The extent of the failure of Ramsey's theorem at successor cardinals

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Ramsey's theorem

The arrow notation

Let $\lambda \to (\lambda)^2_\kappa$ denote the assertion:
For every function $f : [\lambda]^2 \to \kappa$, there exists a subset $H \subseteq \lambda$ s.t.:

- $|H| = \lambda$;
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Sierpiński's theorem is pleasing on its own! It tells us that $[\omega_1]^2$ admits a rather wild 2-valued coloring.

Generalizing Sierpiński

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So, there exists a coloring $f : [\omega_1]^2 \to 2$ such that $[X]^2$ attains all colors for every uncountable $X \subseteq \omega_1$. This raises the question of whether an analogous statement concerning a coloring with more than two colors is valid.

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Question: May the cardinal arithmetic hypothesis be eliminated?

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The rectangular square-bracket relation

Negative square-bracket relation

$\lambda \not\to [\lambda]^2_\kappa$ asserts the existence of a function $f : [\lambda]^2 \to \kappa$ such that for every subset $X \subseteq \lambda$: if $|X| = \lambda$, then $f \restriction [X]^2$ is **onto** κ.

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$\lambda \not\to [\lambda; \lambda]^2_\kappa$ asserts the existence of a function $f : [\lambda]^2 \to \kappa$ s.t. for every subsets X, Y: if $|X| = |Y| = \lambda$, then $f \restriction (X \circledast Y)$ is **onto** κ.

The rectangular square-bracket relation (Cont.)

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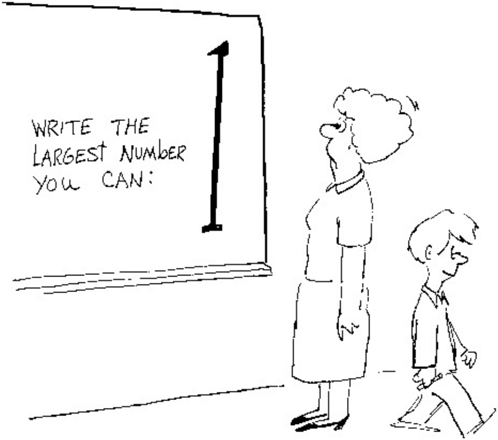
Negative square-bracket for higher cardinals



The rectangular square-bracket relation for higher cardinals

Theorem (Erdös-Hajnal-Rado, 1965)

$2^\lambda = \lambda^+$ *entails* $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$.

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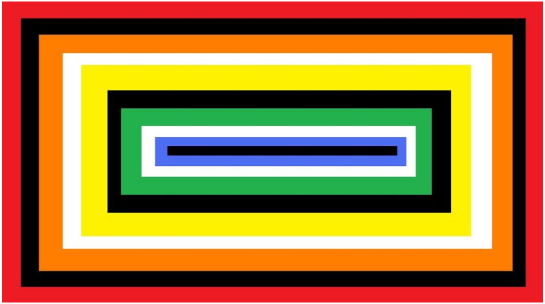
Open Problems

1. Does $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$ hold for every singular cardinal λ?
2. Does $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$ entail $\lambda^+ \not\to [\lambda^+;\lambda^+]^2_{\lambda^+}$?

A Solution to Problem #2



Main result: comparing squares with rectangles

Theorem

The following are equivalent for all cardinals λ, κ:

