Splitting a stationary set: Is there another way?

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This talk is based on a joint work with Maxwell Levine.
Conventions

- \( \kappa \) denotes a regular uncountable cardinal;
- \( \lambda \) denotes an infinite cardinal;
- \( \text{Reg}(\kappa) := \{ \lambda < \kappa \mid \aleph_0 \leq \text{cf}(\lambda) = \lambda \} \);
- \( E^\kappa_\lambda := \{ \alpha < \kappa \mid \text{cf}(\alpha) = \lambda \} \);
- \( E^\kappa_\neq \lambda, E^\kappa_\geq \lambda \) and \( E^\kappa_\succ \lambda \) are defined analogously;
- \( \text{acc}^+(A) := \{ \alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha > 0 \} \).
Partitioning a stationary set

Theorem (Solovay, 1971)

For every stationary $S \subseteq \kappa$, there exists a partition $\langle S_i \mid i < \kappa \rangle$ of $S$ into stationary sets.
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Solovay’s theorem has countless applications in Set Theory. For instance, it plays a role in the proof of strong negative partition relations of the form $\kappa \rightarrow [\kappa]^2_\kappa$, and variations of it are missing for the sought proof that successors of a singular cardinals cannot be Jónsson.
Variations of Solovay’s theorem

Variation I (Brodsky-Rinot, 2019)

For every $\theta \leq \kappa$ and a sequence $\langle S_i \mid i < \theta \rangle$ of stationary subsets of $\kappa$, there exists a cofinal $I \subseteq \theta$ and pairwise disjoint stationary sets $\langle T_i \mid i \in I \rangle$ such that $T_i \subseteq S_i$ for all $i \in I$. 
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Variation I (Brodsky-Rinot, 2019)

For every $\theta \leq \kappa$ and a sequence $\langle S_i \mid i < \theta \rangle$ of stationary subsets of $\kappa$, there exists a cofinal $l \subseteq \theta$ and pairwise disjoint stationary sets $\langle T_i \mid i \in l \rangle$ such that $T_i \subseteq S_i$ for all $i \in l$.

Variation II (Magidor?, 1970’s)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of $S$ into stationary sets such that, for all $i < \lambda^+$, $S_i$ does not reflect.
Variations of Solovay’s theorem

Definition
For $S \subseteq \kappa$, let $\text{Tr}(S) : = \{ \beta \in E^\kappa_{\geq \omega} \mid S \cap \beta \text{ is stationary in } \beta \}.$

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If $\Box_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of $S$ into stationary sets such that, for all $i < \lambda^+$, $S_i$ does not reflect (i.e., $\text{Tr}(S_i) = \emptyset$).
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⊕ Nonreflecting stationary sets are very useful. To exemplify:
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Theorem (Shelah, 1991)

If \( \kappa > \aleph_2 \), and \( E^{\kappa}_{\geq \aleph_2} \) admits a nonreflecting stationary set, then there exists a \( \kappa \)-cc poset whose square is not \( \kappa \)-cc.
Variations of Solovay’s theorem

Variation III (Brodsky-Rinot, 2019)

If $\square(\kappa)$ holds, then for every fat $F \subseteq \kappa$, there is a partition $\langle F_i \mid i < \kappa \rangle$ of $F$ into fat sets such that, for all $i < j < \kappa$, $\text{Tr}(F_i) \cap \text{Tr}(F_j) = \emptyset$.

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If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of $S$ into stationary sets such that, for all $i < \lambda^+$, $S_i$ does not reflect (i.e., $\text{Tr}(S_i) = \emptyset$).

⇑ Nonreflecting stationary sets are very useful. To exemplify:

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$\iff$ Partitions as above are sometime enough:

Theorem (Rinot, 2014)

If $\kappa \geq \aleph_2$, and $\square(\kappa)$ holds, then there exists a $\kappa$-cc poset whose square is not $\kappa$-cc.

$\iff$ Nonreflecting stationary sets are very useful. To exemplify:

Theorem (Shelah, 1991)

If $\kappa > \aleph_2$, and $E^\kappa_{\geq \aleph_2}$ admits a nonreflecting stationary set, then there exists a $\kappa$-cc poset whose square is not $\kappa$-cc.
Is there another way?

