INFINITE COMBINATORIAL TOPOLOGY

ASSAF RINOT AND ROY TEPER

Abstract. We summarize our view on the course given by Dr. Boaz Tsaban at the Weizmann Institute of Science, Fall 2006.

1. 03.11.05

Definition 1.1. Define the *Baire space* to be the family of all functions from $\mathbb{N}$ to $\mathbb{N}$, and denote it by $\mathbb{N}^{\mathbb{N}}$.

Definition 1.2. Assume X is a set. A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal over* X iff it satisfies:

- $\emptyset \in \mathcal{I}$.
- $A \in \mathcal{I} \Longrightarrow \mathcal{P}(A) \subseteq \mathcal{I}$.
- $A, B \in \mathcal{I} \Longrightarrow A \cup B \in \mathcal{I}$.

The ideal is said to be *non-trivial* if, additionally :

- $\big\{\{x\} \mid x \in X\big\} \subseteq \mathcal{I}$.

If $\mathcal{I} \neq \mathcal{P}(X)$ (equivalently, if $X \notin \mathcal{I}$) we say that $\mathcal{I}$ is a *proper ideal*.

Definition 1.3. Assume $\mathcal{I}$ is an ideal over $\mathbb{N}$, for $f, g \in \mathbb{N}^{\mathbb{N}}$, put:

$$f \leq_{\mathcal{I}} g \text{ iff } \{n \in \mathbb{N} \mid f(n) > g(n)\} \in \mathcal{I}.$$

Let $\mathcal{I}_{fin} := \{X \subseteq \mathbb{N} \mid |X| < \aleph_0\}$ be the ideal of finite subsets of $\mathbb{N}$ and $\mathcal{J} := \{\emptyset\}$.

Define two binary relations on $\mathbb{N}^{\mathbb{N}}$: $\leq^* := \leq_{\mathcal{I}_{fin}}$ and $\leq := \leq_{\mathcal{J}}$, i.e., $f \leq^* g$ iff there exists some $m \in \mathbb{N}$ such that $f(n) \leq g(n)$ for all $n > m$, and $f \leq g$ iff $f(n) \leq g(n)$ holds for all n.

Lemma 1.4. *$\langle \mathbb{N}^{\mathbb{N}}, \leq^* \rangle$ is a quasi-ordered set, that is, $\leq^*$ is a reflexive and a transitive binary relation on $\mathbb{N}^{\mathbb{N}}$.*

Definition 1.5. For a set $A \subseteq \mathbb{N}^{\mathbb{N}}$, define the *downward closure* of A:

$$\underline{A} := \{f \in \mathbb{N}^{\mathbb{N}} \mid \exists g \in A(f \leq^* g)\}.$$

Let the *external cofinality* of A be $\mathrm{ecf}(A) := \min\{|D| \mid D \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } A \subseteq \underline{D}\}$.

By "our view" we mean that sometimes we omit material given in class, sometimes we give alternative definitions or proofs, and sometimes we include our own additional propositions. However, we are always *consistent* with the material given in class.

It is obvious that $\text{ecf}(\underline{A}) = \text{ecf}(A) \leq |A|$ for all $A \subseteq \mathbb{N}^\mathbb{N}$.

Definition 1.6. A subset $B \subseteq \mathbb{N}^\mathbb{N}$ is said to be *bounded* iff $\text{ecf}(B) \leq 1$.

As expected, we say that B is *unbounded* iff $\text{ecf}(B) > 1$.

Definition 1.7. A subset $D \subseteq \mathbb{N}^\mathbb{N}$ is said to be *dominating* (or *cofinal*) iff $\underline{D} = \mathbb{N}^\mathbb{N}$.

Definition 1.8. We define three important cardinals:

(i) $\mathfrak{b} := \min\{|B| \mid B \subseteq \mathbb{N}^\mathbb{N} \text{ and } \text{ecf}(B) > 1\}$.
(ii) $\mathfrak{d} := \text{ecf}(\mathbb{N}^\mathbb{N})$.
(iii) $\mathfrak{c} := |\mathbb{N}^\mathbb{N}|$.

Lemma 1.9. $\aleph_0 < \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c} = 2^{\aleph_0}$.

Proof. To see that $\mathfrak{b}$ is uncountable, we pick an arbitrary family $A = \{f_n \in \mathbb{N}^\mathbb{N} \mid n < \omega\}$ and then find some $g \in \mathbb{N}^\mathbb{N}$ witnessing $\text{ecf}(A) = 1$.

Define $g = g_A$ as follows, for all $n \in \mathbb{N}$: $g(n) = \max\{f_i(n) \mid 0 \leq i \leq n\}$. It is now easy to see that $A \subseteq \underline{\{g\}}$ and that $\text{ecf}(A) = 1$.

To see that $\mathfrak{b} \leq \mathfrak{d}$, it suffices to prove that if $D \subseteq \mathbb{N}^\mathbb{N}$ is cofinal, then D is unbounded. Towards a contradiction, assume there exists some dominating $D \subseteq \mathbb{N}^\mathbb{N}$ such that $\text{ecf}(D) = 1$. Pick $g \in \mathbb{N}^\mathbb{N}$ such that $D \subseteq \underline{\{g\}}$. It follows that $\mathbb{N}^\mathbb{N} \subseteq \underline{D} \subseteq \underline{\{g\}}$, i.e., that $\text{ecf}(\mathbb{N}^\mathbb{N}) = 1$, which is an absurd.[1] $\square$

Corollary 1.10. *If CH holds (that is, if $\mathfrak{c} = \aleph_1$), then $\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \aleph_1$.*

It is worth mentioning that an unbounded family is not necessarily cofinal, e.g., take $\{f \in \mathbb{N}^\mathbb{N} \mid \forall n \in \mathbb{N}(f(2n) = 0)\}$.

Lemma 1.11. *There exists a $\mathfrak{b}$-scale, that is, a sequence $\langle f_\alpha \in \mathbb{N}^\mathbb{N} \mid \alpha < \mathfrak{b}\rangle$, such that:*

(a) $\text{ecf}\{f_\alpha \mid \alpha < \mathfrak{b}\} > 1$;
(b) $\alpha < \beta < \mathfrak{b}$ *implies* $f_\alpha \leq^* f_\beta$.

Proof. By definition of $\mathfrak{b}$, we may pick an unbounded family $B = \{g_\alpha \in \mathbb{N}^\mathbb{N} \mid \alpha < \mathfrak{b}\}$.

We now define the $\mathfrak{b}$-scale by induction on $\alpha < \mathfrak{b}$. Put $f_0 := g_0$.

Assume now $\{f_\beta \mid \beta < \alpha\}$ had already been defined. Since $\alpha < \mathfrak{b}$, $\text{ecf}(\{f_\beta \mid \beta < \alpha\}) = 1$, we may pick an exemplifying $h \in \mathbb{N}^\mathbb{N}$. Put $f_\alpha := \max\{g_\alpha, h\}$.[2] End of the construction.

Put $B' := \{f_\alpha \mid \alpha < \mathfrak{b}\}$. Since $g_\alpha \leq^* f_\alpha$ for all relevant α, we get that $B \subseteq \underline{B'}$, thus, $1 < \text{ecf}(B) \leq \text{ecf}(B')$ and property (a) is satisfied. Property (b) follows immediately from the construction. $\square$

[1]For each $f \in \mathbb{N}^\mathbb{N}$: $f \leq^* (f+1)$ and $(f+1) \not\leq^* f$, where $(f+1)(n) = f(n) + 1$ for all $n \in \mathbb{N}$.

[2]Here, max denotes the pointwise-maximum function between two functions of the same domain.

Lemma 1.12. *There exists a $\mathfrak{d}$-scale, that is, a sequence $\langle f_\alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha < \mathfrak{d}\rangle$, such that:*

(a) $\{f_\alpha \mid \alpha < \mathfrak{d}\}$ *is cofinal;*

(b) $\alpha < \beta < \mathfrak{d}$ *implies* $f_\beta \not\leq^* f_\alpha$.

In particular, for all $g \in \mathbb{N}^{\mathbb{N}}$, there exists some $\alpha < \mathfrak{d}$ such that $f_\beta \not\leq^* g$ whenever $\alpha < \beta < \mathfrak{d}$.

Proof. By definition of $\mathfrak{d}$, we may pick a family $D = \{g_\alpha \mid \alpha < \mathfrak{d}\}$ such that $\underline{D} = \mathbb{N}^{\mathbb{N}}$.

We now define the $\mathfrak{d}$-scale by induction on $\alpha < \mathfrak{d}$. Put $f_0 := g_0$.

Assume now $\{f_\beta \mid \beta < \alpha\}$ had already been defined. Since $\alpha < \mathfrak{d}$, we may pick $h_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $h_\alpha \notin \underline{\{f_\beta \mid \beta < \alpha\}}$. Put $f_\alpha := \max\{g_\alpha, h_\alpha\}$. End of the construction.

Just like in the preceding proof, we put $D' := \{f_\alpha \mid \alpha < \mathfrak{b}\}$ and notice that the two properties holds for D'. Being user-friendly, we now give a direct proof for the last property. Fix $g \in \mathbb{N}^{\mathbb{N}}$.

By $\mathbb{N}^{\mathbb{N}} = \underline{D} \subseteq \underline{D'} \subseteq \mathbb{N}^{\mathbb{N}}$, we may pick $\alpha < \mathfrak{d}$ such that $g \leq^* f_\alpha$.

Suppose there exists $\beta > \alpha$ such that $f_\beta \leq^* g$, then, in particular $f_\beta \leq^* f_\alpha$. It follows from $\alpha < \beta$ that $\alpha \in \{\gamma \mid \gamma < \beta\}$ and $f_\beta \in \underline{\{f_\gamma \mid \gamma < \beta\}}$. A moment's reflection make it clear that this implies $h_\beta \in \underline{\{f_\beta\}} \subseteq \underline{\{f_\gamma \mid \gamma < \beta\}}$ which is obviously a contradiction to the choice of h_β. □

Claim 1.13. $\mathfrak{b}$ *is a* regular cardinal*, that is,* $\mathrm{cf}(\mathfrak{b}) = \mathfrak{b}$.

Proof. It is obvious that $\mathrm{cf}(\mathfrak{b}) \leq \mathfrak{b}$, as this is true for any infinite cardinal number.

Fix an increasing sequence of ordinals $\langle \alpha_i < \mathfrak{b} \mid i < \mathrm{cf}(\mathfrak{b})\rangle$ converging to $\mathfrak{b}$. Let $\langle f_\alpha \mid \alpha < \mathfrak{b}\rangle$ be a $\mathfrak{b}$-scale. Put $B := \{f_{\alpha_i} \mid i < \mathrm{cf}(\mathfrak{b})\}$. We shall show that $\mathrm{ecf}(B) > 1$, and then - by definition/minimality of $\mathfrak{b}$ - we would have to conclude that $\mathfrak{b} \leq |B| \leq \mathrm{cf}(\mathfrak{b})$.

Assume there exists some $g \in \mathbb{N}^{\mathbb{N}}$ such that $B \subseteq \underline{\{g\}}$, we reach a contradiction by showing that $f_\alpha \leq^* g$ for all $\alpha < \mathfrak{b}$.

Indeed, pick $\alpha < \mathfrak{b}$ and pick $i < \mathrm{cf}(\mathfrak{b})$ such that $\alpha < \alpha_i$. We get that $f_\alpha \leq^* f_{\alpha_i} \leq^* g$. □

Claim 1.14. $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$.

Proof. Fix a $\mathfrak{d}$-scale $\langle f_\alpha \in \mathbb{N}^{\mathbb{N}} \mid \alpha < \mathfrak{d}\rangle$, and an increasing sequence $\langle \alpha_i \mid i < \mathrm{cf}(\mathfrak{d})\rangle$ converging to $\mathfrak{d}$. Put $B := \{f_{\alpha_i} \mid i < \mathrm{cf}(\mathfrak{b})\}$. We claim that $\mathrm{ecf}(B) > 1$.

Suppose not, and let $g \in \mathbb{N}^{\mathbb{N}}$ be such that $B \subseteq \underline{\{g\}}$. Pick $\alpha < \mathfrak{d}$ such that $g \leq^* f_\alpha$ and $i < \mathrm{cf}(\mathfrak{d})$ such that $\alpha < \alpha_i$. We get from one hand that $B \ni f_{\alpha_i} \leq^* g \leq^* f_\alpha$, while on the other hand $f_{\alpha_i} \not\leq^* f_\alpha$. A contradiction. □

Corollary 1.15. $\aleph_1 \leq \mathrm{cf}(\mathfrak{b}) = \mathfrak{b} \leq \mathrm{cf}(\mathfrak{d}) \leq \mathfrak{d} \leq \mathfrak{c}$.

It is worth mentioning that the latter is all one can *prove*. That's because for all cardinal numbers $\kappa, \lambda, \mu, \theta$ with $\aleph_1 \leq \mathrm{cf}(\kappa) = \kappa \leq \lambda = \mathrm{cf}(\mu) \leq \theta$ and $\mathrm{cf}(\theta) > \aleph_0$, there exists a model of set theory satisfying $\mathfrak{b} = \kappa, \mathfrak{d} = \mu, \mathrm{cf}(\mathfrak{d}) = \lambda$ and $\mathfrak{c} = \theta$.

Definition 1.16 (Menger's Basis property)**.** A metric space $\langle X, d\rangle$ is said to satisfy *Menger's Basis property* iff for each basis $\mathcal{B}$, there exists a sequence $\langle B_n \in \mathcal{B} \mid n \in \mathbb{N}\rangle$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$.

Observation 1.17. *Menger's Basis property is closed hereditary.*[3]

Notation 1.18. *For a metric space $\langle X, d\rangle$, $x \in X$ and $\delta \in \mathbb{R}^+$, let $\mathsf{B}_\delta(x) := \{y \in X \mid d(x, y) < \delta\}$ denote the open ball of radius δ, centered at x.*

Definition 1.19. The *canonical base* for a metric space $\langle X, d\rangle$ is $\{\mathsf{B}_\delta(x) \mid \delta \in \mathbb{R}^+, x \in X\}$.

Fact 1.20. *Suppose $\mathcal{B}$ is a family of open sets in a metric space $\langle X, d\rangle$, satisfying:*

($\star$) *For all relevant x, y, δ with $y \in \mathsf{B}_\delta(x)$, there exists $U \in \mathcal{B}$ satisfying $y \in U \subseteq \mathsf{B}_\delta(x)$.*

Then $\mathcal{B}$ is a basis for $\langle X, d\rangle$.

Lemma 1.21. *A space that satisfies Menger's Basis property is Lindelöf.*

Proof. Suppose $\langle X, d\rangle$ satisfies Menger's Basis property and $\mathcal{U}$ is a given open cover. Put $\mathcal{B} := \{U \cap \mathsf{B}_{\frac{1}{n}}(x) \mid U \in \mathcal{U}, n \in \mathbb{N}^+, x \in X\}$. Since $\mathcal{B}$ is a basis, we can find some $\mathcal{F} \in [\mathcal{B}]^{\aleph_0}$ such that $\bigcup \mathcal{F} = X$. Finally, for each $G \in \mathcal{F}$, pick a single $G' \in \mathcal{U}$ such that $G \subseteq G'$, then $\mathcal{V} := \{G' \mid G \in \mathcal{F}\}$ is a countable subcover of $\mathcal{U}$. $\square$

Corollary 1.22. *The discrete space $\langle X, d\rangle$ satisfies Menger's Basis property iff $|X| \leq \aleph_0$.*

Lemma 1.23. *If $\langle X, d\rangle$ is a compact metric space, then it satisfies Menger's Basis property.*

Proof. Suppose $\mathcal{B}$ is a basis for the space. X is a metric space, thus, it easy to find a family $\{A_n \in \mathcal{B} \mid n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} \mathrm{Diam}(A_n) = 0$.

By compactness, we may pick $\mathcal{U} \in [\mathcal{B}]^{<\omega}$ such that $X = \bigcup \mathcal{U}$. Now, let $\{B_n \mid k \leq n\}$ enumerate $\mathcal{U}$, and for al $n > k$, put $B_n := A_n$. $\square$

Definition 1.24. A space $\langle X, O\rangle$ is said to be σ-compact iff there exists a family of compact subsets $\langle K_n \subseteq X \mid n \in \mathbb{N}\rangle$ such that $X = \bigcup_{n\in\mathbb{N}} K_n$.

It is obvious that a finite union of compact subspaces is compact, hence, we may always assume that the family $\langle K_n \mid n \in \mathbb{N}\rangle$ is increasing with respect to inclusion. For instance $\langle \mathbb{R}, d\rangle$ is σ-compact, as it is the countable union of the compact intervals:

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} [-n, n].$$

[3]A property p is said to be *closed hereditary*, if for any topological space $\langle X, O\rangle$ and any closed subset $Y \subseteq X$: $X \models p$ implies $Y \models p$.

Claim 1.25. *If $\langle X, d\rangle$ is a σ-compact metric space, then it satisfies Menger's Basis property.*

Proof. Suppose $\mathcal{B}$ is a basis for the space. It follows that for all $n \in \mathbb{N}$ and $x \in K_n$, we may find $B_{x,n} \in \mathcal{B}$ with $x \in B_{x,n}$ and $\mathrm{Diam}(B_{x,n}) < \frac{1}{n+1}$. Fix $n \in \mathbb{N}$.

Evidently, $K_n \subseteq \bigcup_{x\in K_n} B_{x,n}$, so by compactness, there exists $f(n) \in \mathbb{N}$ and a family $\{B_{m,n} \in \mathcal{B} \mid m \le f(n)\} \subseteq \{B_{x,n} \mid x \in K_n\}$ s.t. $K_n \subseteq \bigcup_{m\le f(n)} B_{m,n}$ and $\mathrm{Diam}(B_{m,n}) < \frac{1}{n+1}$.

Finally, let $\psi : \mathbb{N} \leftrightarrow \{(m, n) \mid n \in \mathbb{N}, m \le f(n)\}$ be the order-preserving bijection.[4]

We have that $X = \bigcup_{n\in\mathbb{N}} K_n = \bigcup_{n\in\mathbb{N}} \bigcup_{m\le f(n)} B_{m,n} = \bigcup_{n\in\mathbb{N}} B_{\psi(n)}$ and $\lim_{n\to\infty} \mathrm{Diam}(B_{\psi(n)}) = \lim_{n\to\infty} \frac{1}{n+1} = 0$, that is, $\{B_{\psi(n)} \mid n \in \mathbb{N}\}$ witnesses Menger's Basis property. □

Definition 1.26 (Menger's covering)**.** For a topological space $\langle X, O\rangle$, we denote by $S_{fin}(\mathcal{O}, \mathcal{O})$ the property that for any countable sequence of open covers of X, $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X.

Observation 1.27. *Menger's covering is closed hereditary.*

Observation 1.28. *If $\langle X, O\rangle$ satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$, then X is Lindelöf.*

Proof. Suppose $\mathcal{U}$ is an open cover. Put $\mathcal{U}_n := \mathcal{U}$ for all $n \in \mathbb{N}$. For $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ witnessing $S_{fin}(\mathcal{O}, \mathcal{O})$, then $\mathcal{V} := \bigcup \mathcal{F}_n$ is a countable subcover of $\mathcal{U}$. □

Lemma 1.29. *If $\langle X, O\rangle$ is a σ-compact topological space, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. Suppose $X = \bigcup_{n\in\mathbb{N}} K_n$ where each K_n is compact. Assume $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$ is a given family of covers. In particular $K_n \subseteq \bigcup \mathcal{U}_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$.

By compactness, we may pick $\mathcal{F}_n \in [\mathcal{U}_n]^{<\omega}$ such that $K_n \subseteq \bigcup \mathcal{F}_n$.

Evidently, $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X. □

Conjecture 1.30 (Menger)**.** *$S_{fin}(\mathcal{O}, \mathcal{O})$ is equivalent to σ-compactness.*

Observation 1.31. *For a space $\langle X, O\rangle$, and a sequence $\langle \mathcal{B}_n \mid n \in \mathbb{N}\rangle$ of bases to X, TFAE:*

(a) $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

(b) *For any countable sequence of open covers of X, $\langle \mathcal{V}_n \subseteq \mathcal{B}_n \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{V}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X.*

Proof. We assume (b) and prove (a). Suppose $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$ is a given family of covers.

Fix $n \in \mathbb{N}$. Let $\psi_n : O \to \mathcal{P}(\mathcal{B}_n)$ be a function such that $U = \bigcup \psi_n(U)$ for all $U \in O$.[5]

Put $\mathcal{V}_n := \bigcup\{\psi_n(U) \mid U \in \mathcal{U}_n\}$. Clearly, $\mathcal{V}_n \subseteq \mathcal{B}_n$ and $\bigcup \mathcal{V}_n = \bigcup \mathcal{U}_n = X$.

Now, by the hypothesis (b), we yield $\mathcal{F}_n \in [\mathcal{V}_n]^{<\omega}$ for all $n \in \mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X. Finally, for each $n \in \mathbb{N}$ and $G \in \mathcal{F}_n$, pick a single $G' \in \mathcal{U}_n$ such that $G \subseteq G'$ and put $\mathcal{F}'_n := \{G' \mid G \in \mathcal{F}_n\}$. It follows that $|\mathcal{F}'_n| \le |\mathcal{F}_n| < \aleph_0$ and $\bigcup_{n\in\mathbb{N}} \mathcal{F}'_n$ covers X. □

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[4]Recall the lexicographic order on $\mathbb{N} \times \mathbb{N}$: $(m_1, n_1) < (m_2, n_2)$ iff $(n_1 < n_2)$ or $((n_1 = n_2) \wedge (m_1 < m_2))$.

[5]By definition, an open set is a union of basis-elements.

2. 10.11.05

Lemma 2.1. *$S_{fin}(\mathcal{O},\mathcal{O})$ is a topological property, that is, whenever $\langle X_1,O_1\rangle,\langle X_2,O_2\rangle$ are topological spaces, and $f: X_1\rightarrow X_2$ is a continuous surjection, then $X_1\models S_{fin}(\mathcal{O},\mathcal{O})$ implies $X_2\models S_{fin}(\mathcal{O},\mathcal{O})$.*

Proof. Suppose $\langle \mathcal{U}_n\subseteq O_2\mid n\in\mathbb{N}\rangle$ is a family of open covers of X_2. For any relevant n, put $\mathcal{V}_n:=\{f^{-1}[U]\mid U\in\mathcal{U}_n\}$. By continuity of f, $\langle\mathcal{V}_n\subseteq O_1\mid n\in\mathbb{N}\rangle$ is a family of open covers of X_1. If $X_1\models S_{fin}(\mathcal{O},\mathcal{O})$, then there exists a witness in the form of $\langle\mathcal{G}_n\in[\mathcal{V}_n]^{<\omega}\mid n\in\mathbb{N}\rangle$. Finally, put $\mathcal{F}_n:=\{U\mid f^{-1}[U]\in\mathcal{G}_n\}$ and notice that $\langle\mathcal{F}_n\in[\mathcal{U}_n]^{<\omega}\mid n\in\mathbb{N}\rangle$ exemplifies $S_{fin}(\mathcal{O},\mathcal{O})$ for X_2. □

Definition 2.2. For a topological space $\langle X,O\rangle$, put:
- $d(X):=\min\{|D|\mid D\subseteq X \text{is dense in } X\}+\aleph_0$,
- $w(X):=\min\{|\mathcal{B}|\mid \mathcal{B} \text{ is a basis to } \langle X,O\rangle\}+\aleph_0$,
- $L(X):=\min\{\mu\in\text{ICN}\mid \text{ every open cover of } X \text{ contains a subcover of cardinality } \leq\mu\}$.[6]

In the above terminology, a space $\langle X,O\rangle$ is *separable* iff $d(X)=\aleph_0$, is *seocond-countable* iff $w(X)=\aleph_0$, and is *Lindelöf* iff $L(X)=\aleph_0$.

Lemma 2.3. *For any topological space $\langle X,O\rangle$: $d(X)\leq w(X)$ and $L(X)\leq w(X)$.*

Proof. Fix a basis $\mathcal{B}\in[O]^{w(X)}$. For any choice function $f\in\prod_{U\in O}U$, $\text{Im}(f)$ is a dense subset (since its intersection with any non-trivial open sets is never empty). Also $|\text{Im}(f)|\leq w(X)$.

To see that $L(X)\leq w(X)$, fix an open cover $\mathcal{U}$. Pick $\psi: O\rightarrow\mathcal{B}$ such that $U=\bigcup\psi(U)$ for all $U\in O$. Now $\mathcal{V}:=\bigcup\{\psi(U)\mid U\in\mathcal{U}\}\subseteq\mathcal{B}$ is a cover of X and $|\mathcal{V}|\leq|\mathcal{B}|$. For each $G\in\mathcal{V}$, pick $G'\in\mathcal{U}$ such that $G'\subseteq G$.

Finally, $\{G'\mid G\in\mathcal{V}\}\subseteq\mathcal{U}$ is a subcover of cardinality $\leq|\mathcal{B}|=w(X)$. □

To complete the picture, we include the following two observations:

Observation 2.4. *There exists a topological space $\langle X,\tau\rangle$ with $\aleph_0=d(X)<w(X)=\aleph_1$.*

Proof. Take $X:=\omega_1$ and $\tau:=\big\{\{0,\alpha\}\mid\alpha<\omega_1\big\}$. Evidently $\{0\}$ is a dense subset. Notice that if $\mathcal{B}$ is a basis to X, then $\mathcal{B}=\tau$. It follows that $w(X)=\aleph_1$. □

Observation 2.5. *There exists a topological space $\langle X,\tau\rangle$ with $\aleph_0=L(X)<w(X)=\aleph_1$.*

Proof. Put $X:=\omega_1$ and $\tau:=\{\alpha^\uparrow\ \mid\alpha<\omega_1\}$, where $\alpha^\uparrow:=\{\beta<\omega_1\mid\beta>\alpha\}$. Since a basis to this space induces an unbounded set in ω_1 and a countable union of countable sets is countable, $w(X)$ must equal $\aleph_1$. To see that $L(X)=\aleph_0$, fix a cover $\mathcal{U}$ of X.

[6] ICN stands for the class of infinite cardinal numbers.

Put $\gamma := \min\{\alpha < \omega_1 \mid \exists U \in \mathcal{U}(\alpha^{\uparrow} \subseteq U)\}$ and let U_γ be an exemplifying set, i.e., $\gamma^{\uparrow} \subseteq U_\gamma$. Now, for all $\beta < \gamma$ (there are only countable many!), find $U_\beta \in \mathcal{U}$ such that $\beta \in U_\beta$.

It follows that $\{U_\beta \mid \beta \leq \gamma\} \subseteq \mathcal{U}$ is a countable subcover for X. □

It is not by chance that the two spaces mentioned above are not metric:

Lemma 2.6. *If $\langle X, d\rangle$ is a metric space, then $w(X) = d(X) = L(X)$.*

Proof. Fix a dense subset $D \in [X]^{d(X)}$. Put $\mathcal{B} := \{\mathsf{B}_{\frac{1}{n}}(x) \mid x \in D, n \in \mathbb{N}^+\}$. We shall show that $\mathcal{B}$ is a basis, and conclude that $w(X) \leq |\mathcal{B}| = |D| = d(x)$. Fix $y \in X$ and $\delta \in \mathbb{R}^+$. Since D is dense, we may find $x \in D \cap \mathsf{B}_\delta(y)$. Since $x \in \mathsf{B}_\delta(y)$ and the latter is open, then x is an interior point, and hence for a large enough $n \in \mathbb{N}$, we have that $\mathsf{B}_\delta(y) \supseteq \mathsf{B}_{\frac{1}{n}}(x) \in \mathcal{B}$ and we are done.

We now show $d(X) \leq L(X)$. For $n \in \mathbb{N}^+$, it is clear that $\{\mathsf{B}_\delta(x) \mid x \in X, \delta \in (0, \frac{1}{n})\}$ is an open cover of X. Now, by definition of $L(X)$, for all $n \in \mathbb{N}^+$, there exists two families $\{x_{i,n} \in X \mid i < L(X)\}$ and $\{\delta_{i,n} \in (0, \frac{1}{n}) \mid i < L(X)\}$ s.t. $\{\mathsf{B}_{\delta_{i,n}}(x_{i,n}) \mid i < L(X)\}$ covers X.

Put $D := \{x^n_i \mid n \in \mathbb{N}^+, i < L(X)\}$. Evidently, $|D| \leq L(X)$. We are left with showing that D is dense, that is, to show that every member of X is a limit point of D. Fix $y \in X$.

Since the above families covers X, for all $n \in \mathbb{N}^+$, there exists i_n such that $y \in \mathsf{B}_{\delta_{i_n,n}}(x_{i_n,n})$, in particular, $d(y, x_{i_n,n}) < \frac{1}{n}$, hence, $\lim_{n\to\infty} d(y, x_{i_n,i}) = 0$. Since $\{x_{i_n,n} \mid n \in \mathbb{N}^+\} \subseteq D$, then we conclude that y is a limit point of D. □

Definition 2.7. For a topological space $\langle X, O\rangle$, let $I(X) := \{x \in X \mid \{x\} \in O\}$ denote the family of all isolated points of X.