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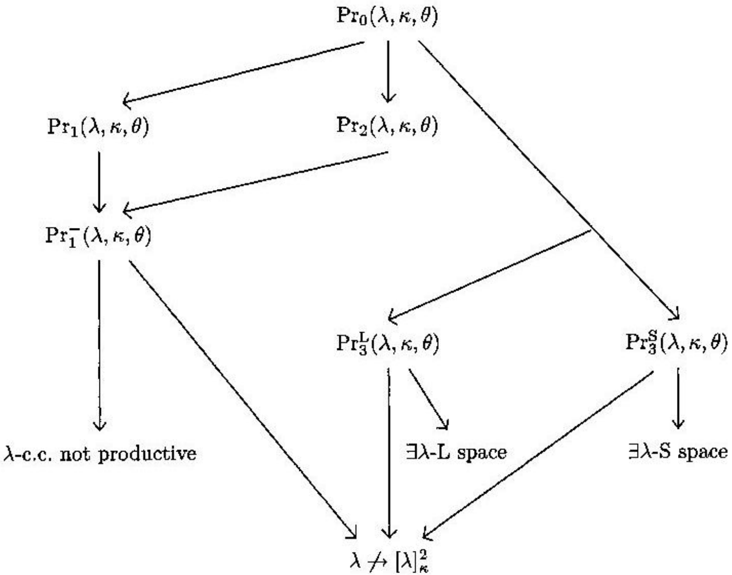
The above is a corollary of the following ZFC theorem.

Main technical result

Every infinite cardinal λ admits a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ s.t.: for every cofinal subsets X, Y of λ^+, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \circledast Y] \supseteq Z \circledast Z$.

Shelah's study of strong colorings



Comparing classic concepts with modern one

Our main technical result was the missing link to the following.

Corollary (Eisworth+Shelah+R.)

TFAE for every uncountable cardinal λ:

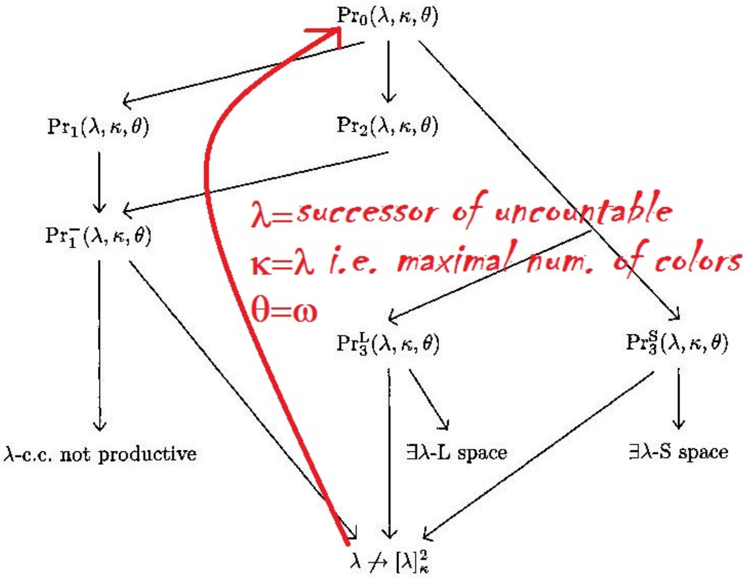
- $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$
- $\mathrm{Pr}_0(\lambda^+, \lambda^+, \omega)$

Definition (Shelah)

$\mathrm{Pr}_0(\lambda^+, \lambda^+, \omega)$ asserts the existence of a function $f : [\lambda^+]^2 \to \lambda^+$ satisfying the following.
For every $n < \omega$, every $g : n \times n \to \lambda^+$, and every collection $\mathcal{A} \subseteq [\lambda^+]^n$ of mutually disjoint sets, of size λ^+,
there exists some $x, y \in A$ with $\max(x) < \min(y)$ such that

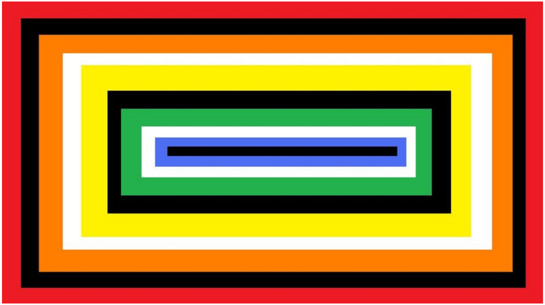
$$f(x(i), y(j)) = g(i, j) \text{ for all } i, j < n.$$

Surprise, Surprise!!



