \gamma \text{ is stationary}\}.$$

Refined Question (final form). Suppose $2^\lambda = \lambda^+$ for a singular λ, and $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects, must $\diamondsuit(S)$ hold?

Jensen's notion of weak square

Fact (Jensen '72). $\square^*_\lambda$ is equivalent to the existence of a special Aronszajn tree of height λ^+.

For the protocol, we also give the original definition:

Definition. For a cardinal λ, $\square^*_\lambda$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that:
(1) for all limit $\alpha < \lambda^+$, C_α is a club of α, otp$(C_\alpha) \leq \lambda$;
(2) $|\{C_\alpha \cap \delta \mid \alpha < \lambda^+\}| \leq \lambda$ for all $\delta < \lambda^+$.

History of the problem, III

Theorem (Shelah, '84). If $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ, and $\square^*_\lambda$ holds, then $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

Theorem (Zeman, 2008). If $2^\lambda = \lambda^+$ for a singular cardinal λ, and $\square^*_\lambda$ holds, then $\diamondsuit(S)$ for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

aims and hopes

✔ Reducing the $\square^*_\lambda$ hypothesis

✔ Studying the effect of cardinals $< \lambda$ to this problem

✔ Studying stronger principles (such as $\diamondsuit^*_{\lambda^+}$), and weaker principles (such as non-saturation)

✔ Obtaining a local information on the validity of $\diamondsuit(S)$ on a particular set, S

✗ Proving "$\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ for every singular cardinal λ" just from GCH

Reducing weak square & obtaining local information

Shelah's weak approachability ideal

Let λ denote a singular cardinal.

Definition. $d : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ is a *distance function* iff
1) $\alpha < \beta < \gamma < \lambda^+$ implies $d(\alpha,\gamma) \leq \max\{d(\alpha,\beta), d(\beta,\gamma)\}$;
2) $\{\alpha < \gamma \mid d(\alpha,\gamma) \leq i\}$ has size $< \lambda$ for all $\gamma < \lambda^+$.

Definition (Shelah). A set $T \subseteq \lambda^+$ is in $I[\lambda^+;\lambda]$ iff there exists a club $C \subseteq \lambda^+$ and a distance function, d, such that for all $\gamma \in T \cap C \cap E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$:

$$\exists A_\gamma \subseteq \gamma \text{ cofinal, with } \sup(d``[A_\gamma]^2) < \mathrm{cf}(\lambda).$$

A relative of approachability ideal

Definition (Shelah). A set $T \subseteq \lambda^+$ is in $I[\lambda^+;\lambda]$ iff there exists a club $C \subseteq \lambda^+$ and a distance function, d, such that for all $\gamma \in T \cap C \cap E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$:

$$\exists A_\gamma \subseteq \gamma \text{ cofinal} \wedge \sup(d``[A_\gamma]^2) < \mathrm{cf}(\lambda).$$

We now consider a local version for a particular $S \subseteq \lambda^+$.

Definition. A set $T \subseteq \mathrm{Tr}(S)$ is in $I[S;\lambda]$ iff there exists a club $C \subseteq \lambda^+$ and a distance function, d, such that for all $\gamma \in T \cap C \cap E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$:

$$\exists S_\gamma \subseteq S \cap \gamma \text{ stationary} \wedge \sup(d``[S_\gamma]^2) < \mathrm{cf}(\lambda).$$

Lemma. If $S \subseteq E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, then $I[S;\lambda] = I[\lambda^+;\lambda] \restriction \mathrm{Tr}(S)$.

Consequences of the new ideal

The new ideal indeed supplies local information on the validity of diamond and related principles.

Theorem. If $I[S;\lambda]$ contains a stationary set, then

$$2^\lambda = \lambda^+ \Rightarrow \diamondsuit(S).$$

Theorem. If $I[S;\lambda]$ contains a stationary set, then $\mathsf{NS}_{\lambda^+} \restriction S$ is non-saturated.

A comparison with weak square

Let λ denote a singular cardinal, and let $S \subseteq \lambda^+$.

Observation. If $I[S;\lambda]$ contains a stationary set, then S reflects.

Proposition. Assume $\square^*_\lambda$. If S reflects, then $I[S;\lambda]$ contains a stationary set.

Theorem. It is relatively consistent with the existence of a supercopmact cardinal that $\square^*_\lambda$ fails, while $I[S;\lambda]$ contains a stationary set for every $S \subseteq \lambda^+$ that reflects.

Stationary Approachability Property

Definition. For a singular cardinal, λ, SAP_λ asserts that $I[S;\lambda]$ contains a stationary set for every $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflects.