As said, partitioning $\kappa$ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.
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Questions

▶ Is it possible to partition $\kappa$ into two reflecting stationary sets?
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- Is it possible to partition $\kappa$ into $\langle S_i \mid i < \kappa \rangle$ such that, for all $i < j < \kappa$, $\text{Tr}(S_i) \cap \text{Tr}(S_j)$ be stationary?
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Definition

$\Pi(S, \theta)$ asserts the existence of a partition $\langle S_i \mid i < \theta \rangle$ of $S$ such that $\bigcap_{i<\theta} \text{Tr}(S_i)$ is stationary.
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Definition

$\Pi(S, \theta, T)$ asserts the existence of a partition $\langle S_i \mid i < \theta \rangle$ of $S$ such that $\cap_{i<\theta} \text{Tr}(S_i) \cap T$ is stationary.
Singular cardinals combinatorics
Scales

Definition
Suppose that $\lambda$ is a singular cardinal, and $\vec{\lambda} = \langle \lambda_i \mid i < \text{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals, converging to $\lambda$. For any two functions $f, g \in \prod \vec{\lambda}$ and $i < \text{cf}(\lambda)$, we write $f <^i g$ to express that $f(j) < g(j)$ whenever $i \leq j < \text{cf}(\lambda)$. 
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**Definition**
Suppose that $\lambda$ is a singular cardinal; $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ is said to be a scale for $\lambda$ iff there exists a sequence $\vec{\lambda}$ as above, such that:

- for every $\beta < \lambda^+$, $f_\beta \in \prod \vec{\lambda}$;
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- for every $g \in \prod \vec{\lambda}$, there exists $\beta < \lambda^+$ such that $g <^* f_\beta$. 
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Theorem (Shelah, 1990’s)

Every singular cardinal $\lambda$ admits a scale.

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*Every singular cardinal* \( \lambda \) *admits a scale.*

Suppose \( \vec{f} \) is a scale in \( \prod \vec{\lambda} \).

An ordinal \( \alpha \in E_{>\text{cf}(\lambda)}^{\lambda^+} \) is said to be **good** if there exist \( i < \text{cf}(\lambda) \) and a cofinal \( A \subseteq \alpha \) such that, for all \( \delta < \gamma \) from \( A \), \( f_\delta <^i f_\gamma \).

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We let $G(\vec{f})$ denote the set of good points with respect to $\vec{f}$.

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We let $G(\vec{f})$ denote the set of good points with respect to $\vec{f}$.

The set of good points is stationary (Shelah, 1990’s)
For every regular $\theta$ with $\text{cf}(\lambda) < \theta < \lambda$, $G(\vec{f}) \cap E^\lambda_\theta$ is stationary.
Scales

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*Every singular cardinal* $\lambda$ *admits a scale.*

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*For every regular* $\theta$ *with* $\text{cf} (\lambda) < \theta < \lambda$, $G(\vec{f}) \cap E_{\theta}^\lambda$ *is stationary.*

The set of good points is robust  
If $\vec{f}, \vec{g}$ are scales in $\prod \lambda$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.
Theorem (Shelah, 1990’s)

Every singular cardinal $\lambda$ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \lambda$.
An ordinal $\alpha \in E_{>\text{cf}(\lambda)}^{\lambda^+}$ is said to be very good if there exist $i < \text{cf}(\lambda)$ and a cofinal club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from $A$, $f_\delta <^i f_\gamma$. 
Theorem (Shelah, 1990’s)

Every singular cardinal $\lambda$ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \lambda$. An ordinal $\alpha \in E_{>\operatorname{cf}(\lambda)}^{\lambda^+}$ is said to be *very good* if there exist $i < \operatorname{cf}(\lambda)$ and a cofinal club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from $A$, $f_\delta <^i f_\gamma$. We let $V(\vec{f})$ denote the set of very good points with respect to $\vec{f}$. 
Scales

Theorem (Shelah, 1990’s)

Every singular cardinal $\lambda$ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \lambda$. An ordinal $\alpha \in E^{\lambda^+}_{>\text{cf}(\lambda)}$ is said to be very good if there exist $i < \text{cf}(\lambda)$ and a cofinal club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from $A$, $f_\delta <^i f_\gamma$. We let $V(\vec{f})$ denote the set of very good points with respect to $\vec{f}$.

