ΠriKry forcings and their iterations

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The subject matter of this talk is Prikry-type forcings

- **Main role:** Generally devised to change cofinalities and blow up the power set of a singular cardinal
  - Due to foundational reasons this needs Very Large Cardinals (Jensen)
- Have found several connections/applications in central areas of Set Theory
  - The Singular Cardinals Problem (Prikry, Magidor, Gitik...)
  - Identity crises phenomena (Magidor, Apter...)
  - Inner Model Theory (Mitchell, Cummings & Schimerling...)
Motivating goal

Theorem

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a cofinality-preserving extension where

1. $\kappa = \sup_{n<\omega} \kappa_n$ is a strong limit cardinal;
2. $\neg \text{SCH}_\kappa$;
3. $\text{Refl}(\langle \omega, \kappa^+ \rangle)$ holds.

Around the same time, it was also proved by Ben-Neria-Hayut-Unger and soon after by Gitik.

Their proof avoids iterated forcing and extends to uncountable cofinality. The novelty in our approach is the iteration scheme for $\Sigma$-Prikry forcings.

Announced by A. Sharon in 2005.
The first representative of this family is the so-called **Prikry forcing**: 

- Let $\kappa$ be a measurable cardinal.
- Let $\mathcal{U}$ be a non-principal, normal and $\kappa$-complete ultrafilter over it (measure).

**Definition (Prikry 1970)**

Prikry forcing $\mathbb{P}_\mathcal{U}$ is the poset whose conditions are pairs $(s, A)$ where

1. $s \in [\kappa]^{<\omega}$ strictly increasing;
2. $A \in \mathcal{U}$ with $\max(s) < \min(A)$.

We will write $(s, A) \leq (t, B)$ iff $s$ end-extends $t$, $s \setminus t \subseteq B$ and $A \subseteq B$. We consider an additional ordering $\leq^* \subseteq \leq$ defined as $(s, A) \leq^* (t, B)$ iff $(s, A) \leq (t, B)$ and $s = t$. 
For each $n < \omega$, let $\mathbb{P}_n$ be the subposet of $\mathbb{P}$ whose conditions $(s, A)$ have $|s| = n$ joint with the trivial condition $1$.

Properties of $\mathbb{P}$

1. $\mathbb{P}$ is $\kappa$-centered, hence cardinals $\geq \kappa^+$ are preserved;
2. $\mathbb{P}$ forces $\text{cof}(\kappa) = \omega$.
3. $\mathbb{P}$ does not add bounded subsets to $\kappa$. In particular, cardinals $\leq \kappa$ are preserved.

(1) and (2) of above are easy to prove but (3) is not so immediate:

1. for each $n < \omega$, $(\mathbb{P}_n, \leq)$ is $\kappa$-closed;
2. $\mathbb{P}$ satisfies the Prikry property.
Prikry property

For each $p \in \mathbb{P}$ and each sentence $\varphi$ in the language of forcing, there is $q \leq^* p$ such that $q$ decides $\varphi$.

In other words, the set $D_\varphi = \{ p \in \mathbb{P} \mid p \parallel \varphi \}$ is $\leq^*$-dense.

Lemma (Prikry)

Prikry forcing has the Prikry property.

Theorem (Prikry)

If there is a measurable cardinal then there is a cardinal-preserving generic extension where the measurable becomes a singular strong limit cardinal of countable cofinality.
Some Examples

1. Prikry forcing (Prikry).
2. Supercompact Prikry forcing (Magidor).
5. Radin forcing (Radin & Woodin)
6. Diagonal Supercompact Magidor forcing (Sinapova)
7. Extender Based Prikry forcing (EBPF) (Gitik & Magidor)
8. Extender Based Radin forcing (Merimovich)
Our project has two goals:

1. Provide an abstract framework which allows a systematic study of Prikry-type forcings
2. Devise a viable iteration scheme for these forcings
The $\Sigma$-Prikry framework

What characterize Prikry-type posets?