Ingredients of the proof

Case 1. Successors of singulars



Successor of singulars — in ZFC

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- Shelah's club guessing theorems, and Eisworth's theorem on the existence of off-center club guessing matrices for singular cardinals of countable cofinality;

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- A generalization of Todorčević method of walks on ordinals, where each ordinal α admits a sequence of clubs, $\langle C^i_\alpha \mid i < \mathrm{cf}(\lambda)\rangle$, rather than a single one;

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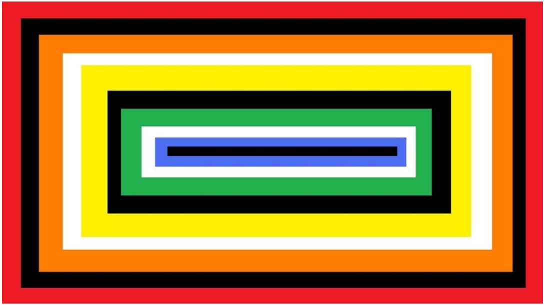
- Oscillation theory of *pcf* scales, plus coding, from which one can get essentially-generic guidelines on which clubs to visit throughout the generalized walks, and moreover, which ordinals to pick from these walks.

Ingredients of the proof

Case 2. Successors of regulars



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Remark

The proofs of 3,4,5 are entirely different, and it was unknown whether a uniform proof of $3+4+5$ exists.

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1. Moore's proof involves the definition of a function $o : [\omega_1]^2 \to \omega$ that witnesses $\omega_1 \not\to [\omega_1;\omega_1]^2_\omega$. (Then, a stretching argument yields $\omega_1 \not\to [\omega_1;\omega_1]^2_{\omega_1}$.)

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4. While $\lambda^+ \nrightarrow [\lambda^+;\lambda^+]^2_\omega$ has been established previously using other functions, the generalized o is the first function that is known to have this successful composition property.

Thank you!

The slides of this talk may be found at the following address:
http://assafrinot.com/talks/asl2012

More on successor of singulars — in ZFC

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If λ *is a singular cardinal of countable cofinality, then* $E^{\lambda^+}_{\omega_1}$ *carries a club-guessing* <u>*matrix*</u> *of a very strong form.*

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Still Open
Whether $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$ hold for all singular λ, in ZFC.

Transforming Rectangles into Squares — in ZFC

Main technical result

For every singular cardinal λ, there exists a function $rts : [\lambda^+]^2 \rightarrow [\lambda^+]^2$ such that for every cofinal subsets X, Y of λ^+, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \circledast Y] \supseteq Z \circledast Z$.

Remark: our proof builds heavily on previous arguments of Shelah, Todorčević, and most notably — Eisworth.

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- Let $\beta_0 := \beta$, and $\beta_{n+1} := \min(C^{\sigma(n)}_{\beta_n} \setminus \alpha)$ for all $n \in \mathrm{dom}(\sigma)$;

Transforming Rectangles into Squares (Cont.)

The definition of *rts*

- Fix a matrix of local clubs $\langle C^i_\alpha \mid \alpha < \lambda^+, i < \mathrm{cf}(\lambda)\rangle$ that incorporates a club-guessing sequence/matrix;
- Fix a function $f : [\lambda^+]^2 \to {}^{<\omega}\mathrm{cf}(\lambda) \times {}^{<\omega}\mathrm{cf}(\lambda)$ with strong coloring properties;
- Given $\alpha < \beta < \lambda^+$, consider $(\sigma, \eta) = f(\alpha, \beta)$;
- Let $\beta_0 := \beta$, and $\beta_{n+1} := \min(C^{\sigma(n)}_{\beta_n} \setminus \alpha)$ for all $n \in \mathrm{dom}(\sigma)$;

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The definition of *rts* is quite natural in this context, and so the main point is to verify that the definition does the job.

Why does *rts* work

- For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type p in the language of the matrix-based walks, let $X_p(\delta) := \{\alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p\}$;

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- Conclude that $rts[X \circledast Y] \supseteq [S_p^X \cap S_p^Y \cap C]^2$ for the club C of ordinals of the form $M \cap \lambda^+$, for elementary submodels $M \prec H_\chi$ of size λ, that contains all relevant objects.