By the previous slide, SAP_λ is strictly weaker than $\square^*_\lambda$.

Remark. For a strong limit singular cardinal, λ, AP_λ is (equivalent to) the assertion that $\lambda^+ \in I[\lambda^+;\lambda]$.

The effect of smaller cardinals

A shift in focus

Instead of studying the validity of $\diamondsuit(S)$, we now focus on finding sufficient conditions for $I[S;\lambda]$ to contain a stationary set.

This yields a linkage between virtually unrelated objects.

Theorem. Assume GCH and that κ is a successor cardinal with no κ^+-Souslin trees.

Then $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds for the class of singular cardinals λ of cofinality κ.

let us explain how small cardinals effects λ..

The effect of smaller cardinals, I

Definition. Assume $\theta > \kappa > \omega$ are regular cardinals.

$R_1(\theta, \kappa)$ asserts that for every function $f : E^\theta_{<\kappa} \to \kappa$, there exists some $j < \kappa$ such that:

$$\{\delta \in E^\theta_\kappa \mid f^{-1}[j] \cap \delta \text{ is stationary}\} \text{ is stationary.}$$

Facts. 1. $\square_\kappa \Rightarrow \neg R_1(\kappa^+, \kappa)$;
2. every stationary subset of $E^{\kappa^{++}}_\kappa$ reflects $\Rightarrow R_1(\kappa^{++}, \kappa^+)$;
3. By Harrington-Shelah '85, $R_1(\aleph_2, \aleph_1)$ is equiconsistent with the existence of a Mahlo cardinal.

The effect of smaller cardinals, II

Theorem. Suppose $\lambda > \mathrm{cf}(\lambda) = \kappa > \omega$;
If there exists a regular $\theta \in (\kappa, \lambda)$ such that $R_1(\theta, \kappa)$ holds, then $I[E^{\lambda^+}_{\mathrm{cf}(\lambda)}; \lambda]$ contains a stationary set.

Corollary. Suppose κ is a regular cardinal and every stationary subset of $E^{\kappa^{++}}_{\kappa}$ reflects.

Then $2^\lambda = \lambda^+ \Rightarrow \diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ for the class of singular cardinals λ of cofinality κ^+.

Corollary. Assume Martin's Maximum (or just PFA^+); $\diamondsuit(E^{\lambda^+}_{\mathrm{cf}(\lambda)})$ holds for every λ strong limit of cofinality ω_1.

The effect of smaller cardinals, III

Definition. Assume $\theta > \kappa > \omega$ are regular cardinals.

$R_2(\theta, \kappa)$ asserts that for every function $f : E^\theta_{<\kappa} \to \kappa$, there exists some $j < \kappa$ such that:

$\{\delta \in E^\theta_\kappa \mid f^{-1}[j] \cap \delta$ is non-stationary$\}$ is non-stationary.

Facts. 1. $R_2(\theta, \kappa) \Rightarrow R_1(\theta, \kappa)$ and hence the strength of $R_2(\kappa^+, \kappa)$ is at least of a Mahlo cardinal.
2. By Magidor '82, $R_2(\aleph_2, \aleph_1)$ is relatively consistent with the existence of a weakly compact cardinal.

Remark. The exact strength of $R_2(\aleph_2, \aleph_1)$ is unknown.

The effect of smaller cardinals, IV

Theorem. Suppose $\lambda > \mathrm{cf}(\lambda) = \kappa > \omega$;
If there exists a regular $\theta \in (\kappa, \lambda)$ such that $R_2(\theta, \kappa)$ holds, then $\mathrm{Tr}(S) \cap E^{\lambda^+}_\theta \in I[S; \lambda]$ for every $S \subseteq \lambda^+$.

Corollary. Suppose $R_2(\theta, \kappa)$ holds.
For every sing. cardinal λ of cofinality κ with $2^\lambda = \lambda^+$:

$$\diamondsuit(S) \text{ holds whenever } \mathrm{Tr}(S) \cap E^{\lambda^+}_\theta \text{ is stationary.}$$

Remark. The $R_2(\cdot, \cdot)$ proof resembles the one of an analogous theorem by Viale-Sharon concerning the weak approachability ideal. The $R_1(\cdot, \cdot)$ proof builds on a fundamental fact from Shelah's *pcf* theory.

The effect of smaller cardinals, V

A surprising link between singular cardinals and smaller cardinals is the following.

Theorem. It is relatively consistent with the existence of two supercompact cardinals that there exists a *cofinality-preserving* forcing of size $\aleph_3$ that introduces a special Aronszajn tree of height $\aleph_{\omega_1+1}$.