1. There is always involved a notion of length.
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2. For all length \( n \), \( \mathbb{P}_n := \{ p \mid \ell(p) = n \} \) is “closed enough” (e.g. in Prikry forcing \( \mathbb{P}_n \) is \( \kappa \)-closed)
What characterizes a Prikry-type forcing?

1. There is always involved a notion of length.
2. For all length $n$, $\mathbb{P}_n := \{p \mid \ell(p) = n\}$ is “closed enough”.
3. There is a notion of minimal extension.
What characterizes a Prikry-type forcing?

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4. Decision by pure extensions (e.g. the Prikry property).
What characterizes a Prikry-type forcing?

1. There is always involved a notion of length $\ell$.
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4. Decision by pure extensions (e.g. the Prikry property).

We want to be able to iterate, so we will in addition require a quite prevalent feature

(*) $P$ has some good chain condition
Definition (Graded poset)

We say that \((\mathbb{P}, \ell)\) is a **graded poset** if \(\mathbb{P} = (P, \leq)\) is a poset, \(\ell : P \to \omega\) is a surjection, and, for all \(p \in P\), the following are true:

- For every \(q \leq p\), \(\ell(q) \geq \ell(p)\);
- There exists \(q \leq p\) with \(\ell(q) = \ell(p) + 1\).

Notation

For a graded poset as above we write

1. \(P_n := \{p \in P \mid \ell(p) = n\}\).
2. \(P^p_n := \{q \in P \mid q \leq p \& \ell(q) = \ell(p) + n\}\).

For ease of notation we sometimes write \(q \leq_n p\) rather than \(q \in P^p_n\).
The $\Sigma$-Prikry framework

1. $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $1$;
2. $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals with $\kappa := \sup_{n<\omega} \kappa_n$;
3. $\mu$ is a cardinal such that $1 \Vdash_{\mathbb{P}} \check{\mu} = \kappa^+$;
4. $\ell : P \to \omega$ and $c : P \to \mu$ are functions;

Definition ($\Sigma$-Prikry forcing)

We say that $(\mathbb{P}, \ell, c)$ is $\Sigma$-Prikry iff all of the following hold:

1. $(\mathbb{P}, \ell)$ is a graded poset;
2. For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{1\}, \leq)$ is $\kappa_n$-directed-closed;
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P^p_0 \cap P^q_0$ is non-empty;
4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{r \leq^n p \mid q \leq^m r\}$ contains a $\leq$-largest condition $m(p, q)$. In the particular case that $m = 0$, we write $w(p, q)$ instead of $m(p, q)$;
Definition (Σ-Prikry forcing)

We say that \((\mathbb{P}, \ell, c)\) is Σ-Prikry iff all of the following hold:

1. \((\mathbb{P}, \ell)\) is a graded poset;
2. For all \(n < \omega\), \(\mathbb{P}_n := (P_n \cup \{1\}, \leq)\) is \(\kappa_n\)-directed-closed;
3. For all \(p, q \in P\), if \(c(p) = c(q)\), then \(P^p_0 \cap P^q_0\) is non-empty;
4. For all \(p \in P\), \(n, m < \omega\) and \(q \leq_{n+m} p\), the set \(\{r \leq^n p \mid q \leq^m r\}\) contains a \(\leq\)-largest condition \(m(p, q)\). In the particular case that \(m = 0\), we write \(w(p, q)\) instead of \(m(p, q)\);
5. For all \(p \in P\), the set \(W(p) := \{w(p, q) \mid q \leq p\}\) has size \(<\mu\);
6. For all \(p' \leq p\) in \(P\), \(q \mapsto w(p, q)\) forms an order-preserving map from \(W(p')\) to \(W(p)\);
7. Suppose that \(U \subseteq P\) is a 0-open set, i.e., \(r \in U\) iff \(P^r_0 \subseteq U\). Then, for all \(p \in P\) and \(n < \omega\), there is \(q \in P^p_0\), such that, either \(P^q_n \cap U = \emptyset\) or \(P^q_n \subseteq U\).
Some clarifications

How $m(p, q)$ and $w(p, q)$ look like?