It is obvious that for all $Y \subseteq X$, if $\exists z \in I(X) \setminus Y$, then $z \notin \overline{Y}$ as well. Hence:

Lemma 2.8. *If $\langle X, O\rangle$ is a topological space and $D \subseteq X$ is a dense subset, then $I(X) \subseteq D$. In particular, $|I(X)| \leq d(X)$.*

Theorem 2.9 (Hurewicz, Lelek)**.** *Suppose $\langle X, d\rangle$ is a metric space. Then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ iff X satisfies Menger's Basis property.*

Proof. (⇒) Suppose $\mathcal{B}$ is a basis for the space. It follows that for all $x \in X$ and $n \in \mathbb{N}$, we may find $B_{x,n} \in \mathcal{B}$ with $x \in B_{x,n}$ and $\mathrm{Diam}(B_{x,n}) < \frac{1}{n+1}$. Now apply $S_{fin}(\mathcal{O}, \mathcal{O})$ to $\langle\{B_{x,n} \mid x \in X\} \mid n \in \mathbb{N}\rangle$ and find $\mathcal{F}_n \in [\{B_{x,n} \mid x \in X\}]^{<\omega}$ such that X is covered by $\mathcal{F}_n$ for all $n \in \mathbb{N}$. The proof now continues in the same fashion of Claim 1.25, we find an enumeration $\{B_n \mid n \in \mathbb{N}\}$ of $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ such that $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$.
(⇐) Fix a family of open covers $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$.

For $x, y \in X$ and $\delta \in \mathbb{R}^+$, put $\mathsf{DB}_\delta(x, y) := \mathsf{B}_\delta(x) \cup \mathsf{B}_\delta(y)$. Let $J(X) := \big\{\{x\} \mid x \in I(X)\big\}$.

For all $n \in \mathbb{N}$, define $\mathcal{V}_n$ to be:

$$\left\{\mathsf{DB}_\delta(x,y) \;\middle|\; x,y \in X, d(x,y) > \frac{1}{n+1}, \delta \in \mathbb{R}^+, \exists\{U',U''\} \in [\mathcal{U}_n]^{\leq 2}\big(\mathsf{DB}_\delta(x,y) \subseteq U' \cup U''\big)\right\}.$$

Claim 2.10. *$\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{V}_n \cup J(X)$ is a basis to $\langle X,d\rangle$.*

Proof. Fix $x \in X, \varepsilon \in \mathbb{R}^+$ and $y \in \mathsf{B}_\varepsilon(x)$. We shall find $U \in \mathcal{B}$ with $y \in U \subseteq \mathsf{B}_\varepsilon(x)$.

Since $J(X) \subseteq \mathcal{B}$, we may assume $y \neq x$. Pick $n \in \mathbb{N}$ large enough such that $d(x,y) > \frac{1}{n+1}$. Now, since $X = \bigcup \mathcal{U}_n$, there exists $\{U',U''\} \in [\mathcal{U}_n]^{\leq 2}$ such that $x \in U', y \in U''$. Since U' is open and $U'' \cap \mathsf{B}_\varepsilon(x)$ is open, we may find some positive $\delta < \varepsilon$ small enough such that $\mathsf{B}_\delta(x) \subseteq U'$ and $\mathsf{B}_\delta(y) \subseteq U'' \cap \mathsf{B}_\varepsilon(x)$. By the choice of δ, we have $\mathsf{B}_\delta(x) \cup \mathsf{B}_\delta(y) \subseteq \mathsf{B}_\varepsilon(x)$.

It now follows that $U := \mathsf{B}_\delta(x) \cup \mathsf{B}_\delta(y) = \mathsf{DB}_\delta(x,y) \in \mathcal{B}$ and $y \in U \subseteq \mathsf{B}_\varepsilon(x)$. □

Assume that X satisfies Menger's basis property. By Lemmas 1.21,2.6,2.8, we may enumerate $I(X) = \{x_i \mid i \in \mathbb{N}\}$. Also, the hypothesis implies the existence of a family $\mathcal{F} = \{B_n \in \mathcal{B} \mid n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$.

Fix $n \in \mathbb{N}$ and let $\mathcal{F}_n := \mathcal{F} \cap \mathcal{V}_n$. Since $\lim_{n\to\infty} \mathrm{Diam}(B_n) = 0$ and $\mathrm{Diam}(U) > \frac{1}{n+1}$ for all $U \in \mathcal{V}_n$, we must conclude that $\mathcal{F}_n$ is finite. Also, by the definition of $\mathcal{B}$ and $\mathcal{F}$:

$$X = \bigcup \mathcal{F} = \bigcup\Big(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \cup J(X)\Big) = \bigcup\bigcup_{n\in\mathbb{N}} \big(\mathcal{F}_n \cup \{x_n\}\big).$$

Now, for all $U \in \mathcal{F}_n$, find $U',U'' \in \mathcal{U}_n$ such that $U \subseteq U' \cup U''$, and also find $G_n \in \mathcal{U}_n$ such that $x_n \in G_n$. Put $\mathcal{F}'_n := \{U',U'' \mid U \in \mathcal{F}_n\} \cup \{G_n\} \subseteq \mathcal{U}_n$.

It easy to see that $|\mathcal{F}'_n| \leq 2 \cdot |\mathcal{F}_n| + 1 < \aleph_0$ and that $\bigcup_{n\in\mathbb{N}} \mathcal{F}'_n$ covers X. □

Corollary 2.11. *Menger's basis property does not depend on the choice of metric for any given metric space.*

Definition 2.12. Suppose I is some index set and $\langle X_i \mid i \in I\rangle$ is a sequence of sets.

The *Cartesian product* of $\langle X_i \mid i \in I\rangle$ is:

$$\prod_{i\in I} X_i = \Big\{f : I \to \bigcup_{i\in I} X_i \mid f(i) \in X_i \text{ for all } i \in I\Big\}$$

In practice, for $x \in \prod_{i\in I} X_i$, we usually write x_i instead of $x(i)$, and x_i is referred as the *i-th coordinate* of x.

The map $\pi_j : \prod_{i\in I} X_i \to X_j$, defined by $\pi_j(x) = x_j$, is called the *projection map* of $\prod_{i\in I} X_i$ on X_j.

Remark: we need the axiom of choice to ensure that the cartesian product of a non-empty collection of non-empty sets is indeed non-empty.

Definition 2.13. Suppose A is some index set. Assume that $\langle\langle X_\alpha, O_\alpha\rangle \mid \alpha \in A\rangle$ is a family of topological spaces. The *product topology* (or *Tychonoff topology*) on $\prod_{\alpha\in A} X_\alpha$ is obtained by taking as a (canonical) base for the space $\langle\prod_{\alpha\in A} X_\alpha, O\rangle$, the family :

$$\mathcal{B} := \left\{ \prod_{\alpha\in A} U_\alpha \middle| \begin{array}{c} U_\alpha \in O_\alpha \text{ for each } \alpha \in A \\ \{\alpha \in A \mid U_\alpha \neq X_\alpha\} \text{ is finite} \end{array} \right\}.$$

Notice that the set $\prod_{\alpha\in A} U_\alpha$, where $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, ..., \alpha_n$, can be written as:

$$\prod U_\alpha = \pi^{-1}_{\alpha_1}(U_{\alpha_1}) \cap \cdots \cap \pi^{-1}_{\alpha_n}(U_{\alpha_n}),$$

Thus, the product topology is precisely that topology which has for a subbase the collection $\{\pi^{-1}_\alpha(U_\alpha) \mid \alpha \in A, U_\alpha \text{ is open in } X_\alpha\}$. Moreover, the sets U_α can be restricted to be taken from some fixed subbases for each of the spaces $\langle X_\alpha, O_\alpha\rangle$ (think why?).

Example 2.14. Consider now the Baire space $\mathbb{N}^\mathbb{N} := \prod_{n\in\mathbb{N}} \mathbb{N}$ where $\mathbb{N}$ is equipped with the discrete topology. A subbase for this product topology is of the form $\{\pi^{-1}_n(\{k\}) \mid n, k \in \mathbb{N}\}$. The canonical base for $\mathbb{N}^\mathbb{N}$ is $\mathcal{B} := \{\sigma^\uparrow \mid \exists I \in [\mathbb{N}]^{<\omega}(\sigma \text{ is a function from } I \text{ to } \mathbb{N})\}$, where $\sigma^\uparrow := \{g \in \mathbb{N}^\mathbb{N} \mid g \restriction \mathrm{dom}(\sigma) = \sigma\}$. It is a nice observation that the following is also a base:

$$\left\{\{(n_1, \dots, n_k)\} \times \mathbb{N}^\mathbb{N} \middle| n_1, \dots, n_k, k \in \mathbb{N}\right\} = \left\{\sigma^\uparrow \middle| k \in \mathbb{N}(\sigma \text{ is a function from } \{1, .., k\} \text{ to } \mathbb{N})\right\}.$$

An easy proposition to formulate is the following,

Proposition 2.15. *The βth projection is continuous and open, and the Tychonoff topology is the weakest topology on $\prod X_\alpha$ for which each projection π_β is continuous.*

Proof. The first part is trivial by definitions. Let O be a topology on the product in which each projection is continuous, then for each β, if U_β is open in X_β, we get that $\pi^{-1}_\beta(U_\beta) \in O$. Thus, the members of a subbase for the Tychonoff topology all belong to O, hence the Tychonoff topology is contained in O. ☐

Definition 2.16. Suppose $\langle X, O\rangle$ is a topological space and some $A \subseteq X$.
- A is G_δ iff it is the countable intersection of open sets.
- A is F_σ iff it is the countable union of closed sets.

Evidently, an open set is G_δ and a closed set is F_σ. In metric spaces, closed set is also G_δ.

Definition 2.17. Let $\langle X, O\rangle$ be a topological space. A set $A \subseteq X$ is *nowhere dense* in X iff $int(\overline{A}) = \emptyset$. A set $A \subseteq X$ is of the *first category* (or *meager*) iff $A = \bigcup_{n\in\mathbb{N}} A_n$ where A_n is nowhere dense for all $n \in \mathbb{N}$. All other subsets of X are said to be of the *second category*.[7]

[7] $int(A)$ stands for the interior of A, that is, the family of all interior points of A.

Remark: It is by definition that A is nowhere dense iff $\overline{A}$ is nowhere dense. Consider a meager set A. Now, $A = \bigcup_{n\geq 1} A_n \subseteq \bigcup_{n\geq 1} \overline{A_n}$, and we conclude that every meager set is a subset of some meager F_σ set.

Fact 2.18. *Suppose $\langle X, O\rangle$ is a topological space and some $A \subseteq X$. Then:*
- $bnd(X \setminus A) = bnd(A)$.[8]
- $\overline{A} = A \cup bnd(A)$.
- $X = int(A) \uplus bnd(A) \uplus int(X \setminus A)$.

Lemma 2.19. *Suppose $\langle X, O\rangle$ is a topological space and some $A \subseteq X$.*
Then A is nowhere dense iff $(X \setminus \overline{A})$ is dense in X.

Proof. Suppose A is nowhere dense. By $X = int(\overline{A}) \cup bnd(\overline{A}) \cup int(X \setminus \overline{A})$, we get that:

$$X = bnd(\overline{A}) \cup int(X \setminus \overline{A}) = bnd(X \setminus \overline{A}) \cup int(X \setminus \overline{A}) = \overline{X \setminus \overline{A}},$$

i.e., that $X \setminus \overline{A}$ is dense in X. The other direction is similar. □

Example 2.20 (The Cantor set)**.** Beginning with the unit interval $I = [0, 1]$, we will define closed subsets $I_1 \supset I_2 \supset \cdots$ in I as follows. We obtain I_1 by removing the interval $(\frac{1}{3}, \frac{2}{3})$ from I. I_2 is obtained by removing from I_1 the intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, having I_{n-1}, I_n is obtained by removing the open middle third of the 2^{n-1} closed intervals that make I_{n-1}.

The Cantor set is obtained by intersecting all these closed sets, $C := \bigcap_{n\in\mathbb{N}} I_n$.

We develop an interesting alternative description of the cantor set. Each $x \in I$ has an expansion $(x_1, x_2, ...)$ in ternary form, that is $x_i \in \{0, 1, 2\}$ for all $i \in \mathbb{N}$, and $x = \sum_{n\in\mathbb{N}} \frac{x_n}{3^n}$. These expressions are unique, except that any number (but 1) expressible in an expansion ending in a sequence of 2's can be re-expressed in an expansion ending in a sequence of 0's. For example, $\frac{1}{3}$ can be written as $(0, 2, 2, 2, ...)$ and also as $(1, 0, 0, 0, ...)$. We agree to use only expressions of the first type. Then the Cantor set is precisely the set of points in I having a ternary expansion without 1's.

The Cantor set is closed, so in order to show that it is nowhere dense we are left with showing that it has no interior. Every base set$(a, b) \subset [0, 1]$ contains some element with 1 in it's ternary decomposition. Hence $(a, b) \not\subseteq C$, thus C is nowhere dense.

Another way of showing that is the following: assume $(a, b) \subset C$ for some $0 \leq a < b \leq 1$. From monotonicity of the Lebesgue measure m, we get that $b - a = m((a, b)) \leq m(C) = 0$, a contradiction. To see that indeed $m(C) = 0$ notice that $m(C) = \lim_{n\to\infty}(\frac{2}{3})^n$.

[8] $bnd(A)$ stands for the boundary of A.

Definition 2.21. Let A be a set in a topological space $\langle X, O\rangle$.

A point $x \in X$ is an *accumulation point* of A iff any $U \in O$ with $x \in U$, satisfies $U \cap A \neq \{x\}$.

A point $x \in A$ is an *isolated point* of A iff $x \in A \setminus A^d$, where A^d is the set of all accumulation points of A

Definition 2.22. A set F is *perfect* iff F is closed, non-empty, and dense in itself; i.e., each point of F is an accumulation point of F (F does not contain isolated points).

Definition 2.23. $x \in X$ is a *Condensation point* of A if $A \cap U_x$ is not countable for all $U_x \in O$ with $x \in U_x$. We denote by $\operatorname{cond}(A)$ the set of all condensation points of A.

Remark: Notice that $I(A) \subseteq A \setminus \operatorname{cond}(A)$.

Theorem 2.24 (Cantor-Bendixson)**.** *Suppose $\langle X, O\rangle$ is a second-countable topological space (i.e., $w(X) = \aleph_0$). Then every closed set F can be written as the decomposition $F = P \uplus N$, where P is perfect, and N is countable.*

Proof. The proof is technical and non-trivial. We will formulate results (and prove some of them) towards the theorem's proof.

Lemma 2.25. *Any topological space $\langle X, O\rangle$ can be decomposed as $X = P \uplus N$, where P is perfect and N is* scattered *(that is, N doesn't contain any set which is dense in itself).*

Proof. Put $\mathcal{A} := \{A \subseteq X \mid A \text{ is dense in itself}\}$ and $P := \bigcup \mathcal{A}$. We claim that P is perfect.

Suppose first that $\bigcup \mathcal{A}$ is not dense in itself, thus, there exists a point $x \in \bigcup \mathcal{A}$ which is isolated in the relative topology of $\bigcup \mathcal{A}$. In particular, $x \in A_0$ for some $A_0 \in \mathcal{A}$.

Now, there is an open set U_x such that $U_x \cap (\bigcup \mathcal{A}) = \{x\}$, therefore $U_x \cap A_0 = \{x\}$, a contradiction to the fact that A_0 is dense in itself.

We know that P is dense in itself and left with showing that P is closed. We will do that by proving that the closure of a set dense in itself, is a set dense in itself.

Assume A is dense in itself and $x \in \overline{A} \setminus A$, that is, $U \cap A \neq \{x\}$ for every open set U containing x. It follows that $\overline{A}$ is dense in itself.

Put $N := X \setminus P$. By the definition of P, N must be scattered. □

Lemma 2.26. $\operatorname{cond}(A)$ *is a closed set and* $\operatorname{cond}(A \cup B) = \operatorname{cond}(A) \cup \operatorname{cond}(B)$.

Lemma 2.27. *In a second-countable space, $A \setminus \operatorname{cond}(A)$ is countable and* $\operatorname{cond}(\operatorname{cond}(A)) = \operatorname{cond}(A)$.

Proof. Fix a countable base $\mathcal{B}$ and a point $x \in A \setminus \operatorname{cond}(A)$. There exists $U_x \in O$ such that $U_x \cap A$ is countable, hence $B_x \cap A$ is countable for all $B_x \in \mathcal{B}$ such that $B_x \subseteq U_x$. Now $\{B_x \mid x \in A \setminus \operatorname{cond}(A)\}$ is countable, thus $A \setminus \operatorname{cond}(A)$ is countable.

Now, $A = (A \setminus \mathrm{cond}(A)) \cup \mathrm{cond}(A)$. Using the previuos lemma we obtain:

$\mathrm{cond}(A) = \mathrm{cond}(A \setminus \mathrm{cond}(A)) \cup \mathrm{cond}(\mathrm{cond}(A)) = \emptyset \cup \mathrm{cond}(\mathrm{cond}(A)) = \mathrm{cond}(\mathrm{cond}(A))$.

□

Define $P := \mathrm{cond}(X)$ and $N := X \setminus P$. By definition, P is dense in itself, and from previous results it is closed, and N is countable. Hence the theorem is proved. □

Definition 2.28. Assume that X and Y are topological spaces. A function $f : X \to Y$ is an *homeomorphism* iff f is a continuous open bijection.

If there exists an homeomorphism from X to Y, we say that X and Y are *homeomorphic*.

Remark: two spaces are homeomorphic if they are equipped with the "same" topology.

Theorem 2.29. *The Baire space* $\mathbb{N}^{\mathbb{N}}$ *is homeomorphic to* $(0,1) \setminus \mathbb{Q}$.

Proof. We break the proof into several lemmas.

Lemma 2.30. *The Baire space is homeomorphic to* $(0,1) \setminus \left\{ \frac{k}{2^n} \mid n \in \mathbb{N}, k < 2^n \right\}$.

Proof. Put $\omega := \mathbb{N} \cup \{0\}$, $D := \left\{ \frac{k}{2^n} \mid n \in \mathbb{N}, k < 2^n \right\}$ and let $A := (0,1) \setminus D$.

Suppose $B \subseteq A$ is a subset of the form $B = (\frac{n}{2^k}, \frac{n+1}{2^k})$ where $n \in \omega, k \in \mathbb{N}$ and $n < 2^k$. For $m \in \omega$, let $B_m := (\frac{n}{2^k} + \frac{m}{2^{k+m}}, \frac{n}{2^k} + \frac{m+1}{2^{k+m+1}})$. Since $D \cap A = \emptyset$, it is easily seen that $B = \biguplus_{m=0}^{\infty} B_m$. For m_1, m_2, we write B_{m_1,m_2} for $(B_{m_1})_{m_2}$, and so forth..

We shall now define an homeomorphism $\psi : A \to \omega^\omega$.[9] Fix $x \in A$.

For notational simplicity, denote $f_x := \psi(x)$. We define $f_x(n)$ by recursion on $n \in \omega$.

For $n = 0$, let $f_x(1)$ be the unique $m \in \omega$ such that $x \in A_m$. For the recursive step, let $f_x(n+1)$ be the unique $m \in \omega$ such that $x \in A_{f(0),..,f(n),m}$.

Evidently, the above defines a bijection. We prove that ψ is open and leave the proof of continuity for the reader, since the idea of the proof is essentially the same.

Pick an open set $U \subseteq A$ and $f \in \psi[U]$. We shall show that f is an interior point of $\psi[U]$. Let $x := \psi^{-1}(f)$. Since x is an interior point of U, we may pick $n \in \omega, k \in \mathbb{N}$ such that $x \in (\frac{n}{2^k}, \frac{n+1}{2^k}) \subseteq U$. Since $\{x\}$ equals the intersection of the decreasing chain $\{A_{f(0),..,f(m)} \mid m \in \omega\}$, there must exist some $m \in \omega$ such that $A_{f(0),..,f(m)} = (\frac{n}{2^k}, \frac{n+1}{2^k})$. Now, put $\sigma := f \restriction \{0,..,m\}$. Clearly, $f \in \sigma^{\uparrow} \subseteq \psi[U]$, where $\sigma^{\uparrow}$ is like in Example 2.14. □

Lemma 2.31 (Cantor)**.** *Any two dense countable sets in* $(0,1)$ *are homeomorphic.*

Proof. Suppose $D = \{d_n\}_{n \geq 1}$ and $E = \{e_n\}_{n \geq 1}$ are dense in $(0,1)$.

We define by induction on $n \in \mathbb{N}$ an increasing chain of partial functions $\{\psi_n : D_n \to E \mid n \in \mathbb{N}\}$ where $D_n \in [D]^n$ for any relevant n.

[9]Clearly ψ would induce an homeomorphism from A to $\mathbb{N}^{\mathbb{N}}$.

Induction base: for $n = 1$, let $D_1 := \{d_1\}$ and $\psi_1(d_1) := e_1$.

Induction hypothesis : ψ_n is order-preserving.

Inductive step: We divide into two case.

For $n+1$ where n is even, Put $j := \min\{j \in \mathbb{N} \mid d_j \notin D_n\}$, and let j_1, j_2 be such that:

$$d_{j_2} := \min\{d \in D_n \mid d > d_j\} \text{ and } d_{j_1} := \max\{d \in D_n \mid d < d_j\}.$$

Now, since E is a dense subset, $\big(\psi_n(d_{j_1}), \psi_n(d_{j_2})\big) \cap E$ is non-empty. So let $i := \min\{i \in \mathbb{N} \mid e_i \in \big(\psi_n(d_{j_1}), \psi_n(d_{j_2})\big)\}$. Let $D_{n+1} := D_n \cup \{d_j\}$ and extend ψ_n to ψ_{n+1} such that $\psi_{n+1}(d_j) = e_i$. By the hypothesis, ψ_n is order-preserving bijection, thus ψ_{n+1} is order-preserving, $e_i \notin \operatorname{Im}(\psi_n)$, and ψ_{n+1} is bijective.

For $n+1$ where n is odd, Put $j := \min\{j \in \mathbb{N} \mid e_j \notin \operatorname{Im}(\psi(D_n)\}$, and let j_1, j_2 be such that:

$$d_{j_2} := \min\{d \in D_n \mid \psi(d) > e_j\} \text{ and } d_{j_1} := \max\{d \in D_n \mid \psi(d) < e_j\}.$$

Now, since D is a dense subset, we may define $i := \min\{i \in \mathbb{N} \mid d_i \in \big(d_{j_1}, d_{j_2}\big)\}$. Let $D_{n+1} := D_n \cup \{d_j\}$ and extend ψ_n to ψ_{n+1} such that $\psi_{n+1}(d_j) = e_i$. End of the construction.

Clearly, the construction ensures that for all $d \in D$ and $e \in E$, there exists some large enough $n \in \mathbb{N}$ such that $d \in \operatorname{dom}(\psi_n)$ and $e \in \operatorname{Im}(\psi_n)$ and we are done by letting $\psi := \bigcup_{n \in \mathbb{N}} \psi_n$.

Finally, since ψ is an order-preserving bijection, then ψ is also an homeomorphism.

□

Lemma 2.32. *The complements of two dense countable sets in* $(0, 1)$ *are homeomorphic.*

Proof. Let D^c and E^c be the complements of some two dense countable sets in $(0, 1)$, and let $\psi : D \to E$ be an homeomorphism.

We shall now define an homeomorphism $\varphi : D^c \to E^c$. Fix $x \in D^c$. Fix a convergent sequence $\{d_n\}_{n \geq 1} \subseteq D$ such that $\lim d_n = x$ and let $\varphi(x) := \lim \psi(d_n)$. Now, $\{\psi(d_n)\}_{n \geq 1}$ is Cauchy in E, but assume that $\lim \psi(d_n) \in E$. ψ is an homeomorphism hence ψ^{-1} is continuous, so $\psi^{-1}(\lim \psi(d_n)) = \lim(\psi^{-1}\psi(d_n)) = \lim d_n = x \notin D$, a contradiction to the fact that the range of ψ^{-1} is D.

φ is well defined (think why?), and since it is an order-preserving bijection, then ψ is an homeomorphism. □

This completes the proof of 2.29. □

3. 24.11.05

We now aim at developing tools to be able to prove the following.

Theorem 3.1 (Luzin). *Assuming CH, there exists a* Luzin set*, that is, an uncountable set $L \subseteq \mathbb{R}$ such that for any meager set $M \subseteq \mathbb{R}$: $|L \cap M| \leq \aleph_0$.*

Definition 3.2. Suppose X is a set. For an ideal $I \subseteq \mathcal{P}(X)$. Put:

- $\mathsf{add}(I) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq I (\bigcup \mathcal{A} \notin I)\}$.
- $\mathsf{cov}(I) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq I (\bigcup \mathcal{A} = X)\}$.
- $\mathsf{cof}(I) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq I \text{ and } \forall B \in I \exists C \in \mathcal{A} (B \subseteq C)\}$.

If $\mathcal{I}$ is a proper ideal, we may also define:

- $\mathsf{non}(I) := \min\{|A| \mid A \subseteq X \text{ and } A \notin I\}$.

Since an ideal is closed under finite unions, always $\mathsf{add}(I) \geq \aleph_0$. If I is a proper ideal, then also $\mathsf{add}(I) \leq \mathsf{cov}(I)$. If I is non-trivial, then also $\mathsf{cov}(I) \leq \mathsf{cof}(I)$.

Intuitively, an ideal is a collection of negligible sets. Two important examples are:

Definition 3.3. Let $\mathcal{M} := \{A \subseteq \mathbb{R} \mid A \text{ is meager }\}$ and $\mathcal{N} := \{A \subseteq \mathbb{R} \mid A \text{ is a null set }\}$.

We also consider $\mathcal{M}_{[0,1]} := \mathcal{M} \cap \mathcal{P}([0,1])$ and $\mathcal{N}_{[0,1]} := \mathcal{N} \cap \mathcal{P}([0,1])$.

Evidently, $\mathcal{M}, \mathcal{N}$ are non-trivial ideals and $\mathsf{add}(\mathcal{M}), \mathsf{add}(\mathcal{N}) \geq \aleph_1$. $|\mathcal{M}| = |\mathcal{N}| = 2^{\mathfrak{c}}$, since the cantor set $C \in \mathcal{M} \cap \mathcal{N}$ is of size $\mathfrak{c}$ and then $\mathcal{P}(C) \subseteq \mathcal{M} \cap \mathcal{N}$. However:

Lemma 3.4. $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{c}$ *and* $\mathsf{cof}(\mathcal{N}) \leq \mathfrak{c}$.

Proof. As mentioned before, any meager set is contained in some F_σ meager set, and there are only $\mathfrak{c}$ many F_σ sets, hence, $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{c}$.

If $A \in \mathcal{N}$, then for all $n \in \mathbb{N}$, there exists some open G_n containing A and of measure $< \frac{1}{n+1}$. It follows that any null set is contained in some G_δ null set, thus, $\mathsf{cof}(\mathcal{N}) \leq \mathfrak{c}$. □

Lemma 3.5. *Assume $\mathcal{I}$ is an ideal over some infinite set X, then* $\mathrm{cf}(\mathsf{add}(\mathcal{I})) = \mathsf{add}(\mathcal{I})$.

If $\mathsf{non}(\mathcal{I})$ *is defined, then* $\mathsf{add}(\mathcal{I}) \leq \mathrm{cf}(\mathsf{non}(\mathcal{I}))$.

If $\mathsf{cof}(\mathcal{I})$ *is infinite, then* $\mathsf{add}(\mathcal{I}) \leq \mathrm{cf}(\mathsf{cof}(\mathcal{I}))$.

Proof. Put $\lambda := \mathsf{add}(\mathcal{I})$, $\kappa := \mathrm{cf}(\lambda)$ and pick a family $\{\lambda_i \in \lambda \mid i < \kappa\}$ with $\sup_{i<\kappa} \lambda_i = \lambda$. Let $\{A_\alpha \in \mathcal{I} \mid \alpha < \lambda\}$ witness $\mathsf{add}(\mathcal{I}) = \lambda$. By the definition of $\mathsf{add}(\mathcal{I})$, for all $i < \kappa$, $B_i := \bigcup_{\alpha<\lambda_i} A_\alpha$ is in $\mathcal{I}$. Now if λ was a singular cardinal, i.e., if $\kappa < \mathsf{add}(\mathcal{I})$, then $\bigcup_{\alpha<\lambda} A_\alpha = \bigcup_{i<\kappa} B_i \in \mathcal{I}$. A Contradiction.

Put $\theta := \mathsf{cof}(\mathcal{I})$ and pick a witness $\mathcal{C} := \{C_\alpha \in \mathcal{I} \mid \alpha < \theta\}$. Also, find $\{\theta_i < \theta \mid i < \tau\}$ witnessing $\tau := \mathrm{cf}(\theta)$. By thinning-out if needed, we may assume non-redundancy of $\mathcal{C}$, i.e.:

$$(\star) \quad \alpha < \beta < \theta \rightarrow C_\beta \nsubseteq C_\alpha.$$

Put $\mathcal{C}' := \{C_{\theta_i} \mid i < \tau\}$. Now, if $\tau < \mathsf{add}(\mathcal{I})$, then $\bigcup \mathcal{C}' \in \mathcal{I}$, and there must exist some $\alpha < \theta$ with $\bigcup \mathcal{C}' \subseteq C_\alpha$. Find $i < \tau$ with $\alpha < \theta_i$, then in particular $C_{\theta_i} \subseteq \bigcup \mathcal{C}' \subseteq C_\alpha$, contradicting $(\star)$.

Put $\mu := \mathsf{non}(\mathcal{I}), \sigma := \mathrm{cf}(\mu)$ and pick some $D \in [X]^\mu$ such that $D \notin \mathcal{I}$. By $|D| = \mu$, there exists a family of sets $\{D_i \in [D]^{<\mu} \mid i < \sigma\}$ such that $D = \bigcup_{i<\sigma} D_i$. Now, by $|D_i| < \mathsf{non}(\mathcal{I})$ for all i, we know that $\{D_i \mid i < \sigma\} \subseteq \mathcal{I}$, thus, if $\sigma < \mathsf{add}(\mathcal{I})$, then $D = \bigcup_{i<\sigma} D_i \in I$. A contradiction.