The effect of smaller cardinals, VI

Theorem. It is relatively consistent with the existence of two supercompact cardinals that there exists a *cofinality-preserving* forcing of size $\aleph_3$ that introduces a special Aronszajn tree of size $\aleph_{\omega_1+1}$.

Idea of the proof: It is possible to kill $\square^*_{\aleph_{\omega_1}}$ in such a way that all that is needed to recover it, is a certain weakening of $R_2(\aleph_2, \aleph_1)$. Now use the fact that, with a right preparation, this particular weakening can be obtained via a cofinality-preserving small forcing.

A stronger guessing principle

A stronger guessing principle, I

Definition (Jensen, '72). For a cardinal λ, $\diamondsuit^*(\lambda^+)$ asserts the existence of a collection $\{\mathcal{A}_\alpha \mid \alpha \in S\}$ with $|\mathcal{A}_\alpha| \leq \lambda$, such that $\{\alpha < \lambda^+ \mid A \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club for all $A \subseteq \lambda^+$.

Theorem (Kunen, mid '70s). $\diamondsuit^*(\lambda^+) \Rightarrow \diamondsuit(S)$ for all stationary $S \subseteq \lambda^+$.

Discussion. Suppose λ is a singular strong limit. Taking into account Shelah's λ-distributive, λ^{++}-c.c. notion of forcing for killing $\diamondsuit(S)$ on $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that does not reflect, if we would like to establish $\diamondsuit^*(\lambda^+)$ from cardinal arithmetic, we need to assume that every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects.

A stronger guessing principle, II

Definition. Refl(S) denotes the assertion that every stationary subset of S reflects.

Theorem. For λ singular, we have:
1. $\mathsf{GCH} + \mathsf{Refl}(E^{\lambda^+}_{\mathsf{cf}(\lambda)}) + \square^*_\lambda \Rightarrow \diamondsuit^*(\lambda^+)$;
2. $\mathsf{GCH} + \mathsf{Refl}(E^{\lambda^+}_{\mathsf{cf}(\lambda)}) + \mathsf{SAP}_\lambda \not\Rightarrow \diamondsuit^*(\lambda^+)$;
3. $\mathsf{GCH} + \mathsf{Refl}(E^{\lambda^+}_{\mathsf{cf}(\lambda)}) + \mathsf{SAP}_\lambda \Rightarrow \diamondsuit(S)$ for every stationary $S \subseteq \lambda^+$.

Remark. here, the non-implication symbol, $\not\Rightarrow$, is a slang for a consistency result modulo the existence of a supercompact cardinal.

Reflection and weak square, I

It is well-known that $\square_\lambda$ entails the existence of a non-reflecting stationary subset of λ^+.

By Cummings-Foreman-Magidor 2001, it is consistent that $\square^*_{\aleph_\omega}$ holds, while every stationary subset of $\aleph_{\omega+1}$ reflects.

Still, we have the following:

Proposition. Assume GCH and $\square^*_\lambda$ for a singular λ.

Adding a λ^+-Cohen set introduces a non-reflecting stationary subset of λ^+.

This gives a new explanation of Shelah's theorem that if $\lambda > \kappa > \mathrm{cf}(\lambda)$ and κ is λ^+-supercompact, then $\square^*_\lambda$ fails.

Reflection and weak square, II

Proposition. Assume GCH and $\square^*_\lambda$ for a singular λ.

Adding a λ^+-Cohen set introduces a non-reflecting stationary subset of λ^+.

Proof. Work in $V[G]$, where G is Add(λ^+,λ^{++})-generic over V. Clearly, $\diamondsuit^*_{\lambda^+}$ fails. By $\square^*_\lambda$ + GCH, and the previous theorem, this must mean that there exists a stationary subset $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that does not reflect. By $|S| = \lambda^+$, we get that $S \in V[G \restriction \mathrm{Add}(\lambda^+,\alpha)]$ for some $\alpha < \lambda^{++}$. Since Add(λ^+,λ^{++}) is homogenous and Add$(\lambda^+,\alpha) \simeq$ Add$(\lambda^+,1)$, we get the conclusion of the theorem. $\square$

Open problems

Open problems

Question 1. For a singular cardinal λ, must $I[E^{\lambda^+}_{\mathrm{cf}(\lambda)};\lambda]$ contain a stationary set?

To compare, Shelah proved that $I[\lambda^+;\lambda] \restriction E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ indeed contains a stationary set.

Question 2. Same as Question 1 for $\mathrm{cf}(\lambda) \leq \omega_1$ under PFA.

Thank you!