For simplicity let us assume that $\mathbb{P}$ is Prikry forcing. Say $p = (s, A)$ and $q = (s \upharpoonright \langle \alpha, \beta \rangle, B)$. Let $m \leq \ell(q) - \ell(p)$.

- Intuitively, $w(p, q)$ is the $\leq$-greatest interpolation between $p$ and $q$ with length $\ell(q)$. In this case, $w(p, q) = (s \upharpoonright \langle \alpha, \beta \rangle, A \setminus \beta + 1)$.

- In general, $m(p, q)$ is the $\leq$-greatest interpolation between $p$ and $q$ with length $\ell(q) - m$. In this case, $1(p, q) = (s \upharpoonright \langle \alpha \rangle, A \setminus \alpha + 1)$ and $2(p, q) = (s, A) = p$.

Convention

For each $n < \omega$ and $p \in \mathbb{P}$, we write $W_n(p) := \{w(p, q) \mid q \leq^n p\}$. Hence, $W(p) = \bigcup_{n<\omega} W_n(p)$. 
Novelties of the $\Sigma$-Prikry framework

$\mu^+$-Linked$_0$-property

For all $p, q \in P$, if $c(p) = c(q)$, then $P_0^p \cap P_0^q$ is non-empty.

Complete Prikry Property

Suppose that $U \subseteq P$ is a 0-open set, i.e., $r \in U$ iff $P_0^r \subseteq U$.
Then, for all $p \in P$ and $n < \omega$, there is $q \in P_0^p$, such that, either
$P_0^q \cap U = \emptyset$ or $P_0^q \subseteq U$.

- The first one is a strong form of $\mu^+$-2-Linkedness, hence a strengthening of the $\mu^+$-cc.
- The second one is inspired by the Complete Ramsey Property.
  Captures two features of Prikry-type forcings: the Prikry Property
  and the Strong Prikry Property (see next slide)
- Both are crucial to define viable iterations of $\Sigma$-Prikry forcings
Proposition

Let \( \mathbb{P} \) be some \( \Sigma \)-Prikry forcing. Then the following are true:

1. \( \mathbb{P} \) has the Prikry property.
2. \( \mathbb{P} \) has the Strong Prikry property; namely, for each dense open set \( D \subseteq \mathbb{P} \) and each \( p \in \mathbb{P} \), there is \( q \leq_0 p \) and \( n < \omega \) such that \( P^q_m \subseteq D \), for each \( m \geq n \).

For the proof we use the key concept of 0-open coloring:

Definition

Let \( (\mathbb{P}, \ell, c) \) be a \( \Sigma \)-Prikry triple. A 0-open coloring \( d : \mathbb{P} \rightarrow \theta \) is a map such that for each pair \( p' \leq_0 p \) of conditions in \( \mathbb{P} \), \( d(p) \in \{0, d(p') \} \). We say that \( H \subseteq \mathbb{P} \) is a set of indiscernibles for \( d \) if for each \( p, q \in H \), \( d(p) = d(q) \), provided \( \ell(p) = \ell(q) \).
CPP yiels the SPP and the PP

**Lemma**

Let $\langle \mathbb{P}, \ell, c \rangle$ be a $\Sigma$-Prikry triple. For each $p \in P$, $n \geq 2$ and each 0-open coloring $d : P \to n$, there is $q \leq^0 p$ such that the set of conditions of $P$ below $q$ is a set of indiscernibles for $d$.

**The CPP yields the PP**

Let $p \in P$ and $\varphi$ a sentence in the language of forcing. Define $d : P \to 3$ as

$$d(r) := \begin{cases} 1, & \text{if } r \models_{\mathbb{P}} \varphi; \\ 2, & \text{if } r \models_{\mathbb{P}} \neg \varphi; \\ 0, & \text{otherwise.} \end{cases}$$

Appeal to the above lemma to find $q \leq^0 p$ such that $P^q$ is a set of indescernibles for $d$. It is not hard to check that $q$ already decides $\varphi$. 
Let \((\mathcal{P}, \ell, c)\) be a \(\Sigma\)-Prikry triple. For each \(p \in P\), \(n \geq 2\) and each 0-open coloring \(d : P \to n\), there is \(q \leq^0 p\) such that the set of conditions of \(P\) below \(q\) is a set of indiscernibles for \(d\).