Recall

If $\vec{f}, \vec{g}$ are scales in $\prod \lambda$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.
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Theorem (Shelah, 1990’s)

Every singular cardinal $\lambda$ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \lambda$.
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We let $V(\vec{f})$ denote the set of very good points with respect to $\vec{f}$.

Recall

If $\vec{f}, \vec{g}$ are scales in $\prod \lambda$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.

Theorem (Cummings-Foreman, 2010)

If $V = L$, then there are scales $\vec{f}, \vec{g}$ in $\prod_{n<\omega} \kappa_n$ for which
$V(\vec{f}) = E_{\omega}^{\kappa_{\omega+1}}$ and $V(\vec{g}) = \emptyset$. 
Very good points are not robust

The following is implicit in the proof of the above-mentioned theorem of Cummings-Foreman concerning \( V = L \):

**Proposition**

Suppose \( \lambda \) is singular, \( T \subseteq \lambda^+ \) is stationary and \( \prod(\lambda^+, \text{cf}(\lambda), T) \). Suppose \( \vec{f} \) is a scale for \( \lambda \), living in some product \( \prod_{i<\text{cf}(\lambda)} \lambda_i \). Then \( T \setminus V(\vec{g}) \) is stationary for some scale \( \vec{g} \) in \( \prod_{i<\text{cf}(\lambda)} \lambda_i \).

**Proof.**

Fix a partition \( \langle S_i \mid i < \text{cf}(\lambda) \rangle \) of \( \lambda^+ \), with \( T' := T \cap \bigcap_{i<\text{cf}(\lambda)} \text{Tr}(S_i) \) stationary. Define \( \langle g_\beta \mid \beta < \lambda^+ \rangle \) by letting \( g_\beta(i) := 0 \) for \( \beta \in S_i \), and \( g_\beta(i) := f_\beta(i) \), otherwise.

Let \( \alpha \in T' \) be arbitrary. To see that \( \alpha \notin V(\vec{g}) \), fix an arbitrary club \( C \subseteq \alpha \) and an index \( i < \text{cf}(\lambda) \).

Let \( \delta := \min(C \cap S_i) \) and \( \gamma := \min(C \cap S_i \setminus (\delta + 1)) \).

Then \( \delta < \gamma \) is a pair of elements of \( C \), while \( g_\delta(i) = 0 = g_\gamma(i) \). \(\square\)
Very good scales

Definition
A scale $\vec{f}$ for a singular cardinal $\lambda$ is said to be very good iff club many $\alpha \in E^{\lambda^+}_{> \text{cf}(\lambda)}$ are very good for $\vec{f}$.
Very good scales

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Conclusion
Suppose $\lambda$ is a singular cardinal and $\Pi(\lambda^+, \text{cf}(\lambda), E^\lambda_{>\text{cf}(\lambda)})$ holds. Then any product $\prod_{i<\text{cf}(\lambda)} \lambda_i$ admitting a scale for $\lambda$, admits yet another scale which is not very good.
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A scale $\vec{f}$ for a singular cardinal $\lambda$ is said to be very good iff club many $\alpha \in E_{\text{cf}(\lambda)}^{\lambda^+}$ are very good for $\vec{f}$.

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Suppose $\lambda$ is a singular cardinal and $\Pi(\lambda^+, \text{cf}(\lambda), E_{\text{cf}(\lambda)}^{\lambda^+})$ holds. Then any product $\prod_{i < \text{cf}(\lambda)} \lambda_i$ admitting a scale for $\lambda$, admits yet another scale which is not very good.

Note
There are numerous ways to consistently get instances of $\Pi(S, \theta, T)$. For instance, in a model of Magidor (1982), $\Pi(S, \aleph_1, T)$ holds for all stationary $S \subseteq E_{\aleph_0}^{\aleph_2}$ and $T \subseteq E_{\aleph_1}^{\aleph_2}$.