□

Corollary 3.6. *Suppose $\mathcal{I}$ is a non-trivial proper ideal over some infinite set X, then:* $\aleph_0 \leq \mathrm{cf}(\mathsf{add}(\mathcal{I})) = \mathsf{add}(\mathcal{I}) \leq \min\left\{ \mathsf{cov}(\mathcal{I}), \mathrm{cf}(\mathsf{non}(\mathcal{I})), \mathrm{cf}(\mathsf{cof}(\mathcal{I}))\right\} \leq \mathsf{cov}(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I}) \leq 2^{|X|}$.

Theorem 3.7. *Assume $\mathcal{I}$ is a non-trivial proper ideal over an infinite set X.*

Suppose $\mathsf{cov}(I) = \mathsf{cof}(I) = \kappa$, then there exists some set $A \subseteq X$ such that $|A| = \kappa$ and for all $B \in \mathcal{I}$, $|B \cap A| < \kappa$.

Proof. Fix $\langle B_\alpha \mid \alpha < \kappa\rangle$ witnessing $\mathsf{cof}(\mathcal{I}) = \kappa$. We define $A = \{a_\alpha \mid \alpha < \kappa\}$ by induction on $\alpha < \kappa$. Assume $\{a_\beta \mid \beta < \alpha\}$ had already been defined. Since $\mathcal{I}$ is non-trivial, $\{a_\beta\} \in \mathcal{I}$ for all $\beta < \alpha$. It follows from $\alpha < \mathsf{cov}(\mathcal{I})$ and properness of $\mathcal{I}$ that $(\bigcup_{\beta<\alpha}\{a_\beta\} \cup \bigcup_{\beta<\alpha} B_\beta) \neq X$, so let us pick $a_\alpha \in X \setminus (\{a_\beta \mid \beta < \alpha\} \cup \bigcup_{\beta<\alpha} B_\beta)$. End of the construction.

Clearly, the construction ensures that $|A| = \kappa$. To see the other property, fix $B \in \mathcal{I}$.

By defining properties of $\langle B_\alpha \mid \alpha < \kappa\rangle$, there exists some $\beta < \kappa$ such that $B \subseteq B_\beta$. By the construction, for all $\alpha < \kappa$ with $\alpha > \beta$, $a_\alpha \in X \setminus B_\beta$ and hence $B \cap A \subseteq \{a_\delta \mid \delta \leq \beta\}$, that is, $|B \cap A| \leq |\beta| < \kappa$. □

Corollary 3.8. *If $\mathfrak{c} = \aleph_1$, then there exists a* Sierpinski set*, that is, an uncountable set $S \subseteq \mathbb{R}$ such that for any null set $N \subseteq \mathbb{R}$: $|S \cap N| \leq \aleph_0$.*

Proof. Trivially, $\mathcal{N}$ is a proper ideal. Applying $\mathsf{add}(\mathcal{N}) \geq \aleph_1$ and Corollary 3.6, we get that:

$$\aleph_1 \leq \mathsf{add}(\mathcal{N}) \leq \mathsf{cov}(\mathcal{N}) \leq \mathsf{cof}(\mathcal{N}) \leq \mathfrak{c} = \aleph_1.$$

□

Corollary 3.9 (Luzin)**.** *If $\mathfrak{c} = \aleph_1$, then there exists a Luzin set.*

Proof. By now, the only missing ingredient is the following. □

Theorem 3.10 (Baire)**.** *$\mathcal{M}$ is a proper ideal.*

Proof. We give a proof in a wider context, e.g., Theorem 3.16. See also Corollary 5.7. □

Thus, we yield the consistency of existence of a Luzin set. It is worth mentioning that the non-existence of a Luzin set is also consistent.

Definition 3.11. A set A is *comeager* iff A^c is meager.

Remark: Assume that A is meager, then there exist a sequence of nowhere dense sets $\{F_i\}_{i\in\mathbb{N}}$ such that $A = \bigcup_{i\geq 1} F_i$, therefore $A \subseteq \bigcup_{i\geq 1} \overline{F_i}$. We conclude that $\bigcap_{i\geq 1} \overline{F_i}^c \subseteq A^c$, where $\{\overline{F_i}^c\}_{i\in\mathbb{N}}$ are dense and open.

Since the converse is also true, we get that a set is comeager iff it contains a G_δ subset, such that each open set in the intersection is dense. We will see that in complete metric spaces, such sets are dense.

Definition 3.12. A metric space is *complete* iff every Cauchy sequence converges.

Lemma 3.13. *Every compact subspace of a metric space is complete.*

Proof. If C is compact, then any sequence from C has a converging subsequence, in particular if the sequence is Cauchy, its (unique) limit is in C. □

Lemma 3.14. *Every closed set in a complete metric space is complete.*

Proof. Assume X is complete, $F \subseteq X$ is closed, and $\{f_n\}_{n\in\mathbb{N}} \subseteq F$ is Cauchy.

$\{f_n\}_{n\in\mathbb{N}} \subseteq X$ is also Cauchy (since the metric on F is induced by the metric on X), thus converges to some $x \in X$. On the other hand, F is closed, so x must be in F. □

Definition 3.15. X is a *Baire space* iff the intersection of any countable family of dense open sets in X is dense.[10]

A generalization of Theorem 3.10 is the following.

Theorem 3.16. *Every complete metric space is a Baire space.*

Proof. Assume $\langle F_i \mid i \in \mathbb{N}\rangle$ is a family of closed and nowhere dense subsets in a complete metric space $\langle X, d\rangle$. We will show that $G := \left(\bigcup F_i\right)^c$ is dense in X.

Pick an arbitrary open ball B. Now, $B \setminus F_1 \neq \emptyset$ (since F_1 is nowhere dense and has no interior), so we pick $x_1 \in B \setminus F_1$. X is metric hence regular, therefore there exist an open ball B_1, such that $x_1 \in B_1 \subseteq \overline{B_1} \subseteq B \setminus F_1$, and $\mathrm{Diam}(B_1) < \frac{1}{2}$. Once again, $B_1 \setminus F_2 \neq \emptyset$, $x_2 \in B_1 \setminus F_2$ is picked and we can find some open ball B_2 that satisfies $x_2 \in B_2 \subseteq \overline{B_2} \subseteq B_1 \setminus F_2$ and $\mathrm{Diam}(B_2) < \frac{1}{3}$.

We continue likewise and construct a downward chain $\{B_n\}_{n\in\mathbb{N}}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$, such that $\mathrm{Diam}(B_n) < \frac{1}{n+1}$, and $x_n \in B_n$ for all $n \in \mathbb{N}$. $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $\overline{B_1}$ which is a complete space, thus converges to some $x \in \overline{B_1}$. Now, $\overline{B_1} \cap F_1 = \emptyset$, thus $x \in B \cap G$.

Finally, since B is an arbitrary ball, we get that G is dense. □

[10] Notice that a Baire space can not be a countable union of nowhere dense sets.

Observation 3.17. *Suppse $\langle X, O\rangle$ is a topological space and $Y \subseteq X$ is such that :*

- $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$*;*
- *If U is an open set containing Y, then $X \setminus U \models S_{fin}(\mathcal{O}, \mathcal{O})$*

then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. Assume X, Y are like in the statement. Let $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$ be a countable family of open covers of X. By $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $\langle \mathcal{U}_{2n} \subseteq O \mid n \in \mathbb{N}\rangle$ being a countable family of open covers of Y, there exists some $\langle \mathcal{F}_{2n} \in [\mathcal{U}_{2n}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n}$ is an open cover of Y. Put $U := \bigcup\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n}$. Finally, since $Y \subseteq U$ and $\langle \mathcal{U}_{2n+1} \subseteq O \mid n \in \mathbb{N}\rangle$ is an open cover of $X \setminus U$, there exists $\langle \mathcal{F}_{2n+1} \in [\mathcal{U}_{2n+1}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n+1}$ is an open cover of $X \setminus U$ and it follows that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X exemplifying $S_{fin}(\mathcal{O}, \mathcal{O})$. □

Definition 3.18. Suppose $\langle X, O\rangle$ is a topological space and κ is an infinite cardinal number. For $Y \subseteq X$, we say that X is *κ-concentrated at Y* iff for any open $U \supseteq Y$: $|X \setminus U| < \kappa$.

Corollary 3.19. *Suppose $\langle X, O\rangle$ is a topological space and $Y \subseteq X$ is such that:*

- $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$*;*
- *X is* concentrated *(i.e. $\aleph_1$-concentrated) at Y.*

then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. By Observation 3.17 and the fact that any countable set satisfies Menger's property. □

In special cases, we can prove a stronger result. We first need another definition.

Definition 3.20. For a topological space $\langle X, O\rangle$, we denote by $S_1(\mathcal{O}, \mathcal{O})$ the property that for any countable sequence of open covers of X, $\langle \mathcal{U}_n \subseteq O \mid n \in \mathbb{N}\rangle$, there exists some $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $X = \bigcup_{n\in\mathbb{N}} U_n$.

Observation 3.21. *Suppose $\langle X, O\rangle$ is a topological space and $Y \subseteq X$ is such that:*

- $Y \models S_1(\mathcal{O}, \mathcal{O})$*;*
- *X is concentrated at Y.*

then $X \models S_1(\mathcal{O}, \mathcal{O})$.

Proof. Same as in Observation 3.17. □

Corollary 3.22. *Suppose $\langle X, O\rangle$ is a topological space and is concentrated at some countable $Y \subseteq X$, then $X \models S_1(\mathcal{O}, \mathcal{O})$.*

It is worth mentioning that $S_1(\mathcal{O},\mathcal{O})$ is indeed stronger than $S_{fin}(\mathcal{O},\mathcal{O})$. $[0,1]\subseteq\mathbb{R}$ is compact, hence, satisfies Menger's property. However, for any family of open covers $\langle \mathcal{U}_n \mid n\in\mathbb{N}\rangle$ with $\mathrm{Diam}(U)<\frac{1}{2^{n+17}}$ for all $n\in\mathbb{N}$ and $U\in\mathcal{U}_n$, we get that $\sum_{n\in\mathbb{N}}\mathrm{Diam}(U_n)<1=\mathrm{Diam}([0,1])$ for all $\langle U_n\in\mathcal{U}_n \mid n\in\mathbb{N}\rangle$. In particular $[0,1]$ cannot satisfy $S_1(\mathcal{O},\mathcal{O})$.

Lemma 3.23. *If $X\subseteq\mathbb{R}$ is uncountable and F_σ (e.g. X is σ-compact), then X contains a perfect set.*

Proof. Assuming $X=\bigcup_{n\in\mathbb{N}}K_n$, where $\langle K_n \mid n\in\mathbb{N}\rangle$ are closed, we know that there must exist some $m\in\mathbb{N}$, with $|K_m|>\aleph_0$, thus, K_m is an uncountable closed set. Applying Theorem 2.24, we conclude that K_m (and hence, also X) contains a perfect subset. □

Theorem 3.24. *Megner's conjecture 1.30 is consistently false.*

Proof. Since the existence of a Luzin set is consistent, it suffices to prove that a Luzin set $L\subseteq\mathbb{R}$ satisfies Menger's property but is not σ-compact.

Claim 3.25. *L is concentrated at some $A\in[L]^{\leq\aleph_0}$.*
In particular, $L\models S_1(\mathcal{O},\mathcal{O})$.

Proof. Since $L\subseteq\mathbb{R}$, we have that $w(L)\leq w(\mathbb{R})\leq\aleph_0$. It follows from Lemma 2.6 that L is separable, so let $A\subseteq L$ be a countable dense subset of L. To see that L is concentrated at A, pick some open set $U\subseteq\mathbb{R}$ with $U\supseteq A$. To see $|L\setminus U|\leq\aleph_0$, notice that $L\setminus U=L\cap(\overline{A}\setminus U)$. Now, $\mathbb{R}\setminus\overline{(\overline{A}\setminus U)}=\mathbb{R}\setminus(\overline{A}\setminus U)=(\mathbb{R}\setminus\overline{A})\cup(\overline{A}\cap U)\supseteq(\mathbb{R}\setminus\overline{A})\cup A$, and the latter is surely dense in $\mathbb{R}$.[11] It follows from Lemma 2.19 that $\overline{A}\setminus U$ is nowhere dense. Recalling that L is a Luzin set, we conclude that $L\cap(\overline{A}\setminus U)$ is countable. □

It follows that $L\models S_{fin}(\mathcal{O},\mathcal{O})$. We are left with showing that L is not σ-compact. Using Lemma 3.23, this reduces to showing that L does not contain a perfect subset. In the following, we prove that any perfect set contains a meager subset of cardinality $\mathfrak{c}$, and hence, L cannot contain a perfect subset. □

Lemma 3.26. *If $P\subseteq\mathbb{R}$ is perfect, then there exists some $X\subseteq P$ such that:*

- *X is perfect;*
- *X is a null set.*
- *X is nowhere dense and homeomorphic to the product space $\{0,1\}^{\mathbb{N}}$;*

In particular, any perfect subset of $\mathbb{R}$ is of cardinality $\mathfrak{c}$.

Proof. We first need the following Observation:

[11]Simply because $\overline{(\mathbb{R}\setminus\overline{A})\cup A}=\overline{(\mathbb{R}\setminus\overline{A})}\cup\overline{A}=\mathbb{R}$.

Observation 3.27. *Suppose $\langle L, \leq\rangle$ is a linearly-ordered set.*

Put $\mathcal{B}_{\leq} := \{(\alpha, \beta) \mid \alpha, \beta \in L, \alpha < \beta\}$,[12] *and let $\langle L, \mathcal{O}_{\leq}\rangle$ be the topological space generated by the base $\mathcal{B}_{\leq}$ (This is called the* interval topology*).*

For any perfect $P \subseteq L$ and a closed interval $I \subseteq L$ with $I \cap P \neq \emptyset$, there exists some closed interval $J \subseteq I$ such that $J \cap P$ is perfect.

Proof. Assume P is perfect and $I = [a, b]$ is an interval with $P \cap I \neq \emptyset$. If $P \cap I$ is perfect, we are done, so assume this is not the case, that is, at least one of the elements a, b are isolated in $P \cap I$ (note that no elements of (a, b) can be isolated in $[a, b] \cap P$). If a is isolated (and b is not), then we can find some $a < c < b$ such that $[c, b] \cap P = I \cap P \setminus \{a\}$, so take $J := [c, b]$. If b is isolated (and a is not), then we can find some $a < d < b$ such that $[a, d] \cap P = I \cap P \setminus \{b\}$, so take $J := [a, d]$. If both a and b are isolated we can find $a < c < d < b$ such that $[c, d] \cap P = I \cap P \setminus \{a, b\}$, so take $J := [c, d]$. □

Assume $P \subset \mathbb{R}$ is a perfect set.

Let $\mathcal{S} := \{s : \{1, .., k\} \to \{0, 1, 2\} \mid k \in \mathbb{N}\}$ denote the family of finite ternary sequences. Define a function $\varphi : \mathcal{S} \to \{I \subseteq \mathbb{R} \mid I \text{ is a closed interval}\}$. By induction on n - the length of $s \in \mathcal{S}$. For $s \in \mathcal{S}$, we sometime write I_s for $\varphi(s)$ whenever defined.

Induction base ($n = 1$): Let $s_0 = \{(1, 0)\}, s_1 = \{(1, 1)\}, s_2 = \{(1, 2)\}$, and find a family of mutually disjoint intervals $\{I_{s_1}, I_{s_2}, I_{s_3}\}$ such that $\text{Diam}(I_{s_i}) < \frac{1}{3}$ and $I_{s_i} \cap P$ is perfect for all $i \in \{0, 1, 2\}$. (E.g. take some interval $I \subseteq P$. Since P is prefect, I is infinite, so split it into three mutually disjoint intervals, and apply the preceding observation on each one of them).

Induction step ($n + 1$): For $s \in \mathcal{S}$ of length n, find a family of mutually disjoint intervals $\mathcal{F} = \{I_{s^\frown 1}, I_{s^\frown 2}, I_{s^\frown 3}\}$ such that $\mathcal{F} \subseteq \mathcal{P}(I_s)$ and $\text{Diam}(I_{s^\frown i}) < (\frac{1}{3})^i$ for all $i \in \{0, 1, 2\}$.

Put $\varphi(s^\frown i) := I_{s^\frown i}$ for all $i \in \{0, 1, 2\}$.

Finally, we define a fucntion $\psi : \{0, 2\}^{\mathbb{N}} \to P$. For $f \in \{0, 2\}^{\mathbb{N}}$, $\cap_{n=1}^{\infty} I_{f \restriction \{1,..,n\}}$ is a single element of P, so let $\psi(f)$ be this single element. Clearly, ψ is one-to-one.

Viewing $\{0, 2\}^{\mathbb{N}}$ as the product of length ω of the discrete space $\{0, 2\}$, we already met the type of arguments justifying why ψ is an homeomorphism on $M := \text{Im}(\psi)$ (see, e.g., Lemma 2.30). Furthere more, it is not hard to see that $int(M) = \emptyset$. Since M is closed, it is also nowhere dense. The choice of diameters in the definition of φ also ensures that M is a null set.

Finally, to see that M is perfect, assume towards a contradiction that there exists some $f \in \{0, 2\}^{\mathbb{N}}$ and interval $(a, b) \subseteq \mathbb{R}$ such that $M \cap (a, b) = \{x\}$ where $x = \psi(f)$. However, by the choice of x, there exists some length $n \in \mathbb{N}$ such that $x \in I_{f \restriction \{1,..,n\}} \subseteq (a, b)$ and $I_{f \restriction \{1,..,n\}} \cap P$ is perfect. A contradiction. □

[12] $(\alpha, \beta) := \{\gamma \in L \mid \alpha < \gamma < \beta\}$ is the open interval. $[\alpha, \beta] := \{\gamma \in L \mid \alpha \leq \gamma \leq \beta\}$ is a closed interval, and so on..

Proposition 3.28. *The Cantor set is homeomorphic to* $\{0,1\}^{\mathbb{N}}$*.*

Remark: Once the proposition is proved, we get that the cantor set is a subspace of the Baire space.

Proof. Fix $x \in C$. $x = \sum_{n\geq 1} \frac{x_n}{3^n}$, where for all $n \in \mathbb{N}$, $x_n \in \{0,2\}$.

Define $\psi : C \to \{0,1\}^{\mathbb{N}}$ by $\psi(x) := \{\frac{x_n}{2}\}_{n\geq 1}$. ψ is obviously a bijection. Using similar methods from the proof of Lemma 2.30, we get that ψ is open and continuous as well. □

A more probabilistic point of view of the set $\{0,1\}^{\mathbb{N}}$ is the following: a coin with equiprobable outcome is tossed endlessly. We define Ω to be all infinite sequences of coin tosses, i.e., $\Omega = [0,1]$ (where *heads* is 1 and *tails* is 0, and we consider the binary representation of elements of $[0,1]$). The event "the first outcome is 0" is of probability 1/2. The event "the first two outcomes are 0" is of probability 1/4, etc.

It follows that $P([a,b)) = b - a$ whenever $0 \leq a \leq b \leq 1$ and a, b are of the form $k/2^n$. Such numbers are dense, and using monotonicity of probability measure we get that $P([a,b)) = b - a$ whenever $0 \leq a \leq b \leq 1$. This is of course the Lebesgue measure.

Example 3.29. Is $\mathbb{Q}$ a G_δ set?

Assume $\mathbb{Q} = \bigcap_{n\geq 1} G_n$ where G_n is open for all $n \in \mathbb{N}$. Obviously, G_n is dense for all $n \in \mathbb{N}$, since $\mathbb{Q} \subseteq G_n$. We get that $\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n\geq 1} G_n^c$ where G_n^c is nowhere dense for all $n \in \mathbb{N}$, thus $\mathbb{R} \setminus \mathbb{Q}$ is meager. But, $\mathbb{Q}$ is also meager, hence $\mathbb{R}$ is meager, a contradiction to Baire's Theorem 3.10.

Definition 3.30. Assume X is a set. A family $F \subseteq \mathcal{P}(X)$ is a *filter over* X iff it satisfies:

- $X \in F$, and $\emptyset \notin F$.
- $A \in F$ and $A \subseteq B \subseteq X \implies B \in F$.
- $A, B \in F \implies A \cap B \in F$.

Intuitively, a filter is a collection of "fat" sets. It is not hard to see that if I is a proper ideal over X, then $I^* := \{X \setminus A \mid A \in I\}$ forms a filter.

It is very often that we call sets that comes from an ideal as "sets of measure zero", sets the comes from a filter as "sets of measure one", and sets that comes from outside a given ideal as "sets of positive measure".

However, this terminology might sometimes be misleading. In the following we show that it is possible for a set to be "of measure zero" from one ideal's point of view, and "of measure one" in the view of another filter.

Proposition 3.31. *$\mathcal{N} \cap \mathcal{M}^* \neq \emptyset$, that is, $\mathbb{R}$ can be decomposed as $\mathbb{R} = D \uplus M$, where M is meager and D is a null set.*

Proof. Write $\mathbb{Q}$ as $\{q_n\}_{n\geq 1}$. Let $\{\varepsilon_k\}_{k\geq 1}$ be a sequence converging to 0. For all $k \in \mathbb{N}$ pick a sequence $\{r_{k,n}\}_{n\geq 1}$ such that $\sum_{n\in\mathbb{N}} r_{k,n} < \varepsilon_k$.

For every $k \in \mathbb{N}$, define $D_k := \bigcup_{n\in\mathbb{N}} \mathsf{B}_{r_{k,n}}(q_n)$. $D := \bigcap_k D_k$ is a null set and is the countable intersection of open dense sets, hence comeager. Now, define $M := \mathbb{R} \setminus D$. □

The example we give next is typical of an existence theorem based on the Baire's theorem. We show that some element of a space must have a given property by showing that the space is second category while the elements which do not have a given property form a set of first category.

Definition 3.32. For an interval $I \subseteq \mathbb{R}$, let $C(I)$ denote the family of all continuous real-valued function on I.

It is a well-known fact that a uniform limit of continuous function is continuous, thus, if we regard $C(I)$ as a metric space with $\rho(f,g) := \sup_{x\in I} |f(x) - g(x)|$ (for all $f, g \in C(I)$, then $\langle C(I), \rho\rangle$ is a complete metric space.

It is nice to see that if $\langle f_1, f_2, ...\rangle$ is a Cauchy sequence in $C(I)$, then, for each $x \in I$, $\{f_n(x)\}_{n\geq 1}$ is a Cauchy sequence of real numbers, hence converges.

Theorem 3.33. *There is a continuous real-valued functions on I (some closed interval) having a derivative at no point.*

Proof. Denote by $\mathcal{D}$ the set of all functions in $C(I)$ having a derivative somewhere.

Define for all $n \in \mathbb{N}$:

$$\mathcal{D}_n := \Big\{ f \in C(I) \Big| \text{ for some } x \in [0, \frac{n-1}{n}], \text{ whenever } h \in (0, 1/n], \Big|\frac{f(x+h) - f(x)}{h}\Big| \leq n \Big\}.$$

If $f \in C(I)$ has a derivative at some point, then for some large enough $n \in \mathbb{N}$, $f \in \mathcal{D}_n$. Hence $\mathcal{D} = \bigcup \mathcal{D}_n$. By showing that $\mathcal{D}_n$ is closed and has no interior (for all n) we will conclude that $C(I) \setminus \mathcal{D}$ is of the second category.

1. $\mathcal{D}_n$ has no interior: Given $f \in \mathcal{D}_n$ we will find a continuous function $g \notin \mathcal{D}_n$ such that $d(f, g) < \varepsilon$, that is, for all $x \in [0, \frac{n-1}{n}]$ there is some $h \in (0, 1/n]$ with $\left|\frac{g(x+h)-g(x)}{h}\right| > n$. Find a polynomial function $P(x)$ on $[0, 1]$ such that $d(f, P) < 1/2$ (that is possible since polynomials functions are dense in $C(I)$ with the uniform metric). Let M be the maximum slope of P in $[0, 1]$, and let $Q(x)$ be a continuous function consisting of straight line segments of slope $\pm(M + n + 1)$ constrained so that $\mid Q(x) \mid < \varepsilon/2$. Now, define $g(x) := P(x) + Q(x)$. Then $d(f, g) < d(f, P) + d(P, Q) < \varepsilon$ and:

$$\Big|\frac{g(x+h) - g(x)}{h}\Big| = \Big|\frac{P(x+h) + Q(x+h) - P(x) - Q(x)}{h}\Big| \geq \Big|\frac{Q(x+h) - Q(x)}{h}\Big| - \Big|\frac{P(x+h) - P(x)}{h}\Big|$$

But for $x \in [0, \frac{n-1}{n}]$, an $h \in (0, 1/n]$ can be found for which the latter is greater than $(M + n + 1) - M = n + 1$. Thus, $g \notin \mathcal{D}_n$.

2. $\mathcal{D}_n$ is closed: The map $e : C(I) \times I \to \mathbb{R}$ defined be $e(f, x) := f(x)$ is continuous. It follows that if h_0 is a fixed element of $(0, 1/n]$, the map $E_{h_0} : C(I) \times [0, \frac{n-1}{n}] \to \mathbb{R}$ defined by $E_{h_0}(f, x) := \left|\frac{f(x+h_0)-f(x)}{h_0}\right|$ is continuous. Thus $E_{h_0}^{-1}[0, n]$ is closed in $C(I) \times [0, \frac{n-1}{n}]$. Define $D_{h_0} := \left\{f \in C(I) \middle| (f, x) \in E_{h_0}^{-1}[0, n), \text{ for some } x \in [0, \frac{n-1}{n}]\right\}$. Then D_{h_0} is closed in $C(I)$. For if $\{f_m\}_m \subseteq D_{h_0}$ where $f_m \to f$, then $\{x_m\}_m \subseteq [0, 1 - 1/n]$ such that $\{f_m, x_m\}_m \subseteq E_{h_0}^{-1}[0, n]$ has a cluster point x. Now, $(f, x) \in E_{h_0}^{-1}[0, n]$, so that $f \in D_{h_0}$.

Now, $\mathcal{D}_n = \bigcap_{h_0 \in (0,1/n]} D_{h_0}$, establishing that $\mathcal{D}_n$ is closed.

□

4. 02.12.05

Observation 4.1. *For any open $U \subseteq \mathbb{N}^{\mathbb{N}}$, $|U| = \mathfrak{c}$ and $\underline{U} = \mathbb{N}^{\mathbb{N}}$.*

Observation 4.2. *For all $g \in \mathbb{N}^{\mathbb{N}}$, $\{f \in \mathbb{N}^{\mathbb{N}} \mid g \leq^* f\}$ is dense in $\mathbb{N}^{\mathbb{N}}$.*

Lemma 4.3. *Suppose $Y \subseteq \mathbb{N}^{\mathbb{N}}$ is a compact subspace, then there exists some $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq g$ for all $f \in Y$.*

Proof. For all $n \in \mathbb{N}$, consider the projection $\pi_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that $\pi_n(f) = f(n)$ for all $f \in \mathbb{N}^{\mathbb{N}}$. By definition of the Baire space, each π_n is continuous and by the hypothesis, Y is compact and it follows that $\pi_n[Y]$ is compact in $\mathbb{N}$. Since any compact subspace of the discrete space $\mathbb{N}$ is finite, we conclude that for all $n \in \mathbb{N}$, there exists some $m_n \in \mathbb{N}$ such that $\pi_n[Y] \subseteq \{1, .., m_m\}$. It other words, the function $g \in \mathbb{N}^{\mathbb{N}}$ defined by $n \mapsto m_n$ has the property that $f \leq g$ for all $f \in Y$ and we are done. □

Observation 4.4. *For all $g \in \mathbb{N}^{\mathbb{N}}$, $D_g := \{f \in \mathbb{N}^{\mathbb{N}} \mid f \leq g\}$ is a closed, nowhere-dense, subspace of $\mathbb{N}^{\mathbb{N}}$.*

Proof. Fix $g \in \mathbb{N}^{\mathbb{N}}$. Assume $h \in \mathbb{N}^{\mathbb{N}} \setminus D_g$. Then there exists some $n \in \mathbb{N}$ such that $h(n) > g(n)$. Then h is in the open set $U = \{f \in \mathbb{N}^{\mathbb{N}} \mid f(n) = h(n)\}$ and $U \subseteq \mathbb{N}^{\mathbb{N}} \setminus D_g$.

To see that $\mathbb{N}^{\mathbb{N}} \setminus D_g$ is dense, we fix a base open set U, and show that $U \cap (\mathbb{N}^{\mathbb{N}} \setminus D_g) \neq \emptyset$. Find $n \in \mathbb{N}$, and $\sigma : \{1, ..n\} \to \mathbb{N}$ such that $U = \sigma^{\uparrow}$. Let $h \in \mathbb{N}^{\mathbb{N}}$ be such that $h \restriction \{1, .., n\} = \sigma$ and $h(k) = g(k) + 1$ for all $k > n$. Clearly, $h \in U \setminus D_g$. □

Corollary 4.5. *For all $g \in \mathbb{N}^{\mathbb{N}}$, $E_g := \{f \in \mathbb{N}^{\mathbb{N}} \mid f \leq^* g\}$ is an F_σ meager subspace of $\mathbb{N}^{\mathbb{N}}$.*

Proof. If σ is a finite sequence of natural numbers, we may consider $sw(\sigma, g) \in \mathbb{N}^{\mathbb{N}}$ such that $sw(\sigma, g)(n) = \sigma(n)$ if $n \in \mathrm{dom}(\sigma)$ and $sw(\sigma, g)(n) = g(n)$ otherwise.

Then $E_g = \bigcup\{D_{sw(\sigma,g)} \mid \sigma$ is a finite sequence of natural numbers $\}$. □

Definition 4.6. Let $\mathcal{I}_{\mathfrak{b}} := \{X \subseteq \mathbb{N}^{\mathbb{N}} \mid \mathrm{ecf}(X) \leq 1\}$.