The CPP yields the SPP

Let \(p \in P\) and \(D\) be an open dense set. Define \(d : P \to 2\) as \(d(r) := 1\) iff \(r \in D\). Appealing to the lemma we get \(q \leq^0 p\) such that \(P^q\) is a set of indiscernibles for \(d\). Since \(D\) is dense, there is \(n < \omega\) and \(r \leq^n q\) such that \(r \in D\). By definition of \(d\), \(d(r) = 1\), hence \(P^q_n \subseteq D\). Finally the openness of \(D\) yields the desired result; that is, \(P^q_m \subseteq D\), for each \(m \geq n\).
Other properties of $\Sigma$-Prikry forcings

Proposition

Let $\mathbb{P} := (P, \leq)$ be some $\Sigma$-Prikry forcing and $p \in P$. Then, the following are true:

1. $\mathbb{P}$ does not add bounded subsets of $\kappa$;
2. For each $\nu \geq \kappa$ regular, and each $p \in P$, if $p \Vdash_{\mathbb{P}} \text{cof}(\nu) < \kappa$ then there is $p' \leq p$ such that $|W(p')| \geq \nu$.
3. Assume $1 \Vdash_{\mathbb{P}} \text{“$\kappa$ is singular”}$. Then, $\mu = \kappa^+$ iff $|W(p)| \leq \kappa$, for each $p \in P$.
4. For each $n < \omega$, $W_n(p)$ is a maximal antichain below $p$.
5. Any two compatible elements of $W(p)$ are comparable. Thus, $(W(p), \supseteq)$ is a tree (the $p$-tree)
6. $c \upharpoonright W(p)$ is injective.
Some examples: Prikry forcing

Definition (Prikry 1970)

Prikry forcing $\mathbb{P}$ is the poset whose conditions are pairs $(s, A)$ where

1. $s \in [\kappa]^{<\omega}$ strictly increasing;
2. $A \in \mathcal{U}$ with $\max(s) < \min(A)$.

$(s, A) \leq (t, B)$ iff $s$ end-extends $t$, $s \setminus t \subseteq B$ and $A \subseteq B$.

Prikry forcing is $\Sigma$-Prikry

1. $\Sigma$ is the constant $\omega$-sequence with value $\kappa$ and $\mu = \kappa^+$;
2. $\ell(s, A) := |s|$;
3. $c(s, A) := s$;
Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals. Suppose that $\mathcal{U}$ is a supercompact measure on $\mathcal{P}_{\kappa_0}(\mu^+)$, and let $\mathcal{U}_n$ be its projection onto $\mathcal{P}_{\kappa_0}(\kappa_n)$.

**Definition (Gitik & Sharon 2008)**

Conditions in $\mathbb{P}$ are sequences $p = \langle x_0^p, \ldots, x_{n-1}^p A_n^p, A_{n+1}^p, \ldots \rangle$ such that the following holds:

1. $x_i \in \mathcal{P}_{\kappa_0}(\kappa_i)$.
2. $x_i \prec x_{i+1}$ (i.e. $\text{otp}(x_i) < \text{otp}(x_{i+1} \cap \kappa_0)$).
3. $A_k \in \mathcal{U}_k$ and $\{ x \in A_k \mid x_{n-1}^p \prec x \} \subseteq A_k$.

The order is the usual: we extend the stems by picking elements from the measure one sets, and then shrink the measure one sets.
Some examples: Gitik-Sharon forcing

Definition (Gitik & Sharon 2008)
Conditions in $\mathbb{P}$ are sequences $p = \langle x_0^p, \ldots, x_{n-1}^p A_n^p, A_{n+1}^p, \ldots \rangle$ such that the following holds:

1. $x_i \in \mathcal{P}_{\kappa_0}(\kappa_i)$.
2. $x_i \prec x_{i+1}$ (i.e. $\text{otp}(x_i) < \text{otp}(x_{i+1} \cap \kappa_0)$).
3. $A_k \in \mathcal{U}_k$ and $\{x \in A_k \mid x_{n-1}^p \prec x\} \subseteq A_k$.