The main point here is to prove instances of $\Pi(S, \theta, T)$ in ZFC.
ZFC results
Main result

Theorem
Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. If $\lambda$ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;

This is trivial
Simply take $\langle E_\mu^\lambda \mid \mu \in \text{Reg}(\aleph_{\theta+1}) \rangle$ and $\langle E_\mu^{\lambda^+} \mid \mu \in \text{Reg}(\lambda) \rangle$. 
Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $\leq \lambda$.

1. If $\lambda$ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;

2. If $\lambda$ is regular, then $\Pi(E_{\mu}^{\lambda^+}, \theta, E_{\theta}^{\lambda^+})$ holds;

This is optimal

If $\Pi(S, \theta, T)$ holds, then $\{\alpha \in T \mid \text{cf}(\alpha) \geq \theta\}$ must be stationary.
Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. If $\lambda$ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;
2. If $\lambda$ is regular, then $\Pi(E_{\mu}^{\lambda^+}, \theta, E_{\theta}^{\lambda^+})$ holds;
3. If $2^\theta \leq \lambda$ and $\theta \neq \text{cf}(\lambda)$, then $\Pi(E_{\mu}^{\lambda^+}, \theta, E_{\theta}^{\lambda^+})$ holds;
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2. If $\lambda$ is regular, then $\Pi(E_{\mu}^{\lambda^+}, \theta, E_{\theta}^{\lambda^+})$ holds;
3. If $2^\theta \leq \lambda$ and $\theta \neq \text{cf}(\lambda)$, then $\Pi(E_{\mu}^{\lambda^+}, \theta, E_{\theta}^{\lambda^+})$ holds;
4. If $\lambda$ is singular and $\theta^{++} \neq \text{cf}(\lambda)$, then $\Pi(E_{\mu}^{\lambda^+}, \theta, E_{\theta^{++}}^{\lambda^+})$ holds;
Main result

Theorem
Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. If $\lambda$ is inaccessible, then $\prod(\lambda, \theta, \lambda)$ and $\prod(\lambda^+, \lambda, \lambda^+)$ hold;
2. If $\lambda$ is regular, then $\prod(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_\theta)$ holds;
3. If $2^\theta \leq \lambda$ and $\theta \neq \text{cf}(\lambda)$, then $\prod(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_\theta)$ holds;
4. If $\lambda$ is singular and $\theta^{++} \neq \text{cf}(\lambda)$, then $\prod(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_{\theta^{++}})$ holds;
5. If $\lambda$ is singular and $\theta^{++} = \text{cf}(\lambda)$, then $\prod(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_{\theta^{+3}})$ holds.

Remark
This follows from Clause (4).
Main result

Theorem
Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. If $\lambda$ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;

2. If $\lambda$ is regular, then $\Pi(E^\lambda_{\mu^+}, \theta, E^{\lambda^+}_\theta)$ holds;

3. If $2^\theta \leq \lambda$ and $\theta \neq \text{cf}(\lambda)$, then $\Pi(E^\lambda_{\mu^+}, \theta, E^{\lambda^+}_\theta)$ holds;

4. If $\lambda$ is singular and $\theta^{++} \neq \text{cf}(\lambda)$, then $\Pi(E^\lambda_{\mu^+}, \theta, E^{\lambda^+}_{\theta^{++}})$ holds;

5. If $\lambda$ is singular and $\theta^{++} = \text{cf}(\lambda)$, then $\Pi(E^\lambda_{\mu^+}, \theta, E^{\lambda^+}_{\theta^{+3}})$ holds.

Remark
Our proof at the level of successors of singulars is indeed different from the standard proofs for partitioning a stationary set. We build on the fact that any singular cardinal admits a scale and that the set of good points of a scale is stationary relative to any cofinality; we also use a combination of Ulam matrices with club-guessing to avoid any cardinal arithmetic hypotheses (Clauses (4) and (5)).
A special case with a simplified proof

Theorem

Let $\lambda$ be a singular cardinal. Let $\mu < \theta$ be regular cardinals with $\text{cf}(\lambda) < \mu < \theta < \lambda$. Then $\Pi(E_\mu^{\lambda^+}, \theta, E_\theta^{\lambda^+})$ holds.
A special case with a simplified proof

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Let $\lambda$ be a singular cardinal. Let $\mu < \theta$ be regular cardinals with $\text{cf}(\lambda) < \mu < \theta < \lambda$. Then $\prod(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_{\theta^{++}})$ holds.