It is by the definition of $\mathfrak{b}$ that $\mathcal{I}_{\mathfrak{b}}$ is a non-trivial proper ideal, $\mathsf{add}(\mathcal{I}_{\mathfrak{b}}) = \mathfrak{b}$, and $\mathcal{I}_{\mathfrak{b}}$ contains exactly all sets that are $\leq^*$-bounded in $\mathbb{N}^{\mathbb{N}}$.

Also notice that $\mathcal{I}_{\mathfrak{b}} = \{X \subseteq \mathbb{N}^{\mathbb{N}} \mid \mathrm{ecf}(X) < \mathfrak{b}\}$ and $\mathsf{cov}(\mathcal{I}_{\mathfrak{b}}) = \mathsf{cof}(\mathcal{I}_{\mathfrak{b}}) = \mathfrak{d}$.

Corollary 4.7. *Suppose that $Z \subseteq \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{b}$-compact topological space, then $Z \in \mathcal{I}_{\mathfrak{b}}$, i.e., there exists some $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for all $f \in Z$.*

In particular (since $\aleph_1 \leq \mathfrak{b}$), any σ-compact subspace of $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-bounded.

Proof. Let $\langle Z_\alpha \subseteq Z \mid \alpha < \kappa\rangle$ witness $\mathfrak{b}$-compactness of Z (in particular, $\kappa < \mathfrak{b}$). For all $\alpha < \kappa$, Theorem 4.3 implies that $Z_\alpha \in \mathcal{I}_{\mathfrak{b}}$ (and even more, but we don't care). Now, by $\kappa < \mathsf{add}(\mathcal{I}_{\mathfrak{b}})$, $Z = \bigcup_{\alpha<\kappa} Z_\alpha \in \mathcal{I}_{\mathfrak{b}}$ and we are done. □

Observation 4.8. $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$.

Proof. Pick a cofinal subset $D \subseteq [\mathbb{N}^\mathbb{N}]^\mathfrak{d}$ and an homeomorphism $\psi : \mathbb{N}^\mathbb{N} \to \mathbb{R}\setminus\mathbb{Q}$. By Corollary 4.5 and $\{\underline{\{f\}} \mid f \in D\} \subseteq \mathcal{I}_\mathfrak{b}$, we have that $\{\psi[\underline{\{f\}}] \mid f \in D\} \subseteq \mathcal{M}$. Finally, since

$$\mathbb{R} = \psi[\mathbb{N}^\mathbb{N}] \cup \mathbb{Q} = \psi[\bigcup_{f\in D} \underline{\{f\}}] \cup \mathbb{Q} = \bigcup\{\psi[\underline{\{f\}}], \mathbb{Q} \mid f \in D\} =: \bigcup A,$$

and $A \in [\mathcal{M}]^\mathfrak{d}$, we conclude that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$. □

Observation 4.9. *There exists $X \in \mathcal{I}_\mathfrak{b}$ with $|X| = \mathfrak{c}$.*

In particular, if $\mathfrak{b} < \mathfrak{c}$, then there exists $X \in \mathcal{I}_\mathfrak{b}$ with $|X| > \mathfrak{b}$.

Proof. Consider $X := \underline{\{f\}}$ where $f : \mathbb{N} \to \{2\}$ is the constant function. □

Theorem 4.10 (Hurewicz)**.** *For all $X \subseteq \mathbb{R}$, TFAE:*

- $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.
- *Any continuous image of X into $\mathbb{N}^\mathbb{N}$ is non-dominating.*

Proof. We omit the proof. Instead, we prove the following two propositions. □

Theorem 4.11. *If $\langle X, O\rangle$ is a topological space and $X \models S_{fin}(\mathcal{O}, \mathcal{O})$, then any continuous image of X into $\mathbb{N}^\mathbb{N}$ is non-dominating.*

Proof. By Lemma 2.1, we may assume that $X \subseteq \mathbb{N}^\mathbb{N}$ and $X \models S_{fin}(\mathcal{O}, \mathcal{O})$. Fix $m \in \mathbb{N}$. Put $\mathcal{U}_m := \{(m, k)^\uparrow \mid k \in \mathbb{N}\}$ where $(m, k)^\uparrow := \{f \in \mathbb{N}^\mathbb{N} \mid f(m) = k\}$ for all $k \in \mathbb{N}$. Evidently, $\mathcal{U}_m$ is an open cover of X (and actually of $\mathbb{N}^\mathbb{N}$). Fix a bijection $\psi : \mathbb{N} \times \mathbb{N} \leftrightarrow \mathbb{N}$. Fix $i \in \mathbb{N}$.

Since $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $\langle \mathcal{U}_{\psi(i,n)} \mid n \in \mathbb{N}\rangle$ is a countable family of open covers of X, there exists some $\langle \mathcal{F}_{\psi(i,n)} \in [\mathcal{U}_{\psi(i,n)}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{\psi(i,n)}$ is an open cover of X.

Define $g : \mathbb{N} \to \mathbb{N}$. For $m \in \mathbb{N}$, let $g(m) := 1 + \max\{k \in \mathbb{N} \mid (m, k)^\uparrow \in \mathcal{F}_m\}$. The definition is good since $\mathcal{F}_m \subseteq \mathcal{U}_m = \{(m, k)^\uparrow \mid k \in \mathbb{N}\}$ and finite. We claim that g witnesses that X is not-dominating. We pick $f \in X$ and show that $\chi_{f,g} := \{m \in \mathbb{N} \mid g(m) \not\leq f(m)\}$ is infinite. We do this by introducing some $h \in \mathbb{N}^\mathbb{N}$ with the property that $\{\psi\big(i, h(i)\big) \mid i \in \mathbb{N}\} \subseteq \chi_{f,g}$.

Fix $i \in \mathbb{N}$. Since $\bigcup_{n\in\mathbb{N}} \mathcal{F}_{\psi(i,n)}$ is an open cover of X, there exists some $n \in \mathbb{N}$ such that $f \in \mathcal{F}_{\psi(i,n)}$, so let $h(i) := n$ for such an n. End of definition. It follows that $f \in \mathcal{F}_{\psi(i,h(i))}$ for all $i \in \mathbb{N}$, and hence $f(\psi(i, h(i))) \leq g(\psi(i, h(i)))-1$. In particular, $\forall i \in \mathbb{N}\big(\psi(i, h(i)) \in \chi_{f,h}\big)$. □

Theorem 4.12 (Recław)**.** *Suppose $\langle X, O\rangle$ is a topological space that has a base $\mathcal{B}$ which is countable and composed only of clopen sets.*

If any continuous image of X into $\mathbb{N}^\mathbb{N}$ is non-dominating, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. By Observation 1.31, we assume a family of open covers of X, $\langle \mathcal{U}_n \subseteq \mathcal{B} \mid n \in \mathbb{N}\rangle$. Since $\mathcal{B}$ is countable, there exists an enumeration $\mathcal{U}_n = \{U_n^m \mid m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. Now, for all $n, m \in \mathbb{N}$, let $V_n^m := U_n^m \setminus \bigcup_{k<m} U_n^k$.

By the hypothesis on $\mathcal{B}$, V_n^m are open for all $n, m \in \mathbb{N}$.

It follows that we may assume for all $n \in \mathbb{N}$ that members of $\mathcal{U}_n$ are mutually-disjoint, thus, for all $x \in X$, there is a unique $f_x \in \mathbb{N}^{\mathbb{N}}$ such that $x \in U_n^{f_x(n)}$ for all $n \in \mathbb{N}$. Finally, let $\psi : X \to \mathbb{N}^{\mathbb{N}}$ be the map $x \mapsto f_x$.

To see that ψ continuous, fix some some $n \in \mathbb{N}$ and $\sigma : \{1, .., n\} \to \mathbb{N}$. We shall show that $\psi^{-1}[\sigma^{\uparrow}]$ is open. Indeed, by definition, $\psi^{-1}[\sigma^{\uparrow}] = \bigcap_{k=1}^{n} U_k^{\sigma(k)}$ which is a finite intersection of open sets, thus, open.

Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness to the fact that $\psi[X]$ is non-dominating. For all $n \in \mathbb{N}$, put $\mathcal{F}_n := \{U_n^1, .., U_n^{g(n)}\}$. We claim that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X. To see this, fix $x \in X$. By definition of g, there must exist some $n \in \mathbb{N}$ with $g(n) \not\leq f_x(n)$, that is, there exists some $k < g(n)$ such that $x \in U_n^k$, and clearly $U_n^k \in \mathcal{F}_n$. It follows that $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{F}_n$. □

Corollary 4.13. *If $X \in [\mathbb{R}]^{<\mathfrak{d}}$, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*

Proof. By ($\Leftarrow$) of Theorem 4.10. □

We now get a result stronger than 3.19, but is only limited to subspaces of the real line.

Corollary 4.14. *Suppose $Y \subseteq X \subseteq \mathbb{R}$ are such that:*

- *$Y \models S_{fin}(\mathcal{O}, \mathcal{O})$;*
- *X is $\mathfrak{d}$-concentrated at Y.*

then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. By Observation 3.17 and the preceding Corollary. □

Corollary 4.15. *If $X \subseteq \mathbb{R}$ is $\mathfrak{d}$-concentrated at some $Y \in [\mathbb{R}]^{<\mathfrak{d}}$, then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.*

Theorem 4.16. *Suppose $X \subseteq \mathbb{R}$ is $\mathfrak{c}$-concentrated at some countable $D \subseteq X$, then X does not contain a perfect subset.*

Proof. Suppose not, and let X be a witness to that. By Lemma 3.26, X contains a closed subspace, C, which is homeomorphic to $\{0, 1\}^{\mathbb{N}}$. Since C is closed and is of cardinality $\mathfrak{c}$, we must conclude that C is $\mathfrak{c}$-concentrated on $C \cap D$, thus it suffices to prove the following. □

Lemma 4.17. *$\{0, 1\}^{\omega}$ is not $\mathfrak{c}$-concentrated at any of its countable subsets.*

Proof. Let $D = \{f_n \mid n \in \omega\}$ be a countable subset of $\{0, 1\}^{\omega}$.

For $n \in \omega$, $U_n := (f_n \restriction \{2n, 2n + 1\})^{\uparrow}$ is an open set containing f_n. It follows that $D \subseteq U$ where $U := \bigcup_{n\in\omega} U_n$. We are left with showing that $\{0, 1\}^{\omega} \setminus U$ is of cardinality $\mathfrak{c}$.

Indeed, for each $a:\omega\to\{0,1\}$, let $f_a:\omega\to\{0,1\}$ be the function satisfying for all $n\in\omega$:

$$f_a(2n+a(n))=f_n(2n+a(n))\quad\text{and:}$$
$$f_a(2n+1-a(n))=1-f_n(2n+1-a(n)).$$

It follows that $\{k\in\omega\mid f_n(k)=f_a(k)\}$ and $\{k\in\omega\mid f_n(k)\neq f_a(k)\}$ are both non-empty for all $n\in\omega$. More importantly, $a\mapsto f_a$ is injective. Thus, $\{f_a\mid a\in{}^{\omega}\{0,1\}\}$ is a subset of $\{0,1\}^\omega$ of cardinality $\mathfrak{c}$ and disjoint from the open set U containing D. □

Corollary 4.18. *If $X\subseteq\mathbb{R}$ is uncountable and $\mathfrak{d}$-concentrated at some $Y\in[\mathbb{R}]^{\aleph_0}$, then X is a counter-example to Menger's conjecture 1.30.*

Proof. By Corollary 4.15, Lemma 3.23 and Theorem 4.16. □

Corollary 4.19. *For all $X\subseteq\mathbb{R}$, if $\aleph_0<|X|<\mathfrak{d}$, then X is a counter-example to Menger's conjecture 1.30.*

In particular, if $\mathfrak{d}>\aleph_1$, then there exists a counter-example to the conjecture.

Theorem 4.20 (Fremlin-Miller)**.** *Menger's conjecture 1.30 is false.*

Proof by Bartoszyński-Tsaban. Let $D\subseteq\mathbb{N}^\mathbb{N}$ be a $\mathfrak{d}$-scale (see Lemma 1.12) and $\psi:\mathbb{N}^\mathbb{N}\leftrightarrow[0,1]\setminus\mathbb{Q}$ be an homemorphism (see Theorem 2.29). Consider $M:=\psi[D]\cup(\mathbb{Q}\cap[0,1])$.

We shall show that M is $\mathfrak{d}$-concentrated at $\mathbb{Q}\cap[0,1]$. Suppose that $U\subseteq\mathbb{R}$ is open and $U\supset(\mathbb{Q}\cap[0,1])$. It follows that:

$$|M\setminus U|=|\psi[D]\cap([0,1]\setminus U)|=|D\cap K|,$$

where $K:=\psi^{-1}([0,1]\setminus U)$.

Since $([0,1]\setminus U)$ is a closed subset of the bounded interval $[0,1]$, it is compact, and hence K is compact. Applying Lemma 4.3 on K, we find some $g\in\mathbb{N}^\mathbb{N}$ such that $K\subseteq\underline{\{g\}}$. Finally, since D is a $\mathfrak{d}$-scale we conclude that $|M\setminus U|=|D\cap K|\leq|D\cap\underline{\{g\}}|<\mathfrak{d}$. □

Similarly, If $B\subseteq\mathbb{N}^\mathbb{N}$ is a $\mathfrak{b}$-scale, then $H:=\psi[B]\cup(\mathbb{Q}\cap[0,1])$ is $\mathfrak{b}$-concentrated at $\mathbb{Q}\cap[0,1]$, thus, $H\subseteq\mathbb{R}$ is another counter-example to Menger's conjecture.

We next give a little background on connectedness.

Definition 4.21. A space X is *disconnected* iff there are disjoint non-empty open sets H,K such that $X=H\cup K$. When no such disconnection exists, X is *connected*.

A space X is *totally disconnected* iff for every $x\in X$ the only connected set containing x is $\{x\}$.

Note that we can replace "open" in the definition by "closed". It is apparent, then, that X is connected iff there are no clopen (open-closed) subsets of X but X itself and $\emptyset$.

The Cantor set, the rationals and the irrationals, are all totally disconnected spaces.

Definition 4.22. A space X is *0-dimensional* iff X has a base consisting of only clopen sets.

Equivalently, X is 0-dimensional iff for each $x \in X$ and a closed set $A \subset X$ not containing x, there is a clopen set containing x and disjoint from A. By this, the following is immediate.

Proposition 4.23. *Every 0-dimensional T_1 space is totally disconnected.*[13]

Lemma 4.24. *If X is a compact, totally disconnected Hausdorff space, then whenever $x \neq y$ in X, there is a clopen set in X containing x but not y.*

Definition 4.25. A space $\langle X, O \rangle$ is locally compact iff whenever $x \notin A$ where A is closed, there is an open set with a compact closure disjoint from A.

Observation 4.26. *If $\langle X, O \rangle$ is a compact topological space and $Y \subseteq X$ is a closed subspace, then Y is compact.*

Corollary 4.27. *Locally compact is an hereditary property.*

Theorem 4.28. *A locally compact, Hausdorff space is 0-dimensional iff it is totally disconnected.*

Proof. It suffices to prove that a locally compact, totally disconnected Hausdorff space is 0-dimensional.

Assume A is a closed set in X, where $x \notin A$. Let U be an open set with compact closure such that $x \in U \subseteq \overline{U} \subseteq A^c$. For each $p \in \overline{U} \setminus U$, let V_p be a clopen subset of $\overline{U}$ containing x but not p. The sets $X \setminus V_p$ form an open cover of $\overline{U} \setminus U$ so a finite subcover exists , say corresponding to the points $p_1, ..., p_n$. Let $V := V_{p_1} \cap \cdots \cap V_{p_n}$. Then V is clopen in $\overline{U}$ containing x and disjoint from $\overline{U} \setminus U$. But then $V \subset U$ and hence is a clopen set in X containing x and disjoint from A. We conclude that X is 0-dimensional. □

[13] X is T_1 iff for every $x \neq y$ in X there is an open set containing x but not y.

5. 08.12.05

Proposition 5.1. $\mathbb{N}^{\mathbb{N}}$ *has a countable base consisting of clopen sets.*

Proof. $\left\{\{(n_1,\dots,n_k)\}\times\mathbb{N}^{\mathbb{N}} \middle| n_1,\dots,n_k,k\in\mathbb{N}\right\}$ is a countable base for $\mathbb{N}^{\mathbb{N}}$ (Recall Example 2.14). The complement of a base set $\{(n_1,\dots,n_k)\}\times\mathbb{N}^{\mathbb{N}}$, is equal to the union of all sets of the form $\{(m_1,\dots,m_k)\}\times\mathbb{N}^{\mathbb{N}}$ where exists $i\le k$ such that $m_i\neq n_i$. This is a union of open sets, hence open. Therefore $\{(m_1,\dots,m_k)\}\times\mathbb{N}^{\mathbb{N}}$ is also closed. □

$\mathcal{B} := \left\{(a,b)\cap(\mathbb{R}\setminus\mathbb{Q}) \mid a,b\in\mathbb{Q}\right\} = \left\{[a,b]\cap(\mathbb{R}\setminus\mathbb{Q}) \mid a,b\in\mathbb{Q}\right\}$ is a countable family of clopen sets, admitting a base to $\mathbb{R}\backslash\mathbb{Q}$. Applying 2.29, we have another proof to Proposition 5.1.

Definition 5.2. Whenever $\langle X,O\rangle$ is a topological space whose topology O is a metric topology[14] (generated by some metric ρ), we say that $\langle X,O\rangle$ is a *metrizable* topological space.

In this case we can say that the metric is *compatible* with the topology.

Lemma 5.3. *Every metric ρ on a set X is equivalent to a bounded metric.*[15]

Proof. There are two standard ways of replacing ρ by a bounded metric: define new functions ρ_1 and ρ_2 on $X\times X$ by

$$\rho_1(x,y) := \min\{1,\rho(x,y)\}$$
$$\rho_2(x,y) := \frac{\rho(x,y)}{1+\rho(x,y)}$$

We will show that ρ_1 is indeed a metric on X, generating the same topology as ρ does. The reader may verify the same for ρ_2.

ρ_1 is a metric:

- $\rho_1(x,y)=\min\{1,\rho(x,y)\}\ge 0$ since $\rho(x,y)\ge 0$.
- $\rho_1(x,y)=0$ iff $\rho(x,y)=0$ and this occur iff $x=y$.
- $\rho_1(x,z)=\min\{1,\rho(x,z)\}\le\min\{1,\rho(x,y)+\rho(y,z)\}\le\min\{1,\rho(x,y)\}+\min\{1,\rho(y,z)\} = \rho_1(x,y)+\rho_1(y,z)$

ρ_1 generates the same topology as ρ does: on one hand, for some $d>0$, $\mathsf{B}^{\rho_1}_d(x)\supseteq \mathsf{B}^{\rho}_{\min\{1,d\}}(x)$. On the other hand, for some $d<1$, $\mathsf{B}^{\rho_1}_d(x)=\mathsf{B}^{\rho}_d(x)$ (where $\mathsf{B}^{\rho_1}_d(x)$ for example is the set $\{y\in X \mid \rho_1(x,y)<d\}$). □

Theorem 5.4. *A product space $\prod_{n\in\mathbb{N}} X_n$ is metrizable iff each space X_n is metrizable.*

[14] Open balls generated by any metric is always a topology base.

[15] Two metrics on a set are equivalent if they generate the same topology.

Proof. ($\Rightarrow$) Each X_n is homeomorphic to a subspace of the product space, hence metrizable.

($\Leftarrow$) Let $\langle\langle X_n, \rho_n\rangle \mid n \in \mathbb{N}\rangle$ be a family of metric spaces with $\mathrm{Im}(\rho_n) \subseteq [0,1]$ for all $n \in \mathbb{N}$.

Define ρ on $X := \prod X_i$ as follows: for $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$

$$\rho(x,y) := \sum_{i\in\mathbb{N}} \frac{\rho_i(x_i, y_i)}{2^i}.$$

It is easily verified to be a metric. We will show that it gives the product topology in X.

Pick $x = (x_1, x_2, ...) \in X$ and assume $B_x \subseteq X$ is an open set containing x. We may assume that B_x is the is of the following form:

$$B_x = \mathsf{B}_{\varepsilon_1}(x_1) \times \cdots \times \mathsf{B}_{\varepsilon_n}(x_n) \times \prod_{k>n} X_k.$$

where $\mathsf{B}_{\varepsilon_i}(x_i) = \{y \in X_i \mid \rho_i(y, x_i) < \varepsilon_i\}$ for all relevant i.

Put $\varepsilon := \min\left(\frac{\varepsilon_1}{2}, ..., \frac{\varepsilon_n}{2^n}\right)$. Now, if $\rho(x,y) < \varepsilon$, then $\rho_i(x_i, y_i) < \varepsilon_i$ for all $i \in \mathbb{N}$, so apparently $\mathsf{B}_\varepsilon(x) \subset B_x$. Thus the product topology on X is weaker that the topology induced by ρ. On the other hand, given $\varepsilon > 0$, we can choose N large enough that $\sum_{i\geq N+1} \frac{1}{2^i} < \varepsilon/2$. Then it is easily verified that $\mathsf{B}_{\frac{\varepsilon}{2N}}(x_1) \times \cdots \times \mathsf{B}_{\frac{\varepsilon}{2N}}(x_N) \times \prod_{k>N} X_k \subset \mathsf{B}_\varepsilon(x)$, hence, the topology induced by ρ is weaker the the product topology. □

Corollary 5.5. *$\mathbb{N}^\mathbb{N}$ is a metric-space.*

Proposition 5.6. *$\mathbb{N}^\mathbb{N}$ is a complete metric space.*

Proof. For $f, g \in \mathbb{N}^\mathbb{N}$, denote by $N(f,g) := \min\{n \in \mathbb{N} \mid f(n) \neq g(n)\}$. Now, define $\rho(f,g) := \frac{1}{N(f,g)}$. As in the proof of Theorem 5.4, ρ is a metric that is compatible with the usual product topology of $\mathbb{N}^\mathbb{N}$.

Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For $K \in \mathbb{N}$, there exists $N_K \in \mathbb{N}$ such that $d(f_l, f_m) < 1/K$ for all $l, m \geq N_K$. By definition of ρ this means that $f_l(n) = f_m(n)$ for all $l, m \geq N_k$ and $n \leq K$.

Define $f \in \mathbb{N}^\mathbb{N}$ as follows: for every $n \in \mathbb{N}$ define $f(n) := f_{N_n}(n)$. Obviously, $d(f_n, f) \to 0$ as $n \to \infty$, concluding that $\mathbb{N}^\mathbb{N}$ is complete. □

Corollary 5.7. *$\mathbb{N}^\mathbb{N}$ is a Baire space.*

Notice that if a space is locally compact, then it is also a Baire space, this is essentially due to Lemma 3.13 and Theorem 3.16.

Now, Since $\mathbb{R}$ is locally compact, and $\mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$,[16] we know that $\mathbb{N}^\mathbb{N}$ is also locally compact.[17] This gives another proof for the preceding Corollary.

[16]**homeomorphic**, not isometric.

[17]Recall Corollary 4.27.

Definition 5.8. Suppose $\langle X, O\rangle$ is a topological space. A family of open sets $\mathcal{U} \subseteq O$ is a *γ-cover* iff $\mathcal{U}$ is infinite, and for all $x \in X$, $\{U \in \mathcal{U} \mid x \notin U\}$ is finite.

Thus, for instance, $\{(-n, n) \mid n \in \mathbb{N}\}$ is a γ-cover of $\mathbb{R}$.

Observation 5.9. *If $\mathcal{U}$ is a γ-cover of some space $\langle X, O\rangle$, then any infinite subset $\mathcal{V} \subseteq \mathcal{U}$ is a γ-cover.*

In particular, any γ-cover contains a countable γ-cover.

Observation 5.10. *Suppose $\mathcal{U} = \{U_n \mid n \in \mathbb{N}\}$ is an open cover of some space $\langle X, O\rangle$, then either $\mathcal{U}$ contains a finite subcover, or that $\mathcal{V} := \{\bigcup_{m\leq n} U_n \mid n \in \mathbb{N}\}$ is a γ-cover of X.*

Proof. If $\mathcal{U}$ does not contain a finite subcover, then $\mathcal{V}$ is infinite, and is clearly a γ-cover. □

Definition 5.11. For a topological space $\langle X, O\rangle$ denote $\mathcal{O} := \{\mathcal{U} \subseteq O \mid \mathcal{U}$ is an open cover of $X\}$ and $\Gamma := \{\mathcal{V} \subseteq O \mid \mathcal{V}$ is an open γ-cover of $X\}$.

Definition 5.12 (Hurewicz)**.** A space $\langle X, O\rangle$ satisfies *Hurewicz's property* or $U_{fin}(\mathcal{O}, \Gamma)$ iff for any sequence of open covers of X, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$, such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ forms a γ-cover of X.

Observation 5.13. *$U_{fin}(\mathcal{O}, \Gamma)$ is a topological property and there also exists an analogue of Observation 1.31 for $U_{fin}(\mathcal{O}, \Gamma)$.*

Proof. Essentially the same proofs of 2.1 and 1.31. □

To compare the definition of $U_{fin}(\mathcal{O}, \Gamma)$ with $S_{fin}(\mathcal{O}, \mathcal{O})$ (Definition 1.26), it is evident that the left hand side set ($\mathcal{O}$ in both cases) is the requirement that $\mathcal{U}_n \in \mathcal{O}$ for all $n \in \mathbb{N}$.

Now, for the right hand side, in the first case we need to generate a γ-cover, that is, a member of Γ, while, on the other, we need to generate an open cover, that is, a member of $\mathcal{O}$. The generation is always based at some finite sets $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$, where S "says" that the object is obtained by taking $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, and U says that the object is obtained by considering $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$.

Observation 5.14. *$X \models S_{fin}(\mathcal{O}, \Gamma)$ implies that any open cover of X contains a γ-cover.*

Consequently, no topological space X satisfies $S_{fin}(\mathcal{O}, \Gamma)$.

Proof. For an open cover $\mathcal{U}$, consider $\langle \mathcal{U}_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$ where $\mathcal{U}_n := \mathcal{U}$ for all $n \in \mathbb{N}$. By the hypothesis, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \subseteq \mathcal{U}$ is a γ-cover.

To see the second assertion, take $\mathcal{U} := \{X\}$. □

Observation 5.15. $U_{fin}(\mathcal{O}, \Gamma) \Rightarrow S_{fin}(\mathcal{O}, \mathcal{O})$.

Proof. We assume a topological space $\langle X, O\rangle$ and $\langle U_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$. By the hypothesis, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \Gamma$.

We claim that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X. Indeed, since $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ covers X, we have:

$$X \subseteq \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{F}_n = \bigcup \bigcup_{n\in\mathbb{N}} \mathcal{F}_n.$$

□

We can now obtain the result of Lemma 1.29 as an application of the preceding together with the following.

Lemma 5.16. *If $\langle X, O\rangle$ is a σ-compact topological space, then $X \models U_{fin}(\mathcal{O}, \Gamma)$.*

Proof. Suppose $\langle K_n \mid n \in \mathbb{N}\rangle$ is an increasing sequence of compact subspaces of X, whose union is X, and $\langle \mathcal{U}_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover of X. By compactness of each factor, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $K_n \subseteq \bigcup \mathcal{F}_n$ for all $n \in \mathbb{N}$. Finally, since $\langle K_n \mid n \in \mathbb{N}\rangle \nearrow X$, we conclude that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ is a γ-cover of X (it is infinite because each $\mathcal{U}_n$ does not contain a finite subcover). □

Conjecture 5.17 (Hurewicz)**.** *$U_{fin}(\mathcal{O}, \Gamma)$ is equivalent to σ-compactness.*

The reader might want to compare the above with Conjecture 1.30. To continue the research, we need the following reduction theorem, an analogue of Theorem 4.10.

Theorem 5.18 (Hurewicz)**.** *For all $X \subseteq \mathbb{R}$, TFAE:*

- *$X \models U_{fin}(\mathcal{O}, \Gamma)$.*
- *Any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-bounded.*

Proof. We omit the proof. Instead, we prove the following two propositions. □

Theorem 5.19. *If $\langle X, O\rangle$ is a topological space and $X \models U_{fin}(\mathcal{O}, \Gamma)$, then any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-bounded.*

Proof. By Observation 5.13, we may assume that $X \subseteq \mathbb{N}^{\mathbb{N}}$ and $X \models U_{fin}(\mathcal{O}, \Gamma)$. Fix $n \in \mathbb{N}$. Put $\mathcal{U}_n := \{(n, k)^{\uparrow} \mid k \in \mathbb{N}\}$. Evidently, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle \in \mathcal{O}$, so let $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ witness $U_{fin}(\mathcal{O}, \Gamma)$. Define $g : \mathbb{N} \to \mathbb{N}$. For $n \in \mathbb{N}$, let $g(n) := 1 + \max\{k \in \mathbb{N} \mid (n, k)^{\uparrow} \in \mathcal{F}_n\}$.

To see that $X \subseteq \underline{\{g\}}$, we pick $f \in X$ and show that $f \leq^* g$.

Since $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \Gamma$, there exists some $N \in \mathbb{N}$, such that $f \in \bigcup \mathcal{F}_n$ for all $n \geq N$, that is, $f(n) \leq g(n)$ for all $n \geq N$, and we are done. □

Theorem 5.20 (Recław)**.** *Suppose $\langle X, O\rangle$ is a topological space that has a base $\mathcal{B}$ which is countable and composed only of clopen sets.*

If any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is $\leq^*$-boudned, then $X \models U_{fin}(\mathcal{O}, \Gamma)$.