The order is the usual: we extend the stems by picking elements from the measure one sets, and then shrink the measure one sets.

GS-poset is $\Sigma$-Prikry

1. $\Sigma$ is the constant $\omega$-sequence with value $\kappa_0$ and $\mu = (\sup_{n<\omega} \kappa_n)^+$.  
2. $\ell(p) := |\langle x_0^p, \ldots, x_{n-1}^p \rangle|$.
3. $c(p) := \langle x_0^p, \ldots, x_{n-1}^p \rangle$.  

Some examples: The Extender-Based Prikry forcing

The set-up

- $\langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence of cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := 2^\mu$;
- $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$;
- for each $n < \omega$, $\kappa_n$ carries a $(\kappa_n, \lambda + 1)$-extender $E_n$.

In particular, for each $n < \omega$, we are assuming that there is an elementary embedding $j_n : V \to M_n$ with $\text{crit}(j) = \kappa_n$ such that $M_n$ is a transitive class, $\kappa_n M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda$.

Definition

For each $n < \omega$, and each $\alpha < \lambda$, define $E_{n,\alpha} := \{ X \subseteq \kappa_n \mid \alpha \in j_n(X) \}$.

For each $\alpha, \beta < \lambda$ write $\alpha \leq_{E_n} \beta$ iff $\alpha \leq \beta$ and there is $\pi_{\beta,\alpha} : \kappa_n \to \kappa_n$ such that $j_n(\pi_{\beta,\alpha})(\beta) = \alpha$. 
Definition

For $n < \omega$, $Q_{n0}$ is defined as follows:

\[(0)_n \quad Q_{n0} := (Q_{n0}, \leq_{n0})\]

where elements of $Q_{n0}$ are triples $p = (a^p, A^p, f^p)$ meeting the following requirements:

1. $f^p$ is a function from some $x \in [\lambda]^{\leq \kappa}$ to $\kappa$;
2. $a^p \in [\lambda]^{<\kappa_n}$, and $a^p$ contains a $\leq_{E_n}$-maximal element, which hereafter is denoted by $mc(a^p)$;
3. $\text{dom}(f^p) \cap a^p = \emptyset$;
4. $A^p \in E_{n, mc(a^p)}$;
5. if $\beta < \alpha$ is a pair in $a$, for all $\nu \in A$, $\pi_{mc(a^p)} \beta(\nu) < \pi_{mc(a^p)} \alpha(\nu)$;
6. if $\alpha, \beta, \gamma \in a$ with $\gamma \leq_{E_n} \beta \leq_{E_n} \alpha$, then, for all $\nu \in \pi_{mc(a^p)} \alpha " A$, $\pi_{\alpha \gamma}(\nu) = \pi_{\beta \gamma}(\pi_{\alpha \beta}(\nu))$.

The ordering $\leq_{n0}$ is defined as follows: $(a^p, A^p, f^p) \leq_{n0} (b^q, B^q, g^q)$ iff the following are satisfied:

(i) $f^p \supseteq g^q$,
(ii) $a^p \supseteq b^q$,
(iii) $\pi_{mc(a^p)mc(b^q)} " A^p \subseteq B^q$.
Definition

For $n < \omega$, $Q_{n1}$ is defined as follows:

\[(1)_n \quad Q_{n1} := (Q_{n1}, \leq_{n1}), \text{ where } Q_{n1} := \bigcup \{x \kappa_n \mid x \in [\lambda]^{\leq \kappa}\} \text{ and } \leq_{n1} := \supseteq.\]

Essentially $Q_{n1}$ is Cohen forcing $\text{Add}(\kappa^+, \lambda)$.