Proof. Fix a scale $\vec{f}$ for $\lambda$ in some product $\prod_{i < \text{cf}(\lambda)} \lambda_i$.
By Shelah’s theorem, $T_0 := E^{\lambda^+}_{\theta^{++}} \cap G(\vec{f})$ is stationary.

Claim 1
There exist $i < \text{cf}(\lambda)$, $\zeta \in E^\lambda_{\theta^{++}}$, a stationary $T_1 \subseteq T_0$, and a sequence $\langle S^1_\alpha | \alpha \in T_1 \rangle$ such that, for all $\alpha \in T_1$:

- $S^1_\alpha$ is a stationary subset of $E^\alpha_\mu$;
- $\langle f_\beta(i) | \beta \in S^1_\alpha \rangle$ is strictly increasing and converging to $\zeta$.

Proof. By Fodor’s lemma, it suffices to prove that for each $\alpha \in T_0$, there is $i < \text{cf}(\lambda)$ and a stationary $S \subseteq E^\alpha_\mu$ on which $\beta \mapsto f_\beta(i)$ is strictly increasing.
Proof of Claim 1

Let \( \alpha \in T_0 \) be arbitrary. We shall find \( i < \text{cf}(\lambda) \) and a stationary \( S \subseteq E_{\mu}^\alpha \) on which \( \beta \mapsto f_{\beta}(i) \) is strictly increasing.

For each \( \gamma < \beta < \alpha \), pick \( i_{\gamma, \beta} < \text{cf}(\lambda) \) such that \( f_\gamma < i_{\gamma, \beta} \ f_\beta \).

As \( \alpha \in T_0 \) is a good point, let us also fix \( i' < \text{cf}(\lambda) \) and a cofinal \( A \subseteq \alpha \) such that, for all \( \delta < \gamma \) from \( A \), \( f_\delta < i' \ f_\gamma \).

Consider \( S' := \text{acc}^+(A) \cap E_{\mu}^\alpha \), which is a stationary subset of \( E_{\mu}^\alpha \).

As \( \mu > \text{cf}(\lambda) \), for each \( \beta \in S' \), we may pick a cofinal \( a_\beta \subseteq A \cap \beta \) and \( i_\beta < \text{cf}(\lambda) \) such that, for all \( \gamma \in a_\beta \), \( i_{\gamma, \beta} = i_\beta \).

As \( \theta^{++} > \text{cf}(\lambda) \), we may pick a stationary \( S \subseteq S' \) and \( i < \text{cf}(\lambda) \) such that, for all \( \beta \in S \), \( \max\{i_\beta, i', i_{\beta, \text{min}(A \setminus \beta+1)}\} = i \).

To see that \( i \) and \( S \) are as sought, let \( \epsilon < \beta \) be arbitrary elements of \( S \). Consider \( \delta := \min(A \setminus \epsilon + 1) \) and \( \gamma := \min(a_\beta \setminus \delta + 1) \).

Clearly, \( \epsilon < \delta < \gamma < \beta \) and \( f_\epsilon < i_{\epsilon, \min(A \setminus \epsilon+1)} \ f_\delta < i' \ f_\gamma < i_\beta \ f_\beta \).

In particular, \( f_\epsilon < i \ f_\beta \), so that \( f_\epsilon(i) < f_\beta(i) \), as sought. \( \square \)

Fix \( i, \zeta \), and \( \langle S_\alpha^1 \mid \alpha \in T_1 \rangle \) as in Claim 1.
Step 2: Find a function $g$

**Claim 2**

There are $g : E^+ \rightarrow \theta^{++}$ and a sequence $\langle S^2_\alpha | \alpha \in T_1 \rangle$ such that, for all $\alpha \in T_1$:

- $S^2_\alpha$ is a stationary subset of $S^1_\alpha$ (hence, of $E^\alpha_\mu$);
- $\langle g(\beta) | \beta \in S^2_\alpha \rangle$ is strictly increasing (hence, cofinal in $\theta^{++}$).