Proof. By Observation 5.13, we assume a family of open covers of X, $\langle \mathcal{U}_n \subseteq \mathcal{B} \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover. Since $\mathcal{B}$ is countable, there exists an enumeration $\mathcal{U}_n = \{U_n^m \mid m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. We may also assume that members of $\mathcal{U}_n$ are mutually-disjoint for all $n \in \mathbb{N}$, thus, for all $x \in X$, there is a unique $f_x \in \mathbb{N}^{\mathbb{N}}$ such that $x \in U_n^{f_x(n)}$ for all $n \in \mathbb{N}$. Finally, let $\psi : X \to \mathbb{N}^{\mathbb{N}}$ be the map $x \mapsto f_x$.

Since ψ is continuous, we may pick $g \in \mathbb{N}^{\mathbb{N}}$ witnessing that $\psi[X]$ is $\leq^*$-boudned. For all $n \in \mathbb{N}$, put $\mathcal{F}_n := \{U_n^1, .., U_n^{g(n)}\}$. To see that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ is a γ-cover, fix $x \in X$. By definition of g, there exists some $N \in \mathbb{N}$ such that $f_x(n) \leq g(n)$ for all $n \geq N$, and hence, $x \in \bigcup \mathcal{F}_n$ for all $n > N$. As usual, $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ is infinite because each $\mathcal{U}_n$ does not contain a finite subcover. □

Corollary 5.21. *If $X \in [\mathbb{R}]^{<\mathfrak{b}}$, then $X \models U_{fin}(\mathcal{O}, \Gamma)$.*

Proof. By ($\Leftarrow$) of Theorem 5.18. □

The next is similar to Corollary 4.19.

Corollary 5.22. *Any uncountable $X \in [\mathbb{R}]^{<\mathfrak{b}}$ is a counter-example to Hurewicz's conjecture. In particular, Hurewicz's conjecture 5.17 is consistently false.*

Proof. Suppose $\mathfrak{b} > \aleph_1$ (this assumption is consistent) and $X \in [\mathbb{R}]^{<\mathfrak{b}}$ is uncountable. If X was σ-compact, then by Lemma 3.23, it had contained a perfect subset and by Lemma 3.26, X had to contained a set of size $\mathfrak{c}$, contradicting $|X| < \mathfrak{b} \leq \mathfrak{c}$. □

Observation 5.23. *Consistently, there exists $X \subseteq \mathbb{N}^{\mathbb{N}}$ such that:*

(a) $X \models S_{fin}(\mathcal{O}, \mathcal{O})$,
(b) $X \not\models U_{fin}(\mathcal{O}, \Gamma)$ *(and in particular, X is not σ-compact).*

Thus, consistently: Menger's conjecture 1.30 has a counter-example already inside $\mathbb{N}^{\mathbb{N}}$, and Observation 5.14 cannot be improved.

Proof. Put $\mathcal{J} := \{Y \subseteq \mathbb{N}^{\mathbb{N}} \mid Y \text{ is meager }\}$. By Corollary 5.7, $\mathcal{J}$ is a proper ideal. Assume $c = \aleph_1$ (this is consistent), or even the weaker assumption that $\mathsf{cov}(\mathcal{J}) = \mathsf{cof}(\mathcal{J})$.

For $X := A$, the set given by Theorem 3.7 by taking $I = \mathcal{J}$, the same argument of the proof of Claim 3.25 shows that A is $\mathsf{cov}(\mathcal{J})$-concentrated on one of its countable (dense) subsets. Now, since $\mathcal{I}_{\mathfrak{b}} \subseteq \mathcal{J}$, we have $\mathsf{cov}(\mathcal{J}) \leq \mathsf{cov}(\mathcal{I}_{\mathfrak{b}}) = \mathfrak{d}$. Thus, we noticed that there exists $D \in [X]^{\aleph_0}$ such that X is $\mathfrak{d}$-concentrated at D, and hence $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.

To see that $X \not\models U_{fin}(\mathcal{O}, \Gamma)$, notice that $X \notin \mathcal{J}$ implies $X \notin \mathcal{I}_{\mathfrak{b}}$ and recall Theorem 5.19. □

With the notation the above proof, it is very interesting to notice that even if $\mathfrak{c} = \aleph_1$ (and hence $\mathfrak{b} = \mathfrak{d}$), then still, somehow, the diagonalization process of Theorem 3.7 will generate here $X \subseteq [\mathbb{N}^{\mathbb{N}}]$ (of cardinality $\mathfrak{b} = \mathfrak{d}$), which is $\leq^*$-unbounded, but not $\leq^*$-dominating.

Definition 5.24. A function $f : X \to Y$ between two topological spaces is a *Borel function* iff the preimage of an open set (in Y) is Borel (in X).

Thus, Borel function is a weakening of continuous function.

Theorem 5.25 (Kuratowski)**.** *If $S \subseteq [0,1] \subseteq \mathbb{R}$ and $f : S \to \mathbb{N}^{\mathbb{N}}$ is a Borel function, then there exists an extension $g : [0,1] \to \mathbb{N}^{\mathbb{N}}$ such that g is a Borel function and $g \restriction S = f$.*

Theorem 5.26 (Luzin)**.** *If $f : [0,1] \to \mathbb{N}^{\mathbb{N}}$ is a Borel function, then for every $\varepsilon > 0$, there exists some closed subset $F \subseteq [0,1]$ such that $f \restriction F$ is continuous and F is of Lebesgue measure $\geq 1 - \varepsilon$.*

Proof. It suffices to assume that f is measurable and $\mathrm{Rng}(f)$ is a topological space with a countable basis $\{B_i \mid i \in \mathbb{N}\}$.

Fix $\varepsilon > 0$. Fix $i \in \mathbb{N}$. Since f is measurable, $f^{-1}[B_i]$ is a measurable set, and we may pick an open set $G_i \subset [0,1]$ and a closed set $F_i \subset [0,1]$ such that $F_i \subset f^{-1}[B_i] \subset G_i$, and the Lebesgue measure of $G_i \setminus F_i$ is at most $\varepsilon/2^i$.[18]

$G := \bigcup_{i\in\mathbb{N}}(G_i \setminus F_i)$ is open and of Lesbegue measure smaller than ε. Denote $F := G^C$, G's complement in $[0,1]$. Now, for all $i \in \mathbb{N}$, $F \cap G_i = F \cap F_i$, implying that $F \cap f^{-1}[B_i] = F \cap G_i$ is open in F, meaning that f is continuous on F. □

The next is similar to Theorem 3.24.

Theorem 5.27. *A Sierpinski subset of $[0,1]$ is a counter-example to Hurewicz's conjecture. In particular, Hurewicz's conjecture 5.17 is consistently false.*

Proof. Let $S \subseteq [0,1]$ be a Sierpinski set. The consistency of existence of such set follows, e.g., from $\mathfrak{c} = \aleph_1$ and the proof of Corollary 3.8 applied to $\mathcal{N}_{[0,1]}$ instead of to $\mathcal{N}$.

Claim 5.28. *S is not σ-compact.*

Proof. If S was σ-compact, then by Lemma 3.23, it had contained a perfect subset and by Lemma 3.26, S had to contain a null set of size $\mathfrak{c}$, contradicting the fact that S is Sierpinski set. □

We now use Theorem 5.18 to prove that $S \models U_{fin}(\mathcal{O}, \Gamma)$.

Claim 5.29. *Assume $\psi : S \to \mathbb{N}^{\mathbb{N}}$ is a Borel function, then $\psi[S] \in \mathcal{I}_{\mathfrak{b}}$.*

[18]One of the many equivalent ways to define a measurable set is the following: $A \subset \mathbb{R}$ is measurable iff for every $\varepsilon > 0$ there exist an open set G and a closed set F such that $F \subset A \subset G$ and the Lebesgue measure of $G \setminus F$ is not more than ε.

Proof. Let $\varphi : [0,1] \to \mathbb{N}^{\mathbb{N}}$ be an extension of ψ given by Theorem 5.25. Let $\langle C_n \subseteq [0,1] \mid n \in \mathbb{N}\rangle$ be like in Theorem 5.26 applied to φ, with $\mu(C_n) > 1 - \frac{1}{n+1}$ for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, the choice of C_n implies that $\varphi[C_n]$ is compact. It follows $\varphi[\bigcup_{n\in\mathbb{N}} C_n] = \bigcup_{n\in\mathbb{N}} \varphi[C_n]$ is σ-compact, and in particular, $\psi[S \cap \bigcup_{n\in\mathbb{N}} C_n] \in \mathcal{I}_{\mathfrak{b}}$. (Recall Lemma 4.7.)

We are left with showing that $\psi[S \setminus \bigcup_{n\in\mathbb{N}} C_n] \in \mathcal{I}_{\mathfrak{b}}$, but this is trivial, because $\bigcup_{n\in\mathbb{N}} C_n$ is of measure 1 and S is a Sierpinski set, so, $S \setminus \bigcup_{n\in\mathbb{N}} C_n$ is countable. □

□

With the notation of the preceding proof, notice that it suffices to assume that S has the property that any intersection of S with a null set is of cardinality $< \mathfrak{b}$, that is, the proof can be carried out flawlessly had we assumed that $S \subseteq [0,1]$ is the set given by Theorem 3.7, whenever $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$.

Definition 5.30. A *compactification* of a space X is a pair (K, h), where K is compact, $h : X \to h(X) \subset K$ is an homeomorphism, and $\overline{h(X)} = K$

We will sometimes simply say that K *is a compactification of* X. In many cases, h will be an inclusion map, so that $X \subset K$.

Definition 5.31. A space $\langle X, O\rangle$ is *locally-compact* iff for all $x \in X$, there exists an open $U \subseteq X$, with $x \in U$ and $\overline{U}$ compact.

Definition 5.32 (Alexandrov compactification)**.** Let $\langle X, O\rangle$ be locally-compact, noncompact Hausdorff space, and $p \notin X$. Define $\langle X^*, O^*\rangle$ by letting $X^* := X \cup \{p\}$ and:

$$O^* := O \cup \big\{\{p\} \cup (X \setminus K) \mid K \subseteq X \text{ is compact}\big\}.$$

We call X^* the *one-point compactification of* X.

Observations:

- Verifying that $\langle X^*, O^*\rangle$ is indeed a topological space is easy.
- X^* is compact. Assume $\{U_s\}_{s\in S}$ is an open cover of X^*.

  It follows that there exist some $s_p \in S$ with $p \in U_{s_p}$, that is, $U_{s_p} = \{p\} \cup (X \setminus K)$ where K is compact in X. Now, $\{U_s\}_{s\in S\setminus s_p}$ is an open cover of K, so there is a finite subcover $\{U_{s_1}, ..., U_{s_n}\}$. We conclude that $\{U_{s_p}, U_{s_1}, ..., U_{s_n}\}$ is a cover of X^*.
- X is open in X^* since X is open in itself.
- X is dense in X^*. Showing that $\{p\}$ is not open will do. Assume that $\{p\}$ is open, meaning $\{p\} = \{p\} \cup (X \setminus X)$ where X is compact. A contradiction, since X is noncompact.
- X^* is Hausdorff. Consider two distinct points x, x' in X^*. If both are in X then we are done since X is Hausdorff. So, assume $x' = p$. X is locally compact, that is, there

is an open set $x \in U_x$ such that $\overline{U_x}$ is compact in X, therefore $V_p := \{p\} \cup (X \setminus \overline{U_x})$ is open and $U_x \cap V_p = \emptyset$.

Example 5.33. (1) Consider the real line $\mathbb{R}$, and define $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ with the topology as described. Now, this is actually a space homeomorphic to S^1, the unit sphere in $\mathbb{R}^2$, which is obviously compact.

(2) Actually, the one-point compactification of $\mathbb{R}^n$ is S^n.

Theorem 5.34 (Alexander)**.** *Assume $\langle X, O\rangle$ is a topological space and $\mathcal{S}$ is some subbase for the topology on X.*

If every cover of X with elements of $\mathcal{S}$ has a finite subcover, then X is compact.

Proof. For the sake of the proof, we shall use the following notation:

A collection $\mathcal{U}$ of open sets is $\mathbb{B}$ iff it is not a cover. It is $\mathbb{B}_{fin}$ iff it does not have a finite subcover. We say that a $\mathbb{B}_{fin}$ collection $\mathcal{U}$ is *maximal* iff there exists some open set U such that $\mathcal{U} \cup \{U\}$ is not $\mathbb{B}_{fin}$.

Evidently, $\mathbb{B} \Rightarrow \mathbb{B}_{fin}$, and $\langle X, O\rangle$ is compact iff $\mathbb{B}_{fin} \Rightarrow \mathbb{B}$ for all $\mathcal{U} \subseteq O$.

Lemma 5.35. *Every $\mathbb{B}_{fin}$ collection can be extended to a maximal $\mathbb{B}_{fin}$ collection.*

Proof. Assume $\mathcal{U}_0$ is $\mathbb{B}_{fin}$. Let $\mathcal{A} := \{\mathcal{U} \mid \mathcal{U}_0 \subseteq \mathcal{U} \subseteq O \text{ is } \mathbb{B}_{fin}\}$. $\mathcal{A}$ is clearly non-empty. Naturally, $\langle \mathcal{A}, \subseteq\rangle$ is a partially ordered-set. Now, recall Zorn's Lemma:

Lemma 5.36 (Zorn)**.** *If $\langle P, \leq\rangle$ is a non-empty poset with the property:*
($\star$) *For all $C \subseteq P$ such that $\langle C, \leq\rangle$ is linearly-ordered, there exists some $y \in P$ such that $x \leq y$ for all $x \in C$.*

Then, $\langle P, \leq\rangle$ contains a maximal element m, that is, $m \not< x$ for all $x \in P$.

Clearly, to complete the proof, it suffices to show that the hypothesis of Zorn's Lemma holds. Let $\{\mathcal{U}_i\}_{i\in I} \subseteq \mathcal{A}$ (where I is some index set) be a chain, and define $\mathcal{U} := \bigcup_{i\in I} \mathcal{U}_i$.

Assume now that $\mathcal{U}$ is not $\mathbb{B}_{fin}$, that is, there are $\{U_k\}_{k\leq n} \subset \mathcal{U}$ such that $X = \bigcup_{k\leq n} U_k$. Since there is an increasing sequence $\langle i_k \in I \mid 1 \leq k \leq n\rangle$ such that $U_k \in \mathcal{U}_{i_k}$, we get that $\mathcal{U}_{i_n}$ is $\mathbb{B}_{fin}$. A contradiction. □

So, assume now that $\mathcal{U}$ is a maximal $\mathbb{B}_{fin}$ extension of $\mathcal{U}_0$.

Let J be an arbitrary index set. For all $j \in J$ assume $V_j \notin \mathcal{U}$ is an open set, then there are $\{U_{j_k}\}_{k\leq n_j}$ all in $\mathcal{U}$ such that $V_j \cup \bigcup_{k\leq n_j} U_{k_j} = X$. Therefore $\left(\bigcap_j V_j\right) \cup \left(\bigcup_j \bigcup_{k\leq n_j} U_k\right) = X$. We conclude that there does not exist $U \in \mathcal{U}$ such that $\bigcap_j V_j \subset U$, otherwise $\mathcal{U}$ would not have been $\mathbb{B}_{fin}$. Thus, if $\bigcap_j V_j \subset U$ for some $U \in \mathcal{U}$, then there is $j \in J$ with $V_j \in \mathcal{U}$.

Define $\mathcal{U}' := \mathcal{U} \cap \mathcal{S}$. Let $x \in U \in \mathcal{U}$. There are $\{V_j\}_{j\leq n} \subset \mathcal{S}$ such that $x \in \bigcap_{j\leq n} V_j \subset U$, thus, there is $j \leq n$ such that $V_j \in \mathcal{U}$, therefore $V_j \in \mathcal{U}'$. We conclude that $\bigcup \mathcal{U}' = \bigcup \mathcal{U}$.

Now, assume $X = \bigcup \mathcal{U}$, meaning $X = \bigcup \mathcal{U}'$, but, by the hypothesis, $\mathcal{U}'$ has a subcover for X, therefore so does $\mathcal{U}$, in contradiction to the fact that $\mathcal{U}$ is $\mathbb{B}_{fin}$.

So, $X \neq \bigcup \mathcal{U}$, that is, $\mathcal{U}$ is a $\mathbb{B}$ collection, in particular, $\mathcal{U}_0$ is a $\mathbb{B}$ collection, but we assumed $\mathcal{U}_0$ is $\mathbb{B}_{fin}$.

Since $\mathcal{U}_0$ is an arbitrary $\mathbb{B}_{fin}$ collection, we get that X is compact. □

Theorem 5.37 (Tychonoff)**.** *A nonempty product space is compact iff each factor space (in the product) is compact.*

Proof. ($\Rightarrow$) If the product space is nonempty, then the projection maps are all continuous (see proposition 2.15) and onto, and since the continuous image of a compact space is compact, the result follows.

($\Leftarrow$) Assume $\{X_i\}_{i\in I}$ is a collection of compact spaces, and define $X := \prod_{i\in I} X_i$. Consider the canonical subbase to the topology of X, $\mathcal{S} := \{\pi_i^{-1}[U] \mid i \in I, U \subseteq X_i \text{ is open}\}$.

By Alexander's theorem 5.34, it is sufficient to show that every $\mathbb{B}_{fin}$ collection $\mathcal{U} \subseteq \mathcal{S}$, is also a $\mathbb{B}$ collection, so let us fix such $\mathcal{U}$.

For all $i \in I$, put $\mathcal{U}_i := \{U \subseteq X_i \mid \pi_i^{-1}[U] \in \mathcal{U}\}$.

Lemma 5.38. *For all $i \in I$, $\mathcal{U}_i$ is $\mathbb{B}_{fin}$ in X_i.*

Proof. Assume that $\mathcal{U}_i$ is not $\mathbb{B}_{fin}$ in X_i, then there are $U_1, .., U_n \in \mathcal{U}_i$ such that $\bigcup_{k\leq n} U_k = X_i$, hence $X = \pi_i^{-1}[X_i] = \pi_i^{-1}\left[\bigcup_{k\leq n} U_k\right] = \bigcup_{k\leq n} \pi_i^{-1}[U_k]$. We conclude that $\mathcal{U}$ is $\mathbb{B}_{fin}$. A contradiction. □

Now, since X_i is compact, we must conclude that $\mathcal{U}_i$ is a $\mathbb{B}$ collection (for all $i \in I$), meaning that there exist some $x_i \in X_i \setminus \left(\bigcup \mathcal{U}_i\right)$.

Let $x \in X$ be the only member in X satisfying $\pi_i(x) = x_i$ for all $i \in I$.

Lemma 5.39. *$x \notin \bigcup \mathcal{A}$.*

In particular, $\mathcal{U}$ is a $\mathbb{B}$ collection.

Proof. Assume $x \in \bigcup \mathcal{U}$, then there exists some $U \in \mathcal{U}$ such that $x \in U$, that is, there exists some $i \in I$ and $U_i \subseteq X_I$ such that $x \in U = \pi_i^{-1}[U_i]$

Now, $x \in \pi_i^{-1}[U_i]$ iff $x_i = \pi_i(x) \in U_i$. This is a contradiction to the fact that $x_i \notin \bigcup \mathcal{U}_i$. □

□

It is worth mentioning that Tychonoff's theorem 5.37 is equivalent to the Axiom of Choice (the C of ZFC) which is equivalent to Zorn's Lemma 5.36.

Theorem 5.40 (Scheepers-Just-Miller-Szeptycki)**.** *Hurewicz's conjecture 5.17 is false.*

We omit the original proof. Instead, in the next lecture we shall introduce an alternative, simpler, proof due to Bartoszyński and Tsaban.

6. 15.12.05

We regard the natural numbers as ordinals, that is:

$$0 := \emptyset,\ 1 := 0 \cup \{0\} = \{\emptyset\}\ , 2 := 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\},\ n+1 := n \cup \{n\}.$$

Let $\omega = \{0, 1, 2, ..\}$ be the collection of all natural numbers.

It is not hard to see that $\langle \omega, \in \rangle$ is order isomorphic to $\langle \mathbb{N}, < \rangle$ with the usual order. Now, consider $\omega + 1 := \omega \cup \{\omega\}$.

$\langle \omega + 1, \in \rangle$ is an infinite linearly ordered set that has a maximal element, p, such that $\langle \omega + 1 \setminus \{p\}, \in \rangle$ is isomorphic to $\langle \mathbb{N}, < \rangle$.

Definition 6.1. Let $\omega + 1$ denote the one-point compactification of the discrete space ω.[19]

It is obvious that $\mathcal{B} := \big\{\{n\} \mid n \in \omega\big\}$ forms a basis to the discrete topology on ω.

Since $A \subseteq \omega$ is compact iff A is finite, we conclude that $\widehat{\mathcal{B}} := \big\{\{n\}, (\omega+1) \setminus \{0, .., n\} \mid n \in \omega\big\}$ forms a basis to the compact space $\omega + 1$. We shall regard $\widehat{\mathcal{B}}$ as the canonical base for $\omega + 1$.

Definition 6.2. Consider the *Bartoszyński space*, $(\omega+1)^\omega$ as the product space $\prod_{n\in\omega}(\omega+1)$.

By Theorem 5.37, the space $(\omega + 1)^\omega$ is compact.

Definition 6.3. $\mathbb{N}^{\uparrow\mathbb{N}}$ is the subspace of $\mathbb{N}^\mathbb{N}$ consisting only of strictly increasing functions:

$$\mathbb{N}^{\uparrow\mathbb{N}} := \{f \in \mathbb{N}^\mathbb{N} \mid n < m \to f(n) < f(m)\}.$$

$(\omega + 1)^{\uparrow\omega}$ is the following subspace of $(\omega + 1)^\omega$:

$$(\omega+1)^{\uparrow\omega} := \left\{ f \in (\omega+1)^\omega \middle| n < m \to \left(\begin{array}{c} f(n) < \omega \to f(n) < f(m) \\ f(n) = \omega \to f(m) = \omega \end{array} \right) \right\}.$$

Define $\omega^{\uparrow\omega}$ in the obvious fashion.

Lemma 6.4. *The following spaces are homeomorphic:*

(1) The Baire space, $\mathbb{N}^\mathbb{N}$.
(2) $\mathbb{N}^{\uparrow\mathbb{N}}$;
(3) $\omega^{\uparrow\omega}$;
(4) ω^ω;
(5) $\mathbb{R} \setminus \mathbb{Q}$.

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[19]Recall Definition 5.32.

Proof. Let's see that $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\uparrow\mathbb{N}}$. Take $\psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\uparrow\mathbb{N}}$ such that $\psi(f)(n) = \sum_{k=1}^{n} f(k)$ for all $f \in \mathbb{N}^{\mathbb{N}}$ and $n \in \mathbb{N}$.

ψ is an injection: Consider $f_1, f_2 \in \mathbb{N}^{\mathbb{N}}$ such that $\psi(f_1) = \psi(f_2)$, that is, for all $n \in \mathbb{N}$ $\sum_{k=1}^{n} f_1(k) = \sum_{k=1}^{n} f_2(k)$. We get that $f_1(n) = f_2(n)$ for all $n \in \mathbb{N}$, thus $f_1 = f_2$.

Pick $g \in \mathbb{N}^{\uparrow\mathbb{N}}$, and consider the function f where $f(1) := g(1)$ and $f(n) := g(n) - g(n-1)$. Now $\psi(f) = \sum_{k=1}^{n} f(k) = g(n)$ which means that ψ is onto.

Now pick two close functions $f_1, f_2 \in \mathbb{N}^{\mathbb{N}}$, that is, $f_1(n) = f_2(n)$ for all $n \leq N$ for some $N \in \mathbb{N}$. We get that $\psi(f_1)(n) = \psi(f_2)(n)$ for all $n \leq N$, meaning that $\psi(f_1)$ and $\psi(f_2)$ are close in $\mathbb{N}^{\uparrow\mathbb{N}}$. Therefore ψ is continuous.

In the same way we get that ψ is continuous, proving the necessary. □

Notice that the natural homeomorphisms between (1),(2),(3),(4), are all $\leq^*$-order-preserving, thus, the image of a $\mathfrak{b}$-scale in $\mathbb{N}^{\mathbb{N}}$ under this homeomorphism would be a $\mathfrak{b}$-scale in $\mathbb{N}^{\uparrow\mathbb{N}}$, etc'..

Definition 6.5. Equip $\mathcal{P}(\mathbb{N})$ with a topology by letting $O \subseteq \mathcal{P}(\mathbb{N})$ be open iff

$$\{f \in \{0,1\}^{\mathbb{N}} \mid f^{-1}[\{1\}] \in O\}$$

is an open subset of the Cantor space $\{0,1\}^{\mathbb{N}}$.

We already know how base sets in the Cantor space look like. They are exactly the sets of the form $(\varepsilon_1, ..., \varepsilon_k) \times \{0,1\}^{\mathbb{N}}$, where $k \in \mathbb{N}$, $\varepsilon_1, ..., \varepsilon_k \in \{0,1\}$. We get that base set for the topology of $\mathcal{P}(\mathbb{N})$ is of the form $\{B \subseteq \mathbb{N} \mid B \cap \{1, .., d\} = \{n_1, .., n_k\}\}$, for some $d, k, n_1, ..., n_k \in \mathbb{N}$.

¿From this it is easily seen that $\mathcal{P}(\mathbb{N})$ is homeomorphic to the Cantor space. A nice conclusion is that $\mathcal{P}(\mathbb{N})$ is metric.

Lemma 6.6. *The following spaces are homeomorphic:*

(1) *The Bartoszyński space,* $(\omega+1)^{\omega}$*;*
(2) $(\omega+1)^{\uparrow\omega}$*;*
(3) $\mathcal{P}(\mathbb{N})$*;*
(4) *The Cantor space,* $\{0,1\}^{\mathbb{N}}$*;*
(5) *The Cantor set,* $C \subseteq [0,1]$.

Proof. To see that $(\omega+1)^{\uparrow\omega}$ is homeomorphic to $\mathcal{P}(\mathbb{N})$, take $\psi : (\omega+1)^{\uparrow\omega} \to \mathcal{P}(\mathbb{N})$ such that for all $f \in (\omega+1)^{\uparrow\omega}$:

$$\psi(f) := \{f(n) + 1 \mid (n \in \omega) \wedge (f(n) < \omega)\}.$$

It is easy to se that ψ is a bijection. Proving that it is open and continuous is similar to the proof of 6.4.

□

From now on, whenever useful, we may think of subspaces of the Bartoszyński space as subspaces of the reals.

Definition 6.7. For $f \in \{0,1\}^{\mathbb{N}}$, let $(f \oplus 1) \in \{0,1\}^{\mathbb{N}}$ be such that $(f \oplus 1)(n) = 1 - f(n)$ for all $n < \omega$. For $A \subseteq \mathbb{N}$, let $A^c := \mathbb{N} \setminus A$.

It is obvious that $A \mapsto A^c$ and $f \mapsto (f \oplus 1)$ are automorphisms of $\mathcal{P}(\mathbb{N})$ and $\{0,1\}^{\mathbb{N}}$, respectively.

Definition 6.8. For $\psi : (\omega+1)^{\uparrow\omega} \to \mathcal{P}(\mathbb{N})$ of theorem 6.6, and each $f \in (\omega+1)^{\uparrow\omega}$, denote :

$$f^c := \psi^{-1}(\psi(f)^c).$$

Observation 6.9. *$f \mapsto f^c$ (for all $f \in (\omega+1)^{\uparrow\omega}$) is an automorphism of $(\omega+1)^{\uparrow\omega}$.*

Definition 6.10. For all $n < \omega$ and $\sigma : n \to \omega$, let $q_\sigma \in (\omega+1)^\omega$ be such that:

$$q_\sigma \restriction n = \sigma \text{ and } q_\sigma(m) = \omega \text{ for all } m \geq n.$$

Clearly, if σ is strictly increasing, then q_σ is in $(\omega+1)^{\uparrow\omega}$.

Definition 6.11. Denote $\mathrm{ISeq}(\omega) := \{\sigma : n \to \omega \mid n < \omega, \sigma \text{ is strictly increasing }\}$.

Lemma 6.12. *$Q := \{q_\sigma \mid \sigma \in \mathrm{ISeq}(\omega)\}$ is a countable dense subset of $(\omega+1)^{\uparrow\omega}$.*

Proof. It suffices to show that $U \cap Q \neq \emptyset$ for all non-empty open $U \subseteq (\omega+1)^{\uparrow\omega}$ of the form:

$$U = (\omega+1)^{\uparrow\omega} \cap \pi_0^{-1}[U_1] \cap ... \cap \pi_n^{-1}[U_n],$$

where $n < \omega$ and $\{U_0, .., U_n\} \in [\widehat{\mathcal{B}}]^{<\omega}$.[20]

It is straight-forward to inductively define $\sigma : n+1 \to \omega$ such that $\sigma(k) \in U_k$ for all $k \leq n$, and $\sigma \in \mathrm{ISeq}(\omega)$. It follows that $q_\sigma \in U$ and we are done. □

Notice that the image of Q under the homeomorphism ψ from Lemma 6.6, is $[\mathbb{N}]^{<\omega}$.

Next thing we do, is improving Observation 5.23 to the following.

Theorem 6.13 (Tsaban-Zdomskyy)**.** *There exists $X \subseteq \omega^{\uparrow\omega}$ such that:*

(a) *$X \models S_{fin}(\mathcal{O}, \mathcal{O})$,*
(b) *$X \not\models U_{fin}(\mathcal{O}, \Gamma)$ (and in particular, X is not σ-compact).*

Thus, Menger's conjecture 1.30 has a counter-example inside $\mathbb{N}^{\mathbb{N}}$, and $S_{fin}(\mathcal{O}, \mathcal{O}) \neq U_{fin}(\mathcal{O}, \Gamma)$.