Definition

For $n < \omega$, $Q_n$ is defined as

\[(2)_n \quad Q_n := (Q_{n0} \cup Q_{n1}, \leq_n).\]

The ordering $\leq_n$ is defined as follows: for each $p, q \in Q_n$, $p \leq_n q$ iff

1. either $p, q \in Q_{ni}$ for some $i \in 2$ and $p \leq_{ni} q$, or
2. $p \in Q_{n1}$, $q \in Q_{n0}$ and, for some $\nu \in A$, $p \leq_{n1} q^\sim \langle \nu \rangle$, where

\[q^\sim \langle \nu \rangle := f^q \cup \{ (\beta, \pi_{mc(a^q)}, \beta(\nu)) \mid \beta \in a^q \}.\]
Some examples: The Extender-Based Prikry forcing

**Definition**

The Extender Based Prikry Forcing is the poset $\mathbb{P} := (P, \leq)$ defined by the following clauses:

- Conditions in $P$ are sequences $p = \langle p_n \mid n < \omega \rangle \in \prod_{n<\omega} Q_n$.
- For all $p, q \in P$, $p \leq q$ iff $p_n \leq_n q_n$ for every $n < \omega$.
- For all $p \in P$:
  - There is $n < \omega$ such that $p_n \in Q_{n0}$;
  - For every $n < \omega$, if $p_n \in Q_{n0}$, then $p_{n+1} \in Q_{n0}$ and $a^{p_n} \subseteq a^{p_{n+1}}$.

The Extender-Based Prikry forcing is $\Sigma$-Prikry

1. $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ and $\mu := (\sup_n \kappa_n)^+$.
2. $\ell(p) := \min\{n < \omega \mid p_n \in Q_{n0}\}$.
3. $c$ is more elaborated than in the previous cases.
The function $c$ for the EBPF

Since we are assuming $\mu^\kappa = \mu$ and $2^\mu = \lambda$, let us fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from $\lambda$ to $\mu$ with the property that, for every function $e : x \to \mu$ with $x \in [\lambda]^{\leq \kappa}$, there exists $i < \mu$ with $e \subseteq e^i$.

**Definition**

For every $f \in \bigcup_{n<\omega} Q_{n1}$, let $i(f) := \min\{i < \mu \mid f \subseteq e^i\}$.

For every $p = (a, A, f) \in \bigcup_{n<\omega} Q_{n0}$, let $i(p)$ be the least $i < \mu$ such that:

- for all $\alpha \in a$, $e^i(\alpha) = 0$;
- for all $\alpha \in \text{dom}(f)$, $e^i(\alpha) = f(\alpha) + 1$.

Finally, for every condition $p = \langle p_n \mid n < \omega \rangle$ in $P$, let

$$c(p) := \ell(p)^\uparrow \langle i(p_n) \mid n < \omega \rangle.$$
Let $p, q$ be two conditions in the EBPF with $c(p) = c(q)$. The goal is to show that $p$ and $q$ are compatible as witnessed by a 0-extension of both conditions. More precisely, we want to prove $P_0^p \cap P_0^q \neq \emptyset$.

Set $i$ be this common value of the $c$ function. By definition, $p$ and $q$ have the same length, say $\ell$. Now let $n \geq \ell$. To prove $P_0^p \cap P_0^q \neq \emptyset$ it suffices to check that $a_n^p \cap \text{dom}(f_n^q) = a_n^q \cap \text{dom}(f_n^p) = \emptyset$. Let us just check that $a_n^p \cap \text{dom}(f_n^q) = \emptyset$ as the other equality can be proved similarly.