**Proof.** Fix a club $z$ in $\zeta$ with $\text{otp}(z) = \theta^{++}$. Define $g : E^+ \rightarrow \theta^{++}$ by letting $g(\beta) := \text{otp}(f_\beta(i) \cap z)$ if $f_\beta(i) < \zeta$ and $g(\beta) := 0$, o.w.; To see that $g$ is as sought, let $\alpha \in T_1$ be arbitrary. Let $\pi : \theta^{++} \rightarrow \alpha$ be the inverse collapse of some club in $\alpha$. Clearly, $\bar{S} := \{\bar{\beta} \in \theta^{++} | \pi(\bar{\beta}) \in S^1_\alpha \& (g \circ \pi)^{++} \bar{\beta} \subseteq \bar{\beta}\}$ is stationary.

Let $\bar{B} := \{\bar{\beta} \in \bar{S} | (g \circ \pi)(\bar{\beta}) < \bar{\beta}\}$. For all $\bar{\epsilon} < \bar{\beta}$ from $\bar{S} \setminus \bar{B}$, we have $g(\pi(\bar{\epsilon})) < \bar{\beta} \leq g(\pi(\bar{\beta}))$. Thus, it suffices to show that $S^2_\alpha := \pi[\bar{S} \setminus \bar{B}]$ (which is a subset of $S^1_\alpha$) is stationary.

Suppose not. In particular, $\bar{B}$ is stationary. But then, Fodor’s lemma entails a stationary $\hat{B} \subseteq \bar{B}$ on which $g \circ \pi$ is constant, contradicting the fact that $\langle f_{\pi(\bar{\beta})}(i) | \bar{\beta} \in \hat{B} \rangle$ converges to $\zeta$. \qed
Step 3: An Ulam Matrix

Let $g : E^{\lambda^+}_\mu \to \theta^+$ and $\langle S^2_\alpha \mid \alpha \in T_1 \rangle$ be given by Claim 2. Now, fix an Ulam matrix $\langle A_{\xi,\eta} \mid \xi < \theta^+, \eta < \theta^+ \rangle$ over $\theta^+$, i.e.,

- for all $\xi < \theta^+$, $|\theta^+ \setminus \bigcup_{\eta < \theta^+} A_{\xi,\eta}| \leq \theta^+$;
- for all $\eta < \theta^+$ and $\xi < \xi' < \theta^+$, $A_{\xi,\eta} \cap A_{\xi',\eta} = \emptyset$.

Claim 3

For every $\alpha \in T_1$, there are $\eta < \theta^+$ and $\lambda \in [\theta^+]^{\theta^+}$ such that, for all $\xi \in \lambda$, $g^{-1}[A_{\xi,\eta}] \cap \alpha$ is stationary in $\alpha$.

Proof. Suppose not. Then, for all $\eta < \theta^+$, the set

$x_{\eta} := \{ \xi < \theta^+ \mid g^{-1}[A_{\xi,\eta}] \cap \alpha \text{ is stationary in } \alpha \}$ has size $\leq \theta^+$. So $X := \bigcup_{\eta < \theta^+} x_{\eta}$ has size $\leq \theta^+$, and we may fix $\xi \in \theta^+ \setminus X$.

It follows that for all $\eta < \theta^+$, $g^{-1}[A_{\xi,\eta}] \cap \alpha$ is nonstationary in $\alpha$. Consequently, $g^{-1}[\bigcup_{\eta < \theta^+} A_{\xi,\eta}] \cap \alpha$ is nonstationary in $\alpha$.

However, $\bigcup_{\eta < \theta^+} A_{\xi,\eta}$ contains a tail of $\theta^+$, contradicting the fact that $\langle g(\beta) \mid \beta \in S^2_\alpha \rangle$ is strictly increasing and cofinal in $\theta^+$. □
Step 4: Club-guessing

By Shelah’s club-guessing theorem, we now fix a sequence 
\[ \langle C_{i} \mid i \in E_{\theta}^{\theta^{++}} \rangle \] such that, for every club \( C \subseteq \theta^{++} \), there exists 
\( i \in E_{\theta}^{\theta^{++}} \) such that \( C_{i} \subseteq C \cap i \) and \( \text{otp}(C_{i}) = \theta \).