[20]Here, π_k is the k'th projection from $(\omega+1)^{\uparrow\omega}$ to $\omega+1$, and $\widehat{\mathcal{B}}$ is the canonical base of $\omega+1$ discussed after Definition 6.1.

Proof by Rinot. If $\mathfrak{b} < \mathfrak{d}$, then pick a $\leq^*$-unbounded family $X \in [\omega^{\uparrow\omega}]^{\mathfrak{b}}$.

By Theorems 4.12,5.1, $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ and by Theorem 5.19, $X \not\models U_{fin}(\mathcal{O}, \Gamma)$.

Assume now $\mathfrak{b} = \mathfrak{d}$. Put $A := \{f \in \omega^{\uparrow\omega} \mid f^c \in \omega^{\uparrow\omega}\} = \{f \in \omega^{\uparrow\omega} \mid \omega \setminus \mathrm{Im}(f) \text{ is infinite }\}$.[21]

Pick a dominating family $D \in [\omega^{\uparrow\omega}]^{\mathfrak{d}}$. By $\mathfrak{b} = \mathfrak{d}$, we may apply the proof of Lemma 1.11 and yield a strictly $\leq^*$-increasing sequence $\langle f_\alpha \in A \mid \alpha < \mathfrak{b}\rangle$ such that $\omega^{\uparrow\omega} \subseteq \underline{\{f_\alpha \mid \alpha < \mathfrak{b}\}}$.

We now define a sequence $\{g_\alpha \in A \mid \alpha < \mathfrak{b}\}$ by induction on $\alpha < \mathfrak{b}$. Let $g_0 := f_0$, and assume $\{g_\beta \in A \mid \beta < \alpha\}$ had already been defined.

Since $B := \{g_\beta, f_\beta, f_\beta^c \mid \beta < \alpha\} \subseteq \omega^{\uparrow\omega}$ is of cardinality $< \mathfrak{b}$, we may find some $h \in \omega^{\uparrow\omega}$ such that $B \subseteq \underline{\{h\}}$. Now, by Corollary 4.5, $C_1 := \{f \in \omega^{\uparrow\omega} \mid f \not\leq^* h\}$ is co-meager. It follows from Observation 6.9 that $C_2 := \{f^c \mid f \in \omega^{\uparrow\omega}, f \not\leq^* h\} \subseteq (\omega+1)^{\uparrow\omega}$ is co-meager. Now, since $(\omega+1)^{\uparrow\omega} = \omega^{\uparrow\omega} \cup Q$ and Q is meager, $C_3 := C_2 \setminus Q = \{f \in \omega^{\uparrow\omega} \mid f^c \in \omega^{\uparrow\omega}, f^c \not\leq^* h\}$ is co-meager, so let us pick $g_\alpha \in C_1 \cap C_3$. End of the construction.

Claim 6.14. *For all $h \in \omega^{\uparrow\omega}$:*

- $|\{g_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{h\}}| < \mathfrak{b}$
- $|\{g_\alpha^c \mid \alpha < \mathfrak{b}\} \cap \underline{\{h\}}| < \mathfrak{b}$

Proof. Pick $h \in \omega^{\uparrow\omega}$. By definition of our strictly increasing scale, there exists some $\delta < \mathfrak{b}$ such that $h \leq^* f_\alpha$ whenever $\delta < \alpha < \mathfrak{b}$. Assume $\delta < \alpha < \mathfrak{b}$, then by the choice of g_α, $\{n < \omega \mid f_\alpha(n) \leq g_\alpha(n)\}$ and $\{n < \omega \mid f_\alpha(n) \leq g_\alpha^c(n)\}$ are both infinite. In particular, $g_\alpha \not\leq^* h$ and $g_\alpha^c \not\leq^* h$, thus:

$$\max\left\{\left|\{g_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{h\}}\right|, \left|\{g_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{h\}}\right|\right\} \leq |\delta| < \mathfrak{b}.$$

□

Put $Y := \{g_\alpha \mid \alpha < \mathfrak{b}\} \cup Q$ and let X be the image of Y under the complement operator of Observation 6.9. It is obvious that $X \subseteq \omega^{\uparrow\omega}$. Since X and Y are homeomorphic, we are left with showing that $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $X \not\models U_{fin}(\mathcal{O}, \Gamma)$.

To see that $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$, notice that the same proof of Theorem 4.20 shows that Y is $\mathfrak{b}$-concentrated at Q.[22] Finally, by the preceding claim, X is $\leq^*$-unbounded. It follows from Theorem 5.19 that $X \not\models U_{fin}(\mathcal{O}, \Gamma)$. □

Corollary 6.15. *There exists $B \in [\mathbb{N}^{\mathbb{N}}]^{\mathfrak{b}}$ which is $\leq^*$-unbounded but not $\leq^*$-dominating.*

Further more, B satisfies:

(a) *For all $h \in \mathbb{N}^{\mathbb{N}}$, $|B \cap \underline{\{h\}}| < \mathfrak{b}$.*
(b) *For any continuous function $\psi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, $\psi[B]$ is not $\leq^*$-dominating.*

[21] Here, f^c denotes the image of f under the homeomorphism defined in Observation 6.9.

[22] We simply replaced the compact space $[0, 1]$ with the space $(\omega+1)^{\uparrow\omega}$. Cf the proof of Claim 6.18.

Proof. If $\mathfrak{b} < \mathfrak{d}$, then just pick a $\mathfrak{b}$-scale. Assume now $\mathfrak{b} = \mathfrak{d}$, and consider X of the preceding theorem, then, (a) is satisfied by (the second item of) claim 6.14, and (b) is satisfied by $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ and Theorem 4.11. □

We now show that there is a very high price to pay for functions to be continuous at Q.

Lemma 6.16 (Bartoszyński)**.** *Suppose $\psi : (\omega+1)^{\uparrow\omega} \to \omega^\omega$ is continuous at Q.*

Then there exists an element $g \in \omega^\omega$ such that for all $x \in \omega^{\uparrow\omega}$ and $n < \omega$:

$$x(n) > g(n) \Longrightarrow \psi(x)(n) \leq g(n)$$

Proof. Fix $n < \omega$. For $\sigma \in \mathrm{ISeq}(\omega)$ with $\mathrm{dom}(\sigma) = n$, put $k_\sigma := \psi(q_\sigma)(n)$. Since $(n, k_\sigma)^\uparrow = \{f \in \omega^\omega \mid f(n) = k_\sigma\}$ is open, $\psi(q_\sigma) \in (n, k_\sigma)^\uparrow$, and ψ is continuous on $q_\sigma \in Q$, we conclude that $\psi^{-1}[(n, k_\sigma)^\uparrow]$ contains an open neighberhood of q_σ.

It follows that for all $\sigma \in \mathrm{ISeq}(\omega)$ with $\mathrm{dom}(\sigma) = n$, we may fix a base open $I_\sigma \subseteq (\omega+1)^{\uparrow\omega}$ such that $q_\sigma \in I_\sigma$ and $\psi(x)(n) = \psi(q_\sigma)(n)$ for all $x \in I_\sigma$.

For $A \subseteq (\omega+1)^{\uparrow\omega}$, denote $A \upharpoonright n := \{x \upharpoonright n \mid x \in A\}$.

Since $\{I_\sigma \upharpoonright n \mid \sigma \in \mathrm{ISeq}(\omega), \mathrm{dom}(\sigma) \leq n\}$ is an open cover of the compact product space $\prod_{k<n}(\omega+1)$, we may find $\mathcal{F}_n \in [\mathrm{ISeq}(\omega)]^{<\omega}$ such that $\mathrm{dom}(\sigma) \leq n$ for all $\sigma \in \mathcal{F}_n$ and $\prod_{k<n}(\omega+1) = \bigcup_{\sigma\in\mathcal{F}_n}(I_\sigma \upharpoonright n)$.

Claim 6.17. *For all $\sigma \in \mathcal{F}_n$, there exists $N_\sigma < \omega$ such that for all $x \in (\omega+1)^{\uparrow\omega}$:*

$$\Big((x \upharpoonright n) \in (I_\sigma \upharpoonright n) \text{ and } x(n) > N_\sigma\Big) \Longrightarrow x \in I_\sigma.$$

Proof. Fix $\sigma \in \mathcal{F}_n$. Since I_σ is base open, there exists a family $\langle U_m \in \widehat{\mathcal{B}} \mid m < \omega\rangle$ such that $I_\sigma = \prod_{m<\omega} U_m$. Further more, we may find a minimal $M < \omega$ such and $U_m = \omega + 1$ for all $m \geq M$. Now, if $M \leq n$, then for all $x \in (\omega+1)^{\uparrow\omega}$ with $(x \upharpoonright n) \in (I_\sigma \upharpoonright n)$, we have $x \in I_\sigma$, thus, N_σ is arbitrary. Assume $M > n$.

Since $q_\sigma \in I_\sigma \in \widehat{\mathcal{B}}$ and $q_\sigma(m) = \omega$ for all $m \geq n$, we know that for all m satisfying $n \leq m < M$, there exists a $k_m < \omega$ such that $U_m = (\omega+1) \setminus k_m$.

Put $N_\sigma := \max\{k < \omega \mid \exists m < M\big(U_m = (\omega+1) \setminus k\big)\}$. It follows that if $x \in \omega^{\uparrow\omega}$, $(x \upharpoonright n) \in (I_\sigma \upharpoonright n)$, and $x(n) > N_\sigma$, then $x \in I_\sigma$. (Recall that x is increasing!) □

Finally, define $g \in \omega^\omega$ by $g(n) := \max\{N_\sigma, \psi(q_\sigma(n)) \mid \sigma \in \mathcal{F}_n\}$ for all $n < \omega$.

To see that g works, pick $x \in \omega^{\uparrow\omega}$ and $n < \omega$ with $x(n) > g(n)$.

Since $\prod_{k<n}(\omega+1) = \bigcup_{\sigma\in\mathcal{F}_n}(I_\sigma \upharpoonright n)$, there exists $\sigma \in \mathcal{F}_n$ such that $(x \upharpoonright n) \in (I_\sigma \upharpoonright n)$. Now, since $x(n) > g(n) \geq N_\sigma$, we have that $x \in I_\sigma$. Finally, since $x \in I_\sigma$, we conclude that $\psi(x)(n) = \psi(q_\sigma)(n) \leq g(n)$. □

Lemma 6.18. *Pick a $\mathfrak{b}$-scale, $\langle f_\alpha \in \omega^{\uparrow\omega} \mid \alpha < \mathfrak{b}\rangle$.*

Put $H := B \cup Q \subseteq (\omega+1)^{\uparrow\omega}$, where $B := \{f_\alpha \mid \alpha < \mathfrak{b}\}$. Then H is $\mathfrak{b}$-concentrated at Q.

Proof. Suppose $U \subseteq (\omega+1)^{\uparrow\omega}$ is open containing Q. Since $(\omega+1)^{\uparrow\omega} = \omega^{\uparrow\omega} \cup Q$ and $(\omega+1)^{\uparrow\omega}$ is compact, $(\omega+1)^{\uparrow\omega} \setminus U$ is a compact subspace of ω^ω.

It follows from Lemma 4.3 that there exists some $g \in \omega^\omega$ such that $(\omega+1)^{\uparrow\omega} \setminus U \subseteq \underline{\{g\}}$. In particular, there exists $\delta < \mathfrak{b}$ such that $H \setminus U = (B \cup Q) \setminus U \subseteq \{f_\alpha \mid \alpha < \delta\}$. □

Theorem 6.19. *Pick a $\mathfrak{b}$-scale, $\langle f_\alpha \in \omega^{\uparrow\omega} \mid \alpha < \mathfrak{b}\rangle$, then $H := B \cup Q$ is a counter-example to Hurewicz's conjecture 5.17, where $B := \{f_\alpha \mid \alpha < \mathfrak{b}\}$.*

Proof. By Lemma 6.6 and the preceding, $H \subseteq (\omega+1)^{\uparrow\omega}$ is homeomorphic to a set of reals which is $\mathfrak{b}$-concentrated at one of its countable subsets. It follows from Lemma 3.23 and Theorem 4.16, that H is not σ-compact.

To see that $H \models U_{fin}(\mathcal{O}, \Gamma)$, we use Theorem 5.18. Assume $\psi : H \to \omega^\omega$ is continuous.

Since the homeomorphism discussed in Lemma 6.4 is order-preserving, we may also assume that $\mathrm{Im}(\psi) \subseteq \omega^{\uparrow\omega}$. Let $g \in \omega^\omega$ be like in Lemma 6.16.

Since B is unbounded, there exists some $\alpha < \mathfrak{b}$ such that $f_\alpha \not\leq^* g$, so let us pick $a \in \omega^{\uparrow\omega}$ such that $g(a(n)) < f_\alpha(a(n))$ for all $n < \omega$.

It follows that for all $n < \omega$, $\psi(f_\alpha(n)) \leq \psi(f_\alpha)(a(n)) \leq h(n)$ where $h := g \circ a$.

Now, if $\beta > \alpha$, then there exists $m < \omega$ such that $f_\alpha(n) \leq f_\beta(n)$ whenever $m < n < \omega$, hence, $g(a(n)) < f_\alpha(a(n)) \leq f_\beta(a(n))$ and $\psi(f_\beta(n)) \leq h(n)$ for all but finitely many n's.

We conclude that $\psi[\{f_\beta \mid \beta \geq \alpha\}] \subseteq \underline{\{h\}}$, hence, $\psi[\{f_\beta \mid \beta \geq \alpha\}] \in \mathcal{I}_\mathfrak{b}$. Finally, since $|\{f_\gamma \mid \gamma < \alpha\}| < \mathfrak{b}$, we have that $\psi[\{f_\gamma \mid \gamma < \alpha\}] \in \mathcal{I}_\mathfrak{b}$ and $\mathrm{Im}(\psi) \in \mathcal{I}_\mathfrak{b}$. □

A curious reader might ask himself what happens had we replaced the $\mathfrak{b}$-scale in the preceding theorem with B of Corollary 6.15. A suspicious reader who compare Y of theorem 6.13 with H of the preceding theorem, might even belive he found a contradiction.

However, there is an important difference between the $\mathfrak{b}$-scale and our B, and this lies in the fact that the $\mathfrak{b}$-scale is linearly-ordered by $\leq^*$, while B is not. It is evident that the usage of linearity has a crucial appearance at the end of the proof of the preceding.

7. 29.12.05

Definition 7.1. A set $A \subset \mathbb{R}$ is of Lebesgue measure 0 if for every $\varepsilon > 0$ there is a family of open intervals $\langle I_n \mid n \in \mathbb{N}\rangle$ that covers A and $\sum_{n\in\mathbb{N}} |I_n| < \varepsilon$.

Definition 7.2. A set $A \subset \mathbb{R}$ is of *strong measure zero* (or SMZ) iff for every sequence $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$ there is a family of open intervals $\langle I_n \mid n \in \mathbb{N}\rangle$ that covers A and $|I_n| < \varepsilon_n$ for every $n \in \mathbb{N}$.

Proposition 7.3. SMZ $\Longrightarrow$ *Lebesgue measure zero.*

Proof. Assume the set $A \subset \mathbb{R}$ is of SMZ. Fix $\varepsilon > 0$. Consider the sequence $\langle \varepsilon/2^n \mid n \in \mathbb{N}\rangle$. Since A is SMZ there is a family of open intervals $\langle I_n | n \in \mathbb{N}\rangle$ that covers A and $|I_n| < \varepsilon/2^n$ for all $n \in \mathbb{N}$. Since $\sum_{n\in\mathbb{N}} \varepsilon/2^n = \varepsilon$, we get that A is of Lebesgue measure zero. □

Observation 7.4. *If $A \subseteq \mathbb{R}$ is countable, then A is* SMZ.

Proof. Suppose $A = \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}$ is countable, and $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$ is a sequence of positive reals. For $n \in \mathbb{N}$, let $I_n := (a_n - \frac{\varepsilon_n}{4}, a_n + \frac{\varepsilon_n}{4})$ and observe that $\langle I_n \mid n \in \mathbb{N}\rangle$ works. □

To see that SMZ is much stronger than measure zero, consider for example the Cantor set. We have seen before that it is of measure zero. Is it SMZ? It is obvious that for the sequence $\langle 1/3^n | n \in \mathbb{N}\rangle$, matching open interval cover the Cantor set. Just take $I_1 := (0, 1/3), I_2 = (6/9, 7/9), \ldots$. On the other hand, for the sequence $\langle 1/3^n | n \in \mathbb{N}, n > K \geq 1\rangle$, such family of open intervals that covers the Cantor set can't be obtained (think why?). Therefore it is not SMZ.

Conjecture 7.5 (Borel, 1919)**.** *If $A \subseteq \mathbb{R}$ is* SMZ, *then A is countable.*

Notice that in $\mathbb{R}$, for some open interval $(a, b) \subset \mathbb{R}$, $|(a, b)|$ stands for the length (one dimensional volume) of (a, b), or equivalently, its' diameter. Is it the same in larger metric spaces? Consider for example $\mathbb{R}^2$. The set $[0, 1] \subset \mathbb{R}^2$ is of Lebesgue measure (volume) zero, but the sum of diameters of any open cover consisting with two dimensional "boxes" is not less than 1.[23] The question arises is how to"properly" define SMZ in large metric spaces? Here is the standard way.

Definition 7.6. Suppose $\langle X, d\rangle$ is a metric space.

$A \subseteq X$ is a *strongly null* set iff for any sequence of positive reals, $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$, there is a partition $\{A_n \mid n \in \mathbb{N}\}$ such that $A = \bigcup_{n\in\mathbb{N}} A_n$ and $\text{Diam}(A_n) < \varepsilon_n$ for all $n \in \mathbb{N}$.

In the special case of strongly null sets in $\mathbb{R}$, we shall keep call them SMZ.

[23]A box in $\mathbb{R}^2$ is a base set of the product topology, that is a product of open intervals in $\mathbb{R}$

Observation 7.7. *If $\langle X, d\rangle$ is a discrete metric space, then $A \subseteq X$ is strongly null iff A is countable.*

Lemma 7.8. *A uniformly continuous image of a strongly null set is strongly null.*

Proof. Let $\langle X, \rho_X\rangle, \langle Y, \rho_Y\rangle$ be metric spaces where X is strongly null, and let $f : X \to Y$ be uniformly continuous onto Y.

Fix $\varepsilon > 0$. Since f is uniformly continuous, a $\delta > 0$ exists, such that given an open ball $B \subset X$ with $\mathrm{Diam}_{\rho_X}(B) < \delta$, we end up with $\mathrm{Diam}_{\rho_Y}(f[B]) < \varepsilon$.

Now, consider some sequence $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$. Implementing the last remark we get a corresponding sequence $\langle \delta_n \mid n \in \mathbb{N}\rangle$. X is strongly null, hence there exist a cover consisting of open balls $\langle B_n \subset X \mid n \in \mathbb{N}\rangle$ where $\mathrm{Diam}_{\rho_X}(B_n) < \delta_n$. For all $n \in \mathbb{N}$, $\mathrm{Diam}_{\rho_Y}(f[B_n]) < \varepsilon_n$.

$$Y = f[X] = f\Big[\bigcup_{n\in\mathbb{N}} B_n\Big] \subseteq \bigcup_{n\in\mathbb{N}} f[B_n] \subseteq \bigcup_{n\in\mathbb{N}} B'_n$$

where $B'_n \subset Y$ are open balls of diameter less than ε_n such that $f[B_n] \subseteq B'_n$. □

Definition 7.9. For a metric space $\langle X, d\rangle$, let $\mathcal{SN}_X := \{A \subseteq X \mid A \text{ is a strongly null set }\}$.

In the special case of $\langle \mathbb{R}, |\cdot|\rangle$, we denote $\mathcal{SN} := \mathcal{SN}_{\mathbb{R}} = \{A \subseteq \mathbb{R} \mid A \text{ is SMZ }\}$.

Proposition 7.10. *For any metric space $\langle X, d\rangle$, $\mathcal{SN}_X$ is a σ-ideal.*[24]

Proof. It is obvious that $\emptyset \in \mathcal{SN}_X$.

Consider some $A \in \mathcal{SN}_X$, and let $B \subset A$. Fix $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$, then since $A \in \mathcal{SN}_X$ there is a cover of A consisting of open set $\langle U_n \mid n \in \mathbb{N}\rangle$ with $\mathrm{Diam}(U_n) < \varepsilon_n$ for all $n \in \mathbb{N}$. Since $B \subseteq \bigcup_{n\in\mathbb{N}} U_n$, we conclude that $B \in \mathcal{SN}_X$.

Finally, to see that $\mathcal{SN}_X$ is σ-additive, assume $\langle A_n \in \mathcal{SN}_X \mid n \in \mathbb{N}\rangle$, and fix $\langle \varepsilon_n \mid n \in \mathbb{N}\rangle$. Let $\biguplus_{n\in\mathbb{N}} J_n$ be a partition of $\mathbb{N}$ where J_n is infinite for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. $A_n \in \mathcal{SN}_X$, therefore there is a cover consisting of open sets $\langle U_{n,k} \mid k \in J_n\rangle$ such that $\mathrm{Diam}(U_{n,k}) < \varepsilon_k$ for all $k \in J_n$.

By $\bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} U_{n,k} \supseteq \bigcup_{n\in\mathbb{N}} A_n$, we conclude that $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{SN}_X$. □

We have already seen that $\mathcal{SN} \subseteq \mathcal{N}$. We now show a nice connection between SMZ and connectedness.

Claim 7.11. SMZ $\Rightarrow$ 0-*dimensional.*

Proof. Assume $A \in \mathbb{R}$ is SMZ. Recalling Theorem 4.28, it is enough to show that A is totally disconnected. Assume the contrary, that is, it happens that $x \in I \subset A$ where I is connected and $I \setminus \{x\} \neq \emptyset$. I is then an interval which means of positive measure, a contradiction to the fact that A is null (Proposition 7.3). □

[24] A σ-ideal is an ideal closed to countable unions, i.e., $\mathsf{add}(\mathcal{SN}_X) \geq \aleph_1$.

We now reveal the combinatorics of SMZ.

Definition 7.12 (Rothberger)**.** For $k \in \mathbb{N}$, a space $\langle X, O\rangle$ satisfies *Rothberger's property* or $S_k(\mathcal{O},\mathcal{O})$ iff for any family of open covers of X, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^k \mid n \in \mathbb{N}\rangle$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X.

Observation 7.13. *For a topological space $\langle X, O\rangle$, TFAE:*

(a) $X \models S_1(\mathcal{O},\mathcal{O})$.
(b) $X \models S_k(\mathcal{O},\mathcal{O})$ *for some* $k \in \mathbb{N}$.
(c) $X \models S_f(\mathcal{O},\mathcal{O})$ *for some* $f \in \mathbb{N}^{\mathbb{N}}$*, i.e., for any family of open covers of* X*,* $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$*, there exists a family* $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{f(n)} \mid n \in \mathbb{N}\rangle$*, such that* $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$

Proof. To see (c)⇒(a), fix $f \in \mathbb{N}^{\mathbb{N}}$ such that $X \models S_f(\mathcal{O},\mathcal{O})$.

Pick an arbitrary partition $\langle A_n \in [\mathbb{N}]^{f(n)} \mid n \in \mathbb{N}\rangle$ with $\biguplus_{n\in\mathbb{N}} A_n = \mathbb{N}$.

For all $n \in \mathbb{N}$, let $\mathcal{V}_n := \{\bigcap \mathrm{Im}(g) \mid g \in \prod_{m\in A_n} \mathcal{U}_m\}$.[25] Evidently, each $\mathcal{V}_n$ covers X.

Applying $S_f(\mathcal{O},\mathcal{O})$ to $\langle \mathcal{V}_n \mid n \in \mathbb{N}\rangle$, we get a family $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{f(n)} \mid n \in \mathbb{N}\rangle$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ covers X. Pick $\langle \mathcal{G}_n \in [\prod_{m\in A_n} \mathcal{U}_m]^{f(n)} \mid n \in \mathbb{N}\rangle$ such that $\mathcal{F}_n = \{\bigcap \mathrm{Im}(g) \mid g \in \mathcal{G}_n\}$ for all $n \in \mathbb{N}$. By $|\mathcal{G}_n| = f(n) = |A_n|$, we may enumerate $\mathcal{G}_n = \{g_i \in \prod_{m\in A_n} \mathcal{U}_m \mid i \in A_n\}$.

In this notation, we get that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n = \{\bigcap \mathrm{Im}(g_i) \mid i \in \mathbb{N}\}$.

Finally, since $\bigcap \mathrm{Im}\, g_i \subseteq g_i(i) \in \mathcal{U}_i$ for all $i \in \mathbb{N}$, we get that $\langle g_n(n) \mid n \in \mathbb{N}\rangle$ exemplifies $X \models S_1(\mathcal{O},\mathcal{O})$. □

Observation 7.14. *Assume $\langle X, d\rangle$ is a metric space.*

For all $Y \subseteq X$, $Y \models S_1(\mathcal{O},\mathcal{O})$ implies that Y is strongly null.

Proof. Consider a family of positive reals $\langle \varepsilon_n \in \mathbb{R} \mid n \in \mathbb{N}\rangle$.

Fix a basis $\mathcal{B}$ for $\langle X, d\rangle$, and put $\mathcal{U}_n := \{U \in \mathcal{B} \mid \mathrm{Diam}(U) < \varepsilon_n\}$ for each $n \in \mathbb{N}$. By applying $S_1(\mathcal{O},\mathcal{O})$ of Y to $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, we obtain a family $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $Y \subseteq \bigcup_{n\in\mathbb{N}} U_n$, and obviously, $\mathrm{Diam}(U_n) < \varepsilon_n$ for all $n \in \mathbb{N}$. □

Corollary 7.15. *A Luzin set is an uncountable strongly null set.*

In particular, Borel's conjecture 7.5 is consistently false.

Proof. By Claim 3.25 and the preceding observation. □

Our reader might conjecture that Observation 7.14 can be improved and $S_1(\mathcal{O},\mathcal{O})$ is actually equivalent to strongly null. However, this is not the case. By Proposition 7.10, strongly null is an hereditary property, whereas we have the following.

Observation 7.16. *$S_{fin}(\mathcal{O},\mathcal{O})$ is non-hereditary.*

[25] $g \in \prod_{m\in A_n} \mathcal{U}_m$ means that $\mathrm{dom}(g) = A_n$ and $g(m) \in \mathcal{U}_m$ for all $m \in A_n$.

Proof. $\mathbb{R}$ is σ-compact, thus by Lemma 1.29, $\mathbb{R} \models S_{fin}(\mathcal{O},\mathcal{O})$. However, by Theorem 2.29 $\mathbb{R}\setminus\mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. It follows from Theorem 4.10 that $\mathbb{R}\setminus\mathbb{Q} \not\models S_{fin}(\mathcal{O},\mathcal{O})$. □

Observation 7.17. *$S_1(\mathcal{O},\mathcal{O})$ is consistently non-hereditary.*

Proof. Assume $\mathfrak{d} = \aleph_1$. Consider $M := \psi[D] \cup (\mathbb{Q}\cap[0,1])$ of Theorem 4.20. Then M is $\aleph_1$-concentrated on the countable set $\mathbb{Q}\cap[0,1]$, thus, $M \models S_1(\mathcal{O},\mathcal{O})$. However, By Theorem 4.10, $\psi[D]$ does not even satisfy $S_{fin}(\mathcal{O},\mathcal{O})$ (since $\psi^{-1}[\psi[D]] = D$ is dominating), not to mention $S_1(\mathcal{O},\mathcal{O})$. □

It follows from Observation 4.8 that $(\mathfrak{d} = \aleph_1) \implies (\mathfrak{d} = \mathsf{cov}(\mathcal{M}))$. It will soon be clear that it suffices to assume $\mathfrak{d} = \mathsf{cov}(\mathcal{M})$ to conclude that $\psi[D]\cup(\mathbb{Q}\cap[0,1]) \models S_1(\mathcal{O},\mathcal{O})$.

Definition 7.18. A set $X \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be *guessed* by $g \in \mathbb{N}^{\mathbb{N}}$ iff $\{n\in\mathbb{N} \mid f(n) = g(n)\}$ is infinite for all $f \in X$.

Theorem 7.19. *Suppose $X \subseteq \mathbb{N}^{\mathbb{N}}$. If $|X| < \mathsf{cov}(\mathcal{M})$, then X can be guessed.*

Proof. For all $f \in X$ and $k \in \mathbb{N}$, it is obvious that:

$$A_{f,k} := \left\{g \in \mathbb{N}^{\mathbb{N}} \mid \exists n \in \mathbb{N}\big((n > k) \wedge g(n) = f(n)\big)\right\}$$

is dense open. Clearly, any $g \in \bigcap_{f\in X}\bigcap_{k\in\mathbb{N}} A_{f,k}$ will do, so assume towards a contradiction that $\bigcap_{k\in\mathbb{N}}\bigcap_{f\in X} A_{f,k} = \emptyset$. It follows that $\mathbb{N}^{\mathbb{N}} = \bigcup_{k\in\mathbb{N}}\bigcup_{f\in X} B_{f,k}$, where $B_{f,k} := \mathbb{N}^{\mathbb{N}} \setminus A_{f,k}$ are nowhere dense sets. Identifying $\mathbb{N}^{\mathbb{N}}$ with $\mathbb{R}\setminus\mathbb{Q}$, we get that:

$$\mathbb{R} = \bigcup_{k\in\mathbb{N}}\bigcup_{f\in X} B_{f,k} \cup \bigcup_{q\in\mathbb{Q}}\{q\}$$

is the union of $|X|$ nowhere dense sets, contradicting $|X| < \mathsf{cov}(\mathcal{M})$. □

Theorem 7.20. *If $\langle X, O\rangle$ is a topological space and $X \models S_1(\mathcal{O},\mathcal{O})$, then any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ can be guessed.*

Proof. This essentially is the same proof as of Theorem 4.11. Assume some $X \subseteq \mathbb{N}^{\mathbb{N}}$ with $X \models S_1(\mathcal{O},\mathcal{O})$. Fix $m \in \mathbb{N}$. Put $\mathcal{U}_m := \{(m,k)^{\uparrow} \mid k\in\mathbb{N}\}$ where $(m,k)^{\uparrow} := \{f \in \mathbb{N}^{\mathbb{N}} \mid f(m) = k\}$ for all $k \in \mathbb{N}$. Evidently, $\mathcal{U}_m$ is an open cover of X. Fix a bijection $\psi : \mathbb{N}\times\mathbb{N} \leftrightarrow \mathbb{N}$.

Fix $i \in \mathbb{N}$. Since $X \models S_1(\mathcal{O},\mathcal{O})$ and $\langle \mathcal{U}_{\psi(i,n)} \mid n \in \mathbb{N}\rangle$ is a countable family of open covers of X, there exists $g_i : \psi[\{i\}\times\mathbb{N}] \to \mathbb{N}$ such that $X \subseteq \bigcup_{n\in\mathbb{N}} \Big(\psi(i,n), g\big(\psi(i,n)\big)\Big)^{\uparrow}$.

Let $g : \mathbb{N} \to \mathbb{N}$ be $g := \bigcup_{n\in\mathbb{N}} g_n$. It is evident that g guesses X. □

Theorem 7.21 (Recław)**.** *Suppose $\langle X, O\rangle$ is a topological space that has a base $\mathcal{B}$ which is countable and composed only of clopen sets.*

If any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ can be guessed, then $X \models S_1(\mathcal{O},\mathcal{O})$.

Proof. Assume a family of open covers of X, $\langle \mathcal{U}_n \subseteq \mathcal{B} \mid n \in \mathbb{N}\rangle$. Since $\mathcal{B}$ is countable, there exists an enumeration $\mathcal{U}_n = \{U_n^m \mid m \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. We may also assume for all $n \in \mathbb{N}$ that members of $\mathcal{U}_n$ are mutually-disjoint, thus, for all $x \in X$, there is a unique $f_x \in \mathbb{N}^{\mathbb{N}}$ such that $x \in U_n^{f_x(n)}$ for all $n \in \mathbb{N}$. Finally, let $\psi : X \to \mathbb{N}^{\mathbb{N}}$ be the map $x \mapsto f_x$.

Since ψ is continuous, we may pick $g \in \mathbb{N}^{\mathbb{N}}$ that guesses $\psi[X]$.

For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. To see that $\langle U_n \mid n \in \mathbb{N}\rangle$ covers X. Notice that for each $x \in X$, there exists some $n \in \mathbb{N}$ such that $f_x(n) = g(n)$, i.e., $x \in U_n^{g(n)} = U_n$. □

Corollary 7.22. *For all $X \subseteq \mathbb{R}$, TFAE:*

- $X \models S_1(\mathcal{O}, \mathcal{O})$.
- *Any continuous image of X into $\mathbb{N}^{\mathbb{N}}$ can be guessed.*

Proof. By theorems 7.20,7.21 and 7.11. □

Corollary 7.23. $X \models S_1(\mathcal{O}, \mathcal{O})$ *for all* $X \in [\mathbb{R}]^{<\mathsf{cov}(\mathcal{M})}$.

Corollary 7.24. *If $X \subseteq \mathbb{R}$ is $\mathsf{cov}(\mathcal{M})$-concentrated on one of its countable subsets, then $X \models S_1(\mathcal{O}, \mathcal{O})$.*

Corollary 7.25. *If $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}$, then M of Theorem 4.20 satisfies $S_1(\mathcal{O}, \mathcal{O})$.*

To complete the picture, we mention the following important result.

Theorem 7.26 (Laver)**.** *Borel's conjecture 7.5 is consistent.*

It follows from Corollary 7.15 and the preceding that Borel's Conjecture is independent of the usual axioms of mathematics (ZFC).

Definition 7.27. A set $X \subseteq \mathbb{N}^{\mathbb{N}}$ is *strongly unbounded* iff for all $f \in \mathbb{N}^{\mathbb{N}}$, $|X \cap \underline{\{f\}}| < |X|$.

Intuitively, strongly unbounded sets needs to be "fat" enough to be unbounded, but "slim" enough to be strongly unbounded. For instance, $\mathbb{N}^{\mathbb{N}}$ is indeed unbounded, but it is too "fat" to be strongly-unbounded, recalling Observation 4.9.

Observation 7.28. *There exists strongly unbounded families of cardinality $\mathfrak{b}$ and $\mathfrak{d}$.*

Proof. By Lemmas 1.11 and 1.12. □

Observation 7.29. *Suppose $X \subseteq \mathbb{N}^{\mathbb{N}}$ is a set such that :*

- $\mathrm{cf}\,|X| > \aleph_0$,
- *For all $f \in \mathbb{N}^{\mathbb{N}}$, $|\{g \in X \mid g \leq f\}| < |X|$.*

then, X is strongly unbounded.

Proof. Because $\{g \in \mathbb{N}^{\mathbb{N}} \mid g \leq^* f\}$ can be obtained as the following countable union:

$$\bigcup \big\{ \{g \in \mathbb{N}^{\mathbb{N}} \mid g \leq f'\} \mid f' \in \mathbb{N}^{\mathbb{N}} \exists N \in \mathbb{N}(\forall n \geq N(f'(n) = f(n)) \big\}.$$

□

Let us examine several consequences of Borel's conjecture (BC).

Observation 7.30. *Assuming ZFC+BC, we have:*

(a) $\mathcal{SN} = [\mathbb{R}]^{\leq\omega}$.
(b) *$X \subseteq \mathbb{R}$ satisfies $S_1(\mathcal{O},\mathcal{O})$ iff X is countable.*
(c) *Any (continuous) image of* SMZ *is* SMZ*.*
(d) *There is no Luzin set.*
(e) *For any uncountable cardinal $\kappa \leq \mathsf{cov}(\mathcal{M})$, there is no strongly unbounded family $X \in [\mathbb{N}^{\mathbb{N}}]^{\kappa}$.*
(g) $\mathsf{cov}(\mathcal{M}) < \min\{\mathsf{cof}(\mathcal{M}), \mathfrak{b}\}$*. In particular $\mathfrak{b} > \aleph_1$ and $\neg$CH.*

Proof. (a) is equivalent to BC. (b) follows from Observation 7.14. (c) follows from the fact that an image of a countable set is countable. (d) follows from Corollary 7.15.

(e) If $X \subseteq \mathbb{N}^{\mathbb{N}}$ is strongly-unbounded and $\psi : \mathbb{N}^{\mathbb{N}} \to (0,1) \setminus \mathbb{Q}$ is an homeomorphism, then $\psi[X] \cup (\mathbb{Q} \cap [0,1])$ is $|X|$-concentrated at $\mathbb{Q} \cap [0,1]$. Thus, if also $|X| \leq \mathsf{cov}(\mathcal{M})$, then by Corollary 7.24 and Observation 7.14, $\psi[X] \cup (\mathbb{Q} \cap [0,1])$ is SMZ.

(f) If $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$, then we may apply Theorem 3.7 to obtain a subset of $\mathbb{R}$ which is $\mathsf{cov}(\mathcal{M})$-concentrated at any of its countable dense subsets. Now apply Corollary 7.24 and Observation 7.14.

If $\mathsf{cov}(\mathcal{M}) = \mathfrak{b}$, then Observation 7.28 would have contradict the preceding item.

Finally, by $\mathfrak{b} > \mathsf{cov}(\mathcal{M})$, we have:

$$\mathfrak{c} \geq \mathfrak{b} > \mathsf{cov}(\mathcal{M}) \geq \mathsf{add}(\mathcal{M}) \geq \aleph_1.$$

□

Question 7.31. Is it always true that the continuous image of SMZ is SMZ?

We had already seen that, consistently, SMZ and $S_1(\mathcal{O},\mathcal{O})$ are different properties, e.g., assuming CH, $S_1(\mathcal{O},\mathcal{O})$ is non-hereditary, while $\mathcal{SN}$ is an ideal. To answer our question (negatively), we introduce the following theorem:

Theorem 7.32 (Fremlin-Miller)**.** *For $X \subseteq \mathbb{R}$, TFAE:*

(a) $X \models S_1(\mathcal{O},\mathcal{O})$.
(b) *Any continuous image of X into $\mathbb{R}$ is strongly null.*

Corollary 7.33 (CH)**.** *There exists an* SMZ*,* $X \subseteq \mathbb{R}$*, and a continuous function* $f : X \to \mathbb{R}$*, such that* $f[X]$ *is not* SMZ*.*

Proof. Suppose not. Put $\mathcal{S} := \{X \subseteq \mathbb{R} \mid X \models S_1(\mathcal{O}, \mathcal{O})\}$, then for all $X \subseteq \mathbb{R}$, if X is SMZ, then any continuous image of it into $\mathbb{R}$ is SMZ. It follows from Theorem 7.32 that $\mathcal{S} = \mathcal{SN}$, and in particular, $\mathcal{S}$ is an ideal, contradicting Observation 7.17 with $\mathfrak{d} = \mathfrak{c} = \aleph_1$. □

It happens that the converse of Theorem 7.19 is also true.

Fact 7.34. *There exists* $X \in [\mathbb{N}^{\mathbb{N}}]^{\mathsf{cov}(\mathcal{M})}$ *that cannot be guessed.*

In particular, the minimal cardinality of $A \subseteq \mathbb{R}$ *with* $A \not\models S_1(\mathcal{O}, \mathcal{O})$ *is* $\mathsf{cov}(\mathcal{M})$*.*

Together with Observation 7.30, we obtain that assuming ZFC+BC: $\mathsf{cov}(\mathcal{M}) = \aleph_1 < \mathfrak{b}$.

8. 05.01.06

Observation 8.1 (ZFC+BC)**.** *If $\langle X, d\rangle$ is metric, and $X \models S_1(\mathcal{O},\mathcal{O})$, then $|X| \leq \aleph_0$.*

Proof. By Observation 7.14, every $S_1(\mathcal{O},\mathcal{O})$ metric space is strongly null. Thus, if Borel's conjecture 7.5 holds, then X must be countable. □

If one omits the requirement of metricity, we get the following.

Theorem 8.2 (ZFC)**.** *There exists an unctounable non-metrizable space that satisfies $S_1(\mathcal{O},\mathcal{O})$.*

Proof. Consider $X := \omega_1 + 1$. We equip X with the *interval topology*. Let $\langle X, O\rangle$ be the topological space determined by the base:

$$B := \{\alpha^\uparrow, \alpha^\downarrow, (\beta,\alpha) \mid \beta < \alpha < \omega_1\},$$

where $\alpha^\uparrow := \{\gamma \in X \mid \gamma > \alpha\}$, $\alpha^\downarrow := \{\gamma \in X \mid \gamma < \alpha\}$, $(\beta,\alpha) := \beta^\uparrow \cap \alpha^\downarrow$. We now show that X is concentrated on the singleton $\{\omega_1\}$, concluding that $X \models S_1(\mathcal{O},\mathcal{O})$. Indeed, if U is an open set containing ω_1, then $U \supseteq \alpha^\uparrow$ for some $\alpha < \omega_1$. For such α, we get that $(X \setminus U) \subseteq \alpha + 1$, and in particular, $(X \setminus U)$ is countable. □

We now work towards giving a direct proof to Corollary 7.33.

Lemma 8.3 (Embedding)**.** *Suppose there is a dominating/unbouned/strongly-unbounded family of cardinality κ, and $A \subseteq \{0,1\}^\omega$ is a set of cardinality $\leq \kappa$.*

Then, there there exists a set $B \in [\omega^\omega]^\kappa$ and a continuous function $\phi : \omega^\omega \to \omega^\omega$ such that B is dominating/unbouned/strongly-unbounded (respectively), and $\phi[B] = A$.

Proof. Assume $D = \{f_\alpha \mid \alpha < \kappa\} \in [\omega^\omega]^\kappa$ is unbounded (or dominating, or strongly-unbounded). Let $\{g_\alpha \mid \alpha < \kappa\}$ enumerate A. Put $B := \{h_\alpha \mid \alpha < \kappa\}$, where:

$$h_\alpha(n) := 2f_\alpha(n) + g_\alpha(n) \qquad (\alpha < \kappa, n < \omega)$$

B is evidently unbounded (or dominating, or strongly-unbounded). Finally, define a continuous function $\phi : \omega^\omega \to \omega^\omega$ by letting for all $f \in \omega^\omega$ and $n < \omega$: $\phi(f)(n) = f(n) \mod 2$. □

Lemma 8.4 (Interleaving)**.** *Suppose there is an unbouned/strongly-unbounded family of cardinality κ, and $A \subseteq \omega^\omega$ is a set of cardinality $\leq \kappa$.*

Then, there there exists a set $B \in [\omega^\omega]^\kappa$ and a continuous surjection $\phi : \omega^\omega \to \omega^\omega$ such that B is unbouned/strongly-unbounded (respectively), and $\phi[B] = A$.

Proof. Assume $D = \{f_\alpha \mid \alpha < \kappa\} \in [\omega^\omega]^\kappa$ is unbounded (or strongly-unbounded). Let $\{g_\alpha \mid \alpha < \kappa\}$ enumerate A. Put $B := \{h_\alpha \mid \alpha < \kappa\}$, where:

$$h_\alpha(n) = \begin{cases} f_\alpha(k) & \exists k < \omega(n = 2k) \\ g_\alpha(k) & \exists k < \omega(n = 2k+1) \end{cases}$$

B is evidently unbounded (or strongly-unbounded). Finally, define $\phi : \omega^\omega \to \omega^\omega$ in the obvious way. $\square$

Definition 8.5. Assume κ is a cardinal, and $\mathcal{I}$ is an ideal over some set X.

We say that $\mathcal{I}$ has the *κ-flexability* property iff $\mathcal{I}$ is non-trivial, and whenever $Y \subseteq X$ is κ-concentrated on some $A \in \mathcal{I}$, then $Y \in \mathcal{I}$.

Observation 8.6. *Suppose $\mathcal{I}$ is an ideal over some set X that has the κ-flexability property, then $\mathsf{non}(\mathcal{I}) \geq \kappa$.*

Proof. Fix $A \in [X]^{<\kappa}$. Pick $a \in A$. Since $\mathcal{I}$ is non-trivial, $\{a\} \in \mathcal{I}$. It is now obvious that A is κ-concentrated at $\{a\} \in \mathcal{I}$. $\square$

Observation 8.7. *$\mathcal{N}$ has the $\mathsf{non}(\mathcal{N})$-flexability property.*

$\mathcal{SN}$ has the $\mathsf{non}(\mathcal{SN})$-flexability property.

Proof. Assume Y, A are subsets of $\mathbb{R}$, where $A \in \mathcal{I}$ and Y is $\mathsf{non}(\mathcal{N})$-concentrated at A.

Fix $\varepsilon > 0$. Since $A \in \mathcal{I}$, we may find a family of open sets $\{U_n \mid n \in \mathbb{N}\}$ with $\sum_{n\in\mathbb{N}} \mathrm{Diam}(U_n) < \frac{\varepsilon}{2}$, and $A \subseteq U := \bigcup_{n\in\mathbb{N}} U_n$

Since U is open containing A, $|Y \setminus U| < \mathsf{non}(\mathcal{N})$. In particular, $(Y \setminus U) \in \mathcal{N}$ and we may find a family of open sets $\{V_n \mid n \in \mathbb{N}\}$ such that $(Y \setminus U) \subseteq \bigcup_{n\in\mathbb{N}} V_n$ and $\sum_{n\in\mathbb{N}} \mathrm{Diam}(V_n) < \frac{\varepsilon}{2}$.

The proof for the case of $\mathcal{SN}$ is essentially the same. $\square$

Theorem 8.8. *Assume $\mathcal{J} \subseteq \mathcal{P}(\mathbb{R})$ is a non-trivial, σ-additive, proper ideal.*

Then for any ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ and a cardinal $\kappa \geq \mathsf{non}(\mathcal{J})$ such that:

- *$\mathcal{I}$ has the κ-flexability property;*
- *There exists a strongly-unbounded family of size κ.*

there exists $X \in \mathcal{I}$, and a continuous function $f : X \to \mathbb{R}$ such that $f[X] \notin \mathcal{J}$.

Proof. Pick $A \in [\mathbb{R}]^{\mathsf{non}(\mathcal{J})}$, with $A \notin \mathcal{J}$. If $\{A \cap [z, z+1] \mid z \in \mathbb{Z}\} \subseteq \mathcal{J}$, then by the σ-additivity of $\mathcal{J}$, $A \in \mathcal{J}$. It follows that there exists $z \in \mathbb{Z}$, such that $[z, z+1] \cap A \notin \mathcal{J}$.

For notational simplicity, we assume $A \subseteq [0, 1]$. $\mathcal{J}$ is σ-additive and non-trivial, thus $\mathbb{Q} \in \mathcal{J}$, hence, we may also assume that $A \cap \mathbb{Q} = \emptyset$.

Altogether, we assume $A \subseteq ([0, 1] \setminus \mathbb{Q})$, $|A| = \mathsf{non}(\mathcal{J})$, and $A \notin \mathcal{J}$.

Let $\psi : [0, 1] \setminus \mathbb{Q} \to \omega^\omega$ be an homeomorphism. Put $A' := \psi[A]$. By the interleaving lemma 8.4, there exists a strongly-unbounded $B \in [\omega^\omega]^\kappa$, and a continuous function $\phi : \omega^\omega \to \omega^\omega$ such that $\phi[B] = A'$. Let $X := \psi^{-1}[B]$ and $f := (\psi^{-1} \circ \phi \circ \psi) \restriction X$.

Notice that $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ is a composition of continuous functions, and:

$$f[X] = \psi^{-1}[\phi[\psi[X]] = \psi^{-1}[\phi[B]] = \psi^{-1}[A'] = A \notin \mathcal{J}.$$

We are left with showing that $X \in \mathcal{I}$. Since $\mathcal{I}$ satisfies the κ-flexability property, it suffices to show that X is κ-concentrated at some set from $\mathcal{I}$. By Observation 8.6 and the hypothesis, $\mathsf{non}(\mathcal{I}) \geq \kappa \geq \mathsf{non}(\mathcal{J}) \geq \mathsf{add}(\mathcal{J}) \geq \aleph_1$, thus $\mathbb{Q} \in \mathcal{I}$. Finally, notice that if U is an open set containing $\mathbb{Q}$, then $\psi[[0,1] \setminus U]$ is compact, thus $\leq^*$-bounded, thus $\psi[X \setminus U]$ is a $\leq^*$-bounded subset of the strongly-unbounded set B, and hence, $|X \setminus U| = |\psi[X \setminus U]| < |B| = \kappa$. $\square$

Thus, for instance, if CH holds, we may find a strongly-null subset of $\mathbb{R}$ with a continuous image which is not null. We may also find a strongly-null subset of $\mathbb{R}$ with a continuous image which is not meager. In particular, this set must be uncountable, thus we had obtained an alternative proof to the fact that $\mathsf{CH} \Longrightarrow \neg\mathsf{BC}$.

Proposition 8.9 (CH)**.** *Assume that $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is an ideal that has the $\aleph_1$-flexability property, then for any $Y \subseteq \omega^\omega$, there exists $X \in \mathcal{I}$ and a continuous $f : X \to \omega^\omega$ such that $f[X] = Y$.*

Proof. Fix $Y \subseteq \mathbb{N}^\mathbb{N}$. If Y is countable, this is easy (recall Observation 8.6).

Assume that Y is uncountable. By CH, we may fix a $\mathfrak{b}$-scale $\{f_y \in \omega^\omega \mid y \in Y\}$. Now, by applying the interleaving lemma 8.4, we obtain a set $B \subseteq \omega^\omega$ that interleaves ω^ω inside this scale. In greater details, we obtain a strongly-unbounded set B of size $\mathfrak{b}$, and a continuous function $\phi : \omega^\omega \to \omega^\omega$ such that $\phi[B] = Y$. Let $\psi : [0,1] \setminus \mathbb{Q} \to \omega^\omega$ be an homeomorphism.

Put $X := \psi^{-1}[B]$ and $f = (\phi \circ \psi) \restriction X$. Evidently, f is continuous and $f[X] = Y$.

The standard argument shows that X is $\mathfrak{b}$-concentrated at $\mathbb{Q}$. Finally, it follows from the hypothesis that $\mathbb{Q} \in \mathcal{I}$, $\mathfrak{b} = \aleph_1$ and $X \in \mathcal{I}$. $\square$

Corollary 8.10 (CH)**.** *There exists $X \in \mathcal{SN}$, and a continuous function $f : X \to \mathbb{R}$ such that $f[X] \in \mathcal{SN}^*$, i.e., a strongly-null set whose continuous image is of Lebesgue measure 1.*

Proof. Since $(0,1) \setminus \mathbb{Q}$ is of Lebesgue measure 1 and a continuous image of ω^ω. $\square$

It is worth mentioning that one can prove in ZFC that there exists continuous mapping from the cantor set (=a set of measure zero) onto the unit interval (=a set of measure 1).

Question 8.11. Suppose there exists an arbitrary metric space $\langle X, d\rangle$ which is uncountable and strongly-null, must this indicate the violation of Borel's Conjecture 7.5 ?

Question 8.12 (Miller)**.** Suppose there exists a metric space $\langle X, d\rangle$ which is strongly-null and $|X| = \mathfrak{c}$, must this indicate the existence of $Y \in [\mathbb{R}]^\mathfrak{c}$ which is SMZ ?

The second question is still open. We shall now work towards introducing a positive answer to the first question. The key to the solution of this question is Carlson's lemma. 8.21 which is deeply inspired by Urysohn's Theorem 8.20.

Definition 8.13. A topological space $\langle X, O\rangle$ is T_1 iff $\{x\}$ is a closed subset for all $x \in X$.

Definition 8.14. A T_1 topological space X is *regular* iff whenever A is closed subset of X and $x \notin A$, then there are disjoint open sets U, V with $x \in U$ and $A \subseteq V$.

A T_1 topological space X is *normal* iff whenever A, B are disjoint closed sets in X, then there are disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$.

Notice that a metric space is normal and regular. Actually, we had already took advantage of this property in the proof of Theorem 3.16. Also notice that a normal space is regular, since in a T_1 space points are closed sets.

Observation 8.15. *Suppose $\langle X, O\rangle$ is a topological space such that for any two closed subsets A, B, there exists a continuous function $f : X \to [0,1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$, then X is normal.*

Proof. Fix closed subsets A, B, and let f be like in the hypothesis. Then $f^{-1}[0, 0.5)$ and $f^{-1}(0.5, 1]$ are mutually disjoint open sets, containing A and B respectively. □

Urysohn, in his celebrated lemma, was able to prove the converse:

Lemma 8.16 (Urysohn)**.** *Let X be a normal topological space, and $A, B \subset X$ are disjoint and closed. Then there exist a continuous function $f : X \to [0,1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.*

Proof. Fix an enumeration $\mathbb{Q} \cap [0,1] = \{r_n \mid n \in \mathbb{N}\rangle$ with $r_1 = 1$ and $r_2 = 0$. We will construct a family of open sets $\langle V_r | r \in \mathbb{Q} \cap [0,1]\rangle$ by induction on $n \in \mathbb{N}$. The family will satisfy:

$$r < r' \Longrightarrow \overline{V_r} \subset V_{r'} \qquad (r, r' \in \mathbb{Q} \cap [0,1])$$

Inductinon base $n \in \{1, 2\}$: Put $V_1 = V_{r_1} := B^c$. Since X is normal, the separation $A \subseteq U \subseteq \overline{U} \subset B^c$, where U is open, is possible. Pick such U and let $V_0 = V_{r_2} := U$.

Inductive hypothesis: Assume we had already defined $V_{r_1}, V_{r_2}, ..., V_{r_n}$.

Induction step $n + 1$: Find $m, l \in \mathbb{N}$ such that $r_m := \max\{r_i \mid i \leq n, r_i < r_n\}$ and $r_l := \min\{r_i \mid i \leq n, r_i > r_n\}$ ("closest" rationals to r_{n+1} so far). By the normality of X, an open set U exists such that $V_{r_m} \subseteq U \subset \overline{U} \subseteq V_{r_l}$. Define $V_{r_{n+1}} := U$. End of the construction.

We now define a function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{r | x \in V_r\} & \text{if } x \in V_1 \\ 1 & \text{if } x \in B \end{cases}$$

In order to prove that f is continuous, it is suffice to show that $f^{-1}[0,a)$ and $f^{-1}(b,1]$ are open subsets of X for any $a,b\in\mathbb{R}$. Indeed:

$$f^{-1}[0,a)=\{x|f(x)<a\}=\{x|\exists r\in\mathbb{Q},r<a,x\in V_r\}=\bigcup_{\substack{0\leq r<a\\ r\in\mathbb{Q}}}V_r.$$

This is a union of open sets, thus open.

$$f^{-1}(b,1]=\{x|f(x)>b\}=\{x|f(x)\leq b\}^c=\{x|\forall r'>b,x\in V_{r'}\}^c=\{x|\exists r'>b,x\notin V_{r'}\}=$$
$$\{x|\exists r,r',r'>r>b,x\notin\overline{V_r}\subseteq V_{r'}\}=\bigcup_{b<r\leq 1}\overline{V_r}^c.$$

Again, this is a union of open sets, hence open. □

In order to prove our next theorem we will have to introduce the Hilbert space ℓ_2.

Definition 8.17. A natural extension of finite dimensional euclidian spaces is

$$\ell_2:=\Big\{(x_1,x_2,...)\mid x_i\in\mathbb{R},\sum_{n\in\mathbb{N}}x_n^2<\infty\Big\}.$$

For any two elements $x,y\in\ell_2$, the inner product is defined by $\langle x,y\rangle:=\sum_{n\in\mathbb{N}}x_ny_n$. It is well known that any inner product space is a normed space by defining

$$||x-y||^2:=\langle x-y,x-y\rangle \qquad (x,y\in\ell_2)$$

Notice that ℓ_2 is separable. A countable dense set is $\big\{(x_1,...,x_n,0,0,..)\in\ell_2\mid n\in\mathbb{N},x_i\in\mathbb{Q}\big\}$.

Theorem 8.18 (Urysohn)**.** *A second countable normal space is metrizable.*[26]

Proof. Let X be a second countable normal space, and assume $\mathcal{B}=\{B_j\mid j\in\mathbb{N}\}$ is a countable base for the topology on X. Put $\mathcal{I}:=\big\{(j,i)\in\mathbb{N}\times\mathbb{N}\mid\overline{B_j}\subseteq B_i\big\}$.

For each $(j,i)\in\mathcal{I}$, by applying Urysohn's lemma 8.16, we may pick a continuous function $f_{j,i}:X\to[0,1]$ such that $f_{j,i}[B_i^c]=\{1\}$ and $f_{j,i}[\overline{B_j}]=\{0\}$. Let us enumerate these functions $\langle f_{j,i}\mid(j,i)\in\mathcal{I}\rangle=\langle g_n\mid n\in\mathbb{N}\rangle$ and define a function $G:X\to\ell_2$ by letting for each $x\in\ell_2$:

$$G(x):=\Big(g_1(x),\frac{g_2(x)}{2},...,\frac{g_n(x)}{n},...\Big).$$

Showing that G is a homeomorphism on $G[X]\subseteq\ell_2$ will do, since a subspace of a metrizable space is metrizable.

G is an injection: Fix $x\neq y$ in X. It suffices to find $(j,i)\in\mathcal{I}$ such that $f_{j,i}(x)\neq f_{j,i}(y)$. X is T_1, thus a base set $B_i\in\mathcal{B}$ exists, such that $x\in B_i$ and $y\notin B_i$. Now, since X is normal, a base set $B_j\in\mathcal{B}$ exists, such that $x\in B_j\subseteq\overline{B_j}\subseteq B_i$, hence, $f_{j,i}(x)=1$ and $f_{j,i}(y)=0$.

[26]Recall that a *second countable* topological space is a space with a countable base to its' topology.

G is continuous: Let $x \in X$ and $\varepsilon > 0$. Let N be large enough so that $\sum_{n>N} \frac{1}{n^2} < \varepsilon^2$. The functions $g_1, ..., g_N$ are continuous, therefore there are open sets $U_1, ..., U_N$, containing x, such that $\frac{1}{n^2}|g_n(x) - g_n(x_n)|^2 < \frac{\varepsilon^2}{N}$ whenever $1 \leq n \leq N$ and $x_n \in U_n$. Finally, for every $u \in U := \bigcap_{1\leq n\leq N} U_n$, we have:

$$||G(x) - G(u)||^2 = \sum_{n\in\mathbb{N}} \frac{|g_n(x) - g_n(u)|}{n^2} < 2\varepsilon^2.$$

We get that for every $x \in X$ there existt an open set $x \in U$ such that $G(U) \subseteq \mathsf{B}_{\sqrt{2}\varepsilon}(G(x))$, that is G is continuous.

G is open: Let U be an open subset of X and pick $x \in U$. Since X is regular there are $B_i, B_j \in \mathcal{B}$ such that $x \in B_j \subseteq \overline{B_j} \subseteq B_i$.

Now, $g_n = f_{j,i}$ satisfies $g_n(x) = 0$ and $g_n(U^c) = 1$, therefore, for $y \in U^c$

$$||G(x) - G(y)|| \geq \frac{1}{n^2}|g_n(x) - g_n(y)|^2 = \frac{1}{n^2}.$$

We get that if y satisfies $G(y) \in \mathsf{B}_{\frac{1}{2n}}(G(x))$ than $y \notin U^c$, meaning that $y \in U$ and therefore $\mathsf{B}_{\frac{1}{2n}}(G(x)) \subset G(U)$, hence G is open. □

The previous theorem can be strengthened with some more topological arguments.

Lemma 8.19. *A second countable regular space X is normal.*

Proof. Suppose A and B are mutually-disjoint closed subsets of X.

Assume $\mathcal{B} = \langle D_n | n \in \mathbb{N}\rangle$ is a countable base to X. Fix functions $f : A \to \mathbb{N}, g : B \to \mathbb{N}$ such that:

- For all $x \in A$: $x \in D_{f(x)} \subseteq \overline{D_{f(x)}} \subseteq B^c$;
- For all $y \in B$: $y \in D_{g(y)} \subseteq \overline{D_{g(y)}} \subseteq A^c$.

To see such function exists, fix for instance $x \in A$. Since X is regular, a base set $D_n \in \mathcal{B}$ exists such that $x \in D_n \subseteq \overline{D_n} \subseteq B^c$.

Enumerate $\{U_n \mid n \in \mathbb{N}\} = \{D_{f(x)} \mid x \in A\}$ and $\{V_n \mid n \in \mathbb{N}\} = \{D_{g(y)} \mid y \in B\}$. It follows that $A \subseteq \bigcup_{n\in\mathbb{N}} U_n, B \subseteq \bigcup_{n\in\mathbb{N}} V_n$, and $B \cap \overline{U_n} = \emptyset, A \cap \overline{V_n} = \emptyset$ for all $n \in \mathbb{N}$.

For every $n \in \mathbb{N}$, define $U'_n := U_n \setminus \bigcup_{i\leq n} \overline{V_i}$ and $V'_n := V_n \setminus \bigcup_{i\leq n} \overline{U_i}$.

Notice that $U := \bigcup_{n\in\mathbb{N}} U'_n$ is a union of open sets, thus open. Same for $V := \bigcup_{n\in\mathbb{N}} V'_n$.

Also, by the choice of $\{U_n, V_n \mid n \in \mathbb{N}\}$, $A \subseteq U$ and $B \subseteq V$. We are left with showing that $U \cap V = \emptyset$. Assume that there is x with $x \in U \cap V$, that is, there are $i, j \in \mathbb{N}$ with $x \in U'_i \cap V'_j$. Obviously, $i \neq j$. Actually, if $i < j$, then $x \notin V'_j$, and if $i > j$, then $x \notin U'_i$. Altogether, we get that $U \cap V = \emptyset$. □

Corollary 8.20 (Urysohn)**.** *A second countable regular space is metrizable.*

ℓ_2 is a separable metric space. Urysohn's theorem assures us that a second countable regular space is separable and metrizable. On the other hand, any separable metrizable space is second countable[27] and normal (hence regular), thus the equivalence. knowing that, we get that every separable metrizable space is homeomorphic to some subspace of ℓ_2.

Lemma 8.21 (Carlson)**.** *If $\langle X, d\rangle$ is a separable metric space and $|X| < 2^{\aleph_0}$, then there exists an injection $\psi : X \to \mathbb{R}$ such that $|\psi(x) - \psi(y)| \leq d(x, y)$ for all $x, y \in X$.*

Proof. By Lemma 5.3, we may assume that $\mathrm{Im}(d) \subseteq [0, 1]$.[28] Since X is separable, we may pick a dense subset $\{x_n \mid n < \omega\}$. For each $x \in X$, attach an analytic function on the unit ball, $f_x : \{y \in \mathbb{C} \mid |y| < 1\} \to \mathbb{C}$, by letting:

$$f_x(z) := \sum_{n=0}^{\infty} \frac{d(x, x_n)}{n!} z^n.$$

Since $x \mapsto \langle d(x, x_n) \mid n < \omega\rangle$ is one-to-one, and two analytic functions with different Taylor expension are different, we have that $x \mapsto f_x$ is one-to-one.

Lemma 8.22. *If f, g are two analytic functions, then $A_{f,g} := \{z \mid f(z) = g(z)\}$ is countable.*

Proof. Suppose not, then we could find a compact subset $K \subseteq \mathbb{C}$ such that $K \cap A_{f,g}$ is uncountable. In particular, f and g are two analytic functions that share an accumulation point, and we must have conclude that $f = g$. □

Put $A := \bigcup\{A_{f_x,f_y} \mid x, y \in X, x \neq y\}$. $|A| < 2^{\aleph_0}$ since $|X| < 2^{\aleph_0}$, and it follows that we may pick $r \in [0, \ln(e)] \subseteq \mathbb{R}$ such that $r \notin A$. Define $\psi : X \to \mathbb{R}$ by $\psi(x) := f_x(r)$ for all $x \in X$. ψ is an injection. To see that it satisfies the Lipshitz property, notice that for all $x, y \in X$, we have:

$$|\psi(x)-\psi(y)| = |f_x(r)-f_y(r)| = \Big|\sum_{n=0}^{\infty} \frac{d(x, x_n)}{n!} r^n - \sum_{n=0}^{\infty} \frac{d(y, x_n)}{n!} r^n\Big| = \Big|\sum_{n=0}^{\infty} \frac{d(x, x_n) - d(y, x_n)}{n!} r^n\Big|$$

$$\leq \sum_{n=0}^{\infty} \frac{d(x, y)}{n!} r^n = e^r \cdot d(x, y) \leq e^{\ln(e)} \cdot d(x, y) = d(x, y).$$

□

Theorem 8.23 (Carlson)**.** *If there exists an uncountable metric space which is strongly null, then $\neg$BC.*

[27] Consider all open balls of rational radiuses centered at elements of a countable dense set.

[28] Notice that if $\langle X, d\rangle$ is strongly null, then so is $\langle X, \frac{d}{1+d}\rangle$.

Proof. If $\mathfrak{c} = \aleph_1$, then by corollaries 3.9 and 7.15, $\neg\mathsf{BC}$ and we are done. Assume $\mathfrak{c} > \aleph_1$. Assume that $\langle X, d\rangle$ is an uncountable strongly-null metric space, then for all $Y \in [X]^{\aleph_1}$, $\langle Y, d\rangle$ is a strongly-null metric space of cardinality $< 2^{\aleph_0}$. Had we known that Y is separable, we could use Lemmas 8.21,7.8 to complete the proof. Recalling Lemma 2.6, we are left with proving the following. □

Lemma 8.24. *Assume $\langle X, d\rangle$ is a strongly null metric space, then X is second-countable.*

Proof. By the hypothesis, for all $n \in \mathbb{N}$, we may find $\langle x^n_m \in X \mid m \in \mathbb{N}\rangle$ and $\{\varepsilon^m_n \in (0,\infty) \mid m \in \mathbb{N}\}$ such that $X \subseteq \bigcup_{m\in\mathbb{N}} \mathsf{B}_{\varepsilon^n_m}(x^n_m)$ and $\sum_{m\in\mathbb{N}} \varepsilon^n_m < \frac{1}{n}$. A moment's reflection makes it clear that $\{\mathsf{B}_{\varepsilon^n_m}(x^n_m) \mid n, m \in \mathbb{N}\}$ is a base to X. □

Corollary 8.25. *Suppose $\langle X, d\rangle$ is a metric space and $X \models S_1(\mathcal{O},\mathcal{O})$, then $w(X) = \aleph_0$.*

Proof. By Observation 7.14 and the preceding lemma. □

Definition 8.26. For a topological space $\langle X, O\rangle$, let $o(X) = |O| + \aleph_0$.

Corollary 8.27. *Suppose $\langle X, d\rangle$ is a metric space and $X \models S_1(\mathcal{O},\mathcal{O})$, then $o(X) \leq w(x)^{\aleph_0}$.*

Proof. By the preceding Lemma, we may pick a base $\mathcal{B}$ of cardinality $\aleph_0$, and then any $U \in O$ is of the form $U = \bigcup\mathcal{U}$ for some $\mathcal{U} \subseteq \mathcal{B}$, i.e., $U = \bigcup\mathcal{U}$ for some $\mathcal{U} \in [\mathcal{B}]^{\leq\aleph_0}$. □

We now work towards proving the same for $S_{fin}(O, O)$.

Lemma 8.28. *Suppose $\langle X, d\rangle$ is a metric space, then any open set U is F_σ.*

Proof. Since U is open $U = \bigcup_{i\in I} \mathsf{B}_{r_i}(x_i)$ (where I is some index set and $\mathsf{B}_{r_i}(x_i)$ is an open ball of radius r_i centered at x_i).

For every $i \in I$ fix some sequence $\langle \varepsilon_{i_k} \mid k \in \mathbb{N}\rangle$ such $\varepsilon_{i_k} \to r_i$. Define $F_k := \bigcup_{i\in I} \overline{\mathsf{B}_{\varepsilon_{i_k}}(x_i)}$.

Evidently $U = \bigcup_{k\in\mathbb{N}} F_k$.

□

Lemma 8.29. *The property $S_{fin}(\mathcal{O},\mathcal{O})$ is σ-additive.*

Proof. Suppose $\langle X, O\rangle$ is a metric space, and $\langle X_m \subseteq X \mid m < \omega\rangle$ is a family of subspaces, each satisfies $S_{fin}(\mathcal{O},\mathcal{O})$. We shall show that $\bigcup_{m\in\mathbb{N}} X_m \models S_{fin}(\mathcal{O},\mathcal{O})$.

Assume $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ is a family of open covers of $\bigcup_{n\in\mathbb{N}} X_n$. Put $\mathbb{N} = \biguplus_{m\in\mathbb{N}} A_m$ where each A_m is infinite. For $m \in \mathbb{N}$, by $X_m \models S_{fin}(\mathcal{O},\mathcal{O})$, we may find $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in A_m\rangle$ such that $X_m \subseteq \bigcup\bigcup_{n\in A_m} \mathcal{F}_m$. It follows that $\bigcup_{m\in\mathbb{N}} X_m \subseteq \bigcup\bigcup_{m\in\mathbb{N}} \mathcal{F}_m$. □

Corollary 8.30. *$S_{fin}(\mathcal{O},\mathcal{O})$ is open hereditary to any metric space.*

Proof. By Observation 1.27, $S_{fin}(\mathcal{O},\mathcal{O})$ is closed hereditary. Now apply Lemmas 8.28,8.29.

□

Corollary 8.31. *Suppose $\langle X, d\rangle$ is a metric space and $X \models S_{fin}(\mathcal{O},\mathcal{O})$, then $o(X) \leq w(x)^{\aleph_0}$.*

Proof. Fix a base $\mathcal{B}$ of cardinality $w(X)$. Then for any open set U, there exists some $\mathcal{U} \subseteq \mathcal{B}$ such that $U = \bigcup \mathcal{U}$. Finally, by Corollary 8.30 and Observation 1.28 (applied to U), there exists $V \in [\mathcal{U}]^{\leq \aleph_0}$ such that $U = \bigcup \mathcal{V}$. Thus, we have shown that for each open set U, there exists $\mathcal{V} \in [\mathcal{B}]^{\leq \aleph_0}$ such that $U = \bigcup \mathcal{V}$. $\square$

9. 12.01.06

Definition 9.1. We say that a topological space $\langle X, O\rangle$ satisfies $S_1(\mathcal{A}, \mathcal{B})$ iff for every sequence $\langle \mathcal{U}_n \in \mathcal{A} \mid n \in \mathbb{N}\rangle$, there are $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{B}$.

Definition 9.2. We say that a topological space $\langle X, O\rangle$ satisfies $S_{fin}(\mathcal{A}, \mathcal{B})$ iff for every sequence $\langle \mathcal{U}_n \in \mathcal{A} \mid n \in \mathbb{N}\rangle$, there are $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathcal{B}$.

Definition 9.3. We say that a topological space $\langle X, O\rangle$ satisfies $U_{fin}(\mathcal{A}, \mathcal{B})$ iff for every sequence $\langle \mathcal{U}_n \in \mathcal{A} \mid n \in \mathbb{N}\rangle$ such that $\mathcal{U}_n$ does not contain a finite cover for all $n \in \mathbb{N}$, there are $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \mathcal{B}$.

We will only be interested in $\mathcal{A}, \mathcal{B}$ with $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(O)$ and $\bigcup U = X$ for all $U \in \mathcal{A} \cup \mathcal{B}$.

Observation 9.4. *Suppose $\langle X, O\rangle$ is a topological space, and $\mathcal{O}$ denotes the family of all open covers of X.*

Then $X \models S_{fin}(\mathcal{A}, \mathcal{O})$ iff $X \models U_{fin}(\mathcal{A}, \mathcal{O})$.

Proof. Same proof as in Observation 5.15. □

Observation 9.5 (monotonicity)**.** *If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$ then $\pi(\mathcal{A}_2, \mathcal{B}_1) \Rightarrow \pi(\mathcal{A}_1, \mathcal{B}_1)$ and $\pi(\mathcal{A}_2, \mathcal{B}_1) \Rightarrow \pi(\mathcal{A}_2, \mathcal{B}_2)$, where $\pi \in \{S_1, S_{fin}, U_{fin}\}$.*

Lemma 9.6. *Suppose $\langle X, O\rangle$ is a Lindelöf topological space, $\mathcal{B} \subseteq \mathcal{P}(O)$, and let $\Gamma := \Gamma_X$ denote the family of all γ-covers of X.*[29]

Then $X \models U_{fin}(\Gamma, \mathcal{B})$ iff for all $\mathcal{A}$, a family of open covers of X, $X \models U_{fin}(\mathcal{A}, \mathcal{B})$.

Proof. We would like to prove:

$$\forall \mathcal{A}.X \models U_{fin}(\mathcal{A}, \mathcal{B}) \Rightarrow X \models U_{fin}(\Gamma, \mathcal{B}) \Rightarrow X \models U_{fin}(\mathcal{O}, \mathcal{B}) \Rightarrow \forall \mathcal{A}.X \models U_{fin}(\mathcal{A}, \mathcal{B}).$$

But the only non-trivial implication is $X \models U_{fin}(\Gamma, \mathcal{B}) \Rightarrow X \models U_{fin}(\mathcal{O}, \mathcal{B})$.

Assume $\langle \mathcal{U}_n \in \mathcal{O} \mid n \in \mathbb{N}\rangle$ are given and no $\mathcal{U}_n$ contains a finite cover. Fix $n \in \mathbb{N}$. By Lindelöfness, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$. Let $\mathcal{V}_n := \{V_n^k \mid k \in \mathbb{N}\}$ where $V_n^k := \bigcup_{m\leq k} U_n^m$ for all $k \in \mathbb{N}$. Since $\mathcal{U}_n$ contains no finite cover, we know that $\mathcal{V}_n \in \Gamma$.

By $X \models U_{fin}(\Gamma, \mathcal{B})$, there exists $f : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ such that if we let $\mathcal{F}_n := \{V_n^k \mid k \in f(n)\}$ for all $n \in \mathbb{N}$, then $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\} \in \mathcal{B}$.

Define $g : \mathbb{N} \to [\mathbb{N}]^{<\omega}$ by letting $g(n) := \{m \in \mathbb{N} \mid \exists k \in f(n).m \leq k\}$ for all $n \in \mathbb{N}$. It is evident that $\bigcup \mathcal{G}_n = \bigcup \mathcal{F}_n$ whenever $n \in \mathbb{N}$ and $\mathcal{G}_n := \{U_n^k \mid k \in g(n)\} \in [\mathcal{U}_n]^{<\omega}$. □

Corollary 9.7. *Suppose $\langle X, O\rangle$ is a Lindelöf topological space. Let $\Gamma := \Gamma_X$.*

Then $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ iff $X \models U_{fin}(\mathcal{O}, \mathcal{O})$ iff $X \models U_{fin}(\Gamma, \mathcal{O})$

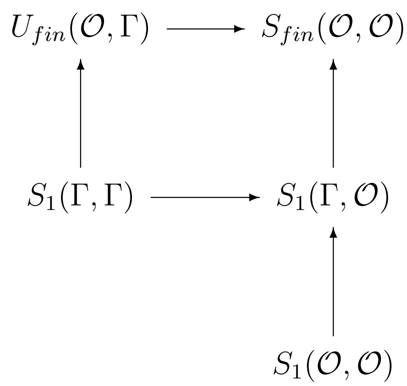
[29]Recall Definition 5.8.

Corollary 9.8. *Suppose $\langle X, O\rangle$ is a Lindelöf topological space. Let $\Gamma := \Gamma_X$.*
Then $X \models U_{fin}(\mathcal{O}, \Gamma)$ iff $X \models U_{fin}(\Gamma, \Gamma)$.

Proposition 9.9. *$S_1(\mathcal{O}, \Gamma)$ is trivial*

Proof. Because it implies $S_{fin}(\mathcal{O}, \Gamma)$. Now recall Observation 5.14. □

The same trick of the proof of Theorem 9.6 will prove that $S_1(\Gamma, \Gamma)$ implies $U_{fin}(\mathcal{O}, \Gamma)$ and that $S_1(\Gamma, \mathcal{O})$ implies $S_{fin}(\mathcal{O}, \mathcal{O})$, thus we obtain the following diagram of implications:



Theorem 9.10 (Scheepers-Just-Miller-Szeptycki)**.** $S_{fin}(\Gamma, \Gamma) = S_1(\Gamma, \Gamma)$.

Proof. Suppose $\langle X, O\rangle$ is a topological space, $\Gamma := \Gamma_X$, and $X \models S_{fin}(\Gamma, \Gamma)$.

Assume $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By the hypothesis, there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \Gamma$. By Observation 5.9, if we pick $\langle U_n \in \mathcal{F}_n \mid n \in \mathbb{N}\rangle$, then also $\{U_n \mid n \in \mathbb{N}\} \in \Gamma$ and we are almost done.

In order to be done, we need to somehow ensure that we indeed selected an element $U_n \in \mathcal{U}_n$ for all $n \in \mathbb{N}$, but this wouldn't happen in the above approach if there exists empty $\mathcal{F}_n$'s. To complete the proof, we need the following. □

We now generalize the idea of Observation 7.13.

Lemma 9.11. *Suppose $\langle X, O\rangle$ is a topological space, then $X \models S_1(\Gamma, \Gamma)$ iff for all $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ there exists $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{\leq 1} \mid n \in \mathbb{N}\rangle$ such that $\{U \mid \exists n \in \mathbb{N}(U \in \mathcal{F}_n)\} \in \Gamma$.*

Proof. Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U^k_n \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $\mathcal{V}_n := \{U^k_1 \cap ... \cap U^k_n \mid k \in \mathbb{N}\}$. Clearly, $\langle \mathcal{V}_n \mid n \in \mathbb{N}\rangle$ is a sequence of γ-covers, so by the hypothesis we find $\mathcal{F}_n \in [\mathcal{V}_n]^{\leq 1}$ for each $n \in \mathbb{N}$.

Let $f : \mathbb{N} \to \mathbb{N} \cup \{\star\}$ be the function such that for all $n \in \mathbb{N}$, $f(n) = \{\star\}$ if $\mathcal{F}_n = \emptyset$, and $\mathcal{F}_n = \{U^{f(n)}_1 \cap ... \cap U^{f(n)}_n\}$, otherwise. Since $\{U \mid \exists n \in \mathbb{N}(U \in \mathcal{F}_n)\} \in \Gamma$, $\mathrm{Im}(f)$ is infinite, and the function $\tilde{f} : \mathbb{N} \to \mathbb{N}$ is well-defined:

$$f(n) := \{f(m) \mid m = \min\{k \geq n \mid f(k) \neq \star\}.$$

For $n \in \mathbb{N}$, put $U_n := U_n^{\tilde{f}(n)}$. It is now obvious that $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ is a witness to $S_1(\Gamma, \Gamma)$.[30] □

Observation 9.12. *Assume $\langle X, O\rangle$ is a topological space, and $\langle U_n \mid n \in \mathbb{N}\rangle$ is a **sequence** of open sets such that $\{n \in \mathbb{N} \mid x \notin U_n\}$ is finite for all $x \in X$.*

*If $X \neq U_n$ for all $n \in N$, then $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\}$ is an infinite **set**, and in particular $\mathcal{U} \in \Gamma$.*

Proof. Suppose not, then by a trivial pigeonhole argument, there exists some $m \in \mathbb{N}$ and infinite $I \subseteq \mathbb{N}$ such that $U_n = U_m$ for all $n \in I$. Since $U_n \neq X$, we may pick $x \in X \setminus U_m$ and conclude that $I \subseteq \{n \in \mathbb{N} \mid x \notin U_n\}$, yielding a contradiction to the finiteness hypothesis. □

[30] More accurately, it is a witness to an instance of $S_1(\Gamma, \Gamma)$, because the family $\langle U_n \mid n \in \mathbb{N}\rangle$ were already given.

10. 19.01.06

Definition 10.1. A property P is a *topological invariant* iff every two homeomorphic spaces either both satisfy P, or they do not satisfy it.

It is easier to think about it in the sense that topological invariants are determined by the topology (which is determined up to an homeomorphism).

For example:

- Completeness of metrics is not a topological invariant. Despite the fact that the spaces $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{R} \setminus \mathbb{Q}$ are homeomorphic, $\mathbb{N}^{\mathbb{N}}$ is a complete space but $\mathbb{R} \setminus \mathbb{Q}$ is not.
- We have already seen that SMZ is not a topological invariant.

Assume X, Y are homeomorphic, and let $\psi : X \to Y$ be an homeomorphism.

- First category is a topological invariant. Assume $M \subset X$ is meager, hence, $M \subseteq \bigcup_{n\in\mathbb{N}} F_n$, where F_n is closed and nowhere dense for every $n \in \mathbb{N}$. Now $\psi[M] \subseteq \psi\big[\bigcup_{n\in\mathbb{N}} F_n\big] = \bigcup_{n\in\mathbb{N}} \psi[F_n]$. Assume that for some $n \in \mathbb{N}$ $\psi[F_n]$ is not nowhere dense, that is, there is an open set $U \subset Y$ such that $U \subset \psi[F_n]$ meaning that $\psi^{-1}[U] \subset F_n$. But ψ^{-1} is continuous, thus $\psi^{-1}[U]$ is open, a contradiction to the fact that F_n is nowhere dense.
- Being a Luzin set is a topological invariant. Let $L \subset X$ be a Luzin set, and assume $M \subset Y$ is meager. knowing the last example $L \cap \psi^{-1}[M]$ is countable, but since ψ is an injection, so is $\psi\big[L \cap \psi^{-1}[M]\big] = \psi[L] \cap M$. The last equality holds since ψ is a bijection. Since M is an arbitrary meager set in Y, we get that $\psi[L]$ is a Luzin set.

Definition 10.2. Suppose P is a topological invariant property, let $\mathsf{non}(P)$ denote the minimal cardinality of a space that does not satisfy property P.

We sometime call $\mathsf{non}(P)$ as *the critical cardinality* of P.

The diagram from page 60 shows implications and has the property that any property $\pi(\mathcal{A}, \mathcal{B})$, where $\pi \in \{S_1, S_{fin}, U_{fin}\}$ and $\{\mathcal{A}, \mathcal{B}\} \subseteq \{\mathcal{O}, \Gamma\}$, is equivalent to one of the properties that appears in the diagram.

We now would like to show that this diagram is succinct, in the sense that there are no more equivalent properties in this diagram. We obtain our goal by analyzing their critical cardinalities.

Observation 10.3. $\mathsf{non}(S_{fin}(\mathcal{O}, \mathcal{O})) = \mathfrak{d}$.

Proof. By Theorem 4.10. □

Observation 10.4. $\mathsf{non}(U_{fin}(\mathcal{O}, \Gamma)) = \mathfrak{b}$.

Proof. By Theorem 5.18. □

Observation 10.5. *$\mathsf{non}(S_1(\mathcal{O},\mathcal{O})) = \mathsf{cov}(\mathcal{M})$.*

Proof. By Corollary 7.22, Theorem 7.19 and Fact 7.34. □

Observation 10.6. *Suppose P, Q are topological properties, and $P \to Q$, that is, for any space X, $X \models P$ only if $X \models Q$. then $\mathsf{non}(P) \leq \mathsf{non}(Q)$.*

Lemma 10.7. *$\mathsf{non}\left(S_1(\Gamma,\mathcal{O})\right) = \mathfrak{d}$.*

Proof. By the preceding observation, by $S_1(\Gamma,\mathcal{O}) \to S_{fin}(\mathcal{O},\mathcal{O})$, and by $\mathsf{non}(S_{fin}(\mathcal{O},\mathcal{O})) = \mathfrak{d}$, it suffices to show that if $\langle X, O\rangle$ is a topological space and $|X| < \mathfrak{d}$, then $X \models S_1(\Gamma, O)$.

Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. For all $x \in X$, define $f_x \in \mathbb{N}^{\mathbb{N}}$, by letting for all $n \in \mathbb{N}$:

$$f_x(n) := \min\{m \in \mathbb{N} \mid \forall k \geq m(x \in U_n^k)\}.$$

By $|X| < \mathfrak{d}$, we may pick some $g \in \mathbb{N}^{\mathbb{N}}$ such that $g \not\leq^* f_x$ for all $x \in X$.

For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. We claim that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{O}$. To see this, fix $x \in X$.

Let $n \in \mathbb{N}$ be such that $f_x(n) < g(n)$, then, by definition of f_x, $x \in U_n^{g(n)} = U_n$. □

Thus, we obtain the analogue of Corollary 7.24.

Corollary 10.8. *If $X \subseteq \mathbb{R}$ is $\mathfrak{d}$-concentrated at one of its countable subsets, then $X \models S_1(\Gamma,\mathcal{O})$.*

Proof. Divide to odds and evens like in the proof of Observation 3.17. □

Lemma 10.9. *$\mathsf{non}(S_1(\Gamma,\Gamma)) = \mathfrak{b}$.*

Proof. By $S_1(\Gamma,\Gamma) \to U_{fin}(\mathcal{O},\Gamma)$, and $\mathsf{non}(U_{fin}(\mathcal{O},\Gamma)) = \mathfrak{b}$, it suffices to show that if $\langle X, O\rangle$ is a topological space and $|X| < \mathfrak{b}$, then $X \models S_1(\Gamma,\Gamma)$.

Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N}\rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. For all $x \in X$, define $f_x \in \mathbb{N}^{\mathbb{N}}$, by letting for all $n \in \mathbb{N}$:

$$f_x(n) := \min\{m \in \mathbb{N} \mid \forall k \geq m(x \in U_n^k)\}.$$

By $|X| < \mathfrak{b}$, we may pick some $g \in \mathbb{N}^{\mathbb{N}}$ such that $\{f_x \mid x \in X\} \subseteq \underline{\{g\}}$. For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. We claim that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{O}$. To see this, fix $x \in X$.

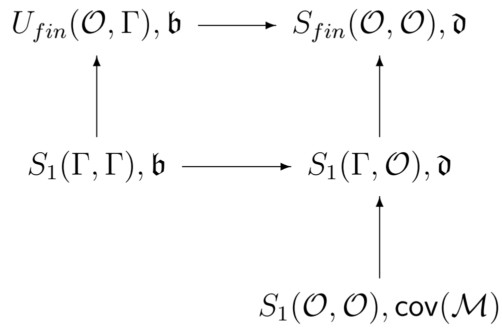
Let $m \in \mathbb{N}$ be such that $f_x(n) \leq g(n)$ for all $m \geq n$, then, by definition of f_x, we have that $x \in U_n^{g(n)} = U_n$ for all $n \geq m$ and we are done. □

By Lemma 1.9, $\mathfrak{b} \leq \mathfrak{d}$, and by Observation 5.9, $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$. Assuming CH they are all equal, but it is also consistent to have $\mathfrak{b} < \mathfrak{d}$ or $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$. Thus:

Corollary 10.10. *$S_1(\Gamma,\Gamma) \nrightarrow S_1(\mathcal{O},\mathcal{O})$, $S_1(\Gamma,\mathcal{O}) \nrightarrow S_{fin}(\mathcal{O},\Gamma)$ and $S_1(\mathcal{O},\mathcal{O}) \nrightarrow U_{fin}(\mathcal{O},\Gamma)$.*

Also recall Theorem 6.13 that shows that $S_{fin}(\mathcal{O},\mathcal{O}) \not\rightarrow U_{fin}(\mathcal{O},\Gamma)$.

Thus, to claim that the diagram is succinct, we still need to seperate $S_1(\Gamma,\Gamma)$ from $U_{fin}(\mathcal{O},\Gamma)$ and $S_1(\Gamma,\mathcal{O})$ from $S_{fin}(\mathcal{O},\mathcal{O})$.



Theorem 10.11 (Scheepers-Just-Miller-Szeptycki)**.** *The cantor space satisfies $S_{fin}(\mathcal{O},\mathcal{O})$ and $U_{fin}(\mathcal{O},\Gamma)$ but does not satisfy $S_1(\Gamma,\mathcal{O})$ and $S_1(\Gamma,\Gamma)$.*

Proof. Let $X := \{0,1\}^{\mathbb{N}}$ be the cantor space. X is compact, so by Lemma 5.16, $X \models U_{fin}(\mathcal{O},\Gamma)$, and hence also $X \models S_{fin}(\mathcal{O},\mathcal{O})$.

To show that $X \models \neg S_1(\gamma,\mathcal{O}) \wedge \neg S_1(\Gamma,\Gamma)$, it suffices to show that $X \not\models S_1(\Gamma,\mathcal{O})$. We first need the following lemma:

Lemma 10.12. *There exist a matrix $A = \langle A_m^n \mid m,n \in \mathbb{N}\rangle$ satisfying :*

(1) *Each element of the matrix is closed subset of the cantor space.*
(2) *Fixing $m \