Indeed, since $c(p) = i$ it follows that $e^i \upharpoonright a_n^p = 0$. On the other hand, as $c(q) = i$, $e^i \upharpoonright \text{dom}(f_n^q) \neq 0$. Both equalities combined finally yield $a_n^p \cap \text{dom}(f_n^q) = \emptyset$, as desired.
More examples

1. Supercompact Prikry forcing (Magidor);
2. AIM forcing (Cummings et al.);

Other candidates to be $\Sigma$-Prikry

1. Tree Prikry forcing;
2. Strongly Compact Gitik-Sharon forcing;
3. Extender Based Prikry forcing with a single extender;
An interlude on iterations of forcing
Some iteration theorems

(1) The $\langle \aleph_0 \rangle$-support iteration of ccc forcing is also ccc $\Rightarrow$ Consistency of $\text{FA}_{2^{\aleph_0}}(\text{ccc}) = MA$ (Solovay-Tennenbaum)

Observation
The above result does not extend to larger supports. Namely, even under the CH, there are countable support iterations of $\aleph_2$-cc $+ \aleph_1$-closed forcing which are not $\aleph_2$-cc (Mitchell).
(I) The $<\mathfrak{N}_0$-support iteration of ccc forcing is also ccc $\Rightarrow$ **Consistency of** $\text{FA}_{2^{\mathfrak{N}_0}}(\text{ccc}) = \text{MA}$ (**Solovay-Tennenbaum**).

(II) Let $\Gamma$ be the family of well-met, $\mathfrak{N}_1$-linked and $\mathfrak{N}_1$-closed forcings. Under the $\text{CH}$, the $<\mathfrak{N}_1$-support iteration of forcings in $\Gamma$ is $\mathfrak{N}_2$-cc $\Rightarrow$ **Consistency of** $\text{FA}_{2^{\mathfrak{N}_1}}(\Gamma) := \text{BA}$ (**Baumgartner**)
(I) The $<\aleph_0$-support iteration of ccc forcing is also ccc $\Rightarrow \text{Consistency of } \text{FA}_{2^{\aleph_0}}(ccc) = MA$ \textbf{(Solovay-Tennenbaum)}.

(II) Let $\Gamma$ be the family of well-met, $\aleph_1$-linked and $\aleph_1$-closed forcings. Under the CH, the $<\aleph_1$-support iteration of forcings in $\Gamma$ is $\aleph_2$-cc $\Rightarrow \text{Consistency of } \text{FA}_{2^{\aleph_1}}(\Gamma) := BA$ \textbf{(Baumgartner)}

(III) Let $\Gamma$ be the family of well-met, $\aleph_2$-stationary-cc and $\aleph_1$-closed forcings with exact upper bounds. Under the CH, the $<\aleph_1$-support iteration of members of $\Gamma$ is $\aleph_2$-stationary-cc $\Rightarrow \text{Consistency of } \text{FA}_{2^{\aleph_1}}(\Gamma)$ \textbf{(Shelah)}
(I) The $<\aleph_0$-support iteration of ccc forcing is also ccc $\Rightarrow$ **Consistency of $FA_{2^{\aleph_0}}(ccc) = MA$ (Solovay-Tennenbaum).**

(II) Let $\Gamma$ be the family of well-met, $\aleph_1$-linked and $\aleph_1$-closed forcings. Under the CH, the $<\aleph_1$-support iteration of forcings in $\Gamma$ is $\aleph_2$-cc $\Rightarrow$ **Consistency of $FA_{2^{\aleph_1}}(\Gamma) := BA$ (Baumgartner)**

(III) Let $\Gamma$ be the family of well-met, $\aleph_2$-stationary-cc and $\aleph_1$-closed forcings with exact upper bounds. Under the CH, the $<\aleph_1$-support iteration of members of $\Gamma$ is $\aleph_2$-stationary-cc $\Rightarrow$ **Consistency of $FA_{2^{\aleph_1}}(\Gamma)$ (Shelah)**

(IV) Let $\Gamma$ be the family of well-met, $\kappa^+$-stationary-cc, $\kappa$-closed and countably parallel closed forcing. Under $\kappa^{<\kappa} = \kappa$, the iteration of $<\kappa$-supported iteration of members of $\Gamma$ is $\kappa^+$-stationary-cc **Consistency of $FA_{2^\kappa}(\Gamma)$ (Cummings et. al)**
Goal
Solve problems about singular cardinals and their successors.

Strategy
Find an analogous iteration theorem for $\kappa$ being a successor of a singular cardinal.

To be continued in the next lecture