By Claim 3, for every \( \alpha \in T_{1} \), let us fix \( \eta_{\alpha} < \theta^{+} \) and \( x_{\alpha} \in [\theta^{++}]^{\theta^{++}} \) such that, for all \( \xi \in x_{\alpha} \), \( g^{-1}[A_{\xi,\eta_{\alpha}}] \cap \alpha \) is stationary in \( \alpha \).

Then, fix \( i_{\alpha} \in E_{\theta}^{\theta^{++}} \) such that \( C_{i_{\alpha}} \subseteq \text{acc}^{+}(x_{\alpha}) \cap i_{\alpha} \) and \( \text{otp}(C_{i_{\alpha}}) = \theta \).

By Fodor’s lemma, fix a stationary \( T_{2} \subseteq T_{1} \), \( \eta < \theta^{+} \) and \( i \in E_{\theta}^{\theta^{++}} \) such that, for all \( \alpha \in T_{2} \), \( \eta_{\alpha} = \eta \) and \( i_{\alpha} = i \).

As the elements of \( \langle A_{\xi,\eta} \mid \xi < \theta^{++} \rangle \) are pairwise disjoint, we may fix a function \( h : E_{\mu}^{\lambda^{+}} \to \theta \) such that, for all \( \beta < \lambda^{+} : \)

\[(g(\beta) \in A_{\xi,\eta} \& \xi < \nu) \implies h(\delta) = \sup(\text{otp}(C_{i} \cap \xi)).\]
Step 5: Verification

For each $i < \theta$, let $S_i := h^{-1}\{i\}$.
We claim that $\langle S_i \mid i < \theta \rangle$ witnesses $\prod(E^\lambda_\mu, \theta, E^\lambda_\theta)$. Furthermore:

Claim 4
\[
\bigcap_{i < \theta} \text{Tr}(S_i) \cap E^\lambda_\theta \text{ covers the stationary set } T_2.
\]

Proof. Fix arbitrary $\alpha \in T_2$ and $i < \theta$. We shall find a stationary subset $S' \subseteq E^\alpha_\mu$ such that $h[S'] = \{i\}$.
As $i < \theta = \text{otp}(C_\ell)$, let $\xi'$ denote the unique element of $C_\ell$ such that $\text{otp}(C_\ell \cap \xi') = i$. Then, put $\xi := \min(x_\alpha \setminus (\xi' + 1))$.
As $C_\ell \subseteq \text{acc}^+(x_\alpha)$, we have that $[\xi', \xi) \cap C_\ell = \{\xi\}$.
Consequently, $\text{otp}(C_\ell \cap \xi) = \text{otp}(C_\ell \cap (\xi' + 1)) = i + 1$.
As $\eta = \eta_\alpha$ and $\xi \in x_\alpha$, the set $S' := g^{-1}[A_{\xi, \eta}] \cap \alpha$ is a stationary subset of $E^\alpha_\mu$. Finally, for each $\beta \in S'$, we have $g(\beta) \in A_{\xi, \eta}$, meaning that $h(\beta) = \sup(\text{otp}(C_\ell \cap \xi)) = \sup(i + 1) = i$, as sought.

\[\text{qed}\]
A finer result

We also have a finer result that apply for arbitrary stationary $S \subseteq \lambda^+$ (rather than $S = E^\lambda_\mu$).

Theorem
Suppose $\theta < \lambda$ are infinite cardinals with $\theta \neq \text{cf}(\lambda)$ and $2^\theta \leq \lambda$. For all subsets $S, T$ of $\lambda^+$ with a stationary $\text{Tr}(S) \cap T \cap E^\lambda_\theta$, any of the following implies that $\prod(S, \theta, T)$ holds:

1. $\lambda$ is regular;

2. $\lambda$ is a singular cardinal admitting a good scale.

Good scale
A scale $\vec{f}$ for $\lambda$ such that club many $\alpha \in E^\lambda_{>\text{cf}(\lambda)}$ are good for $\vec{f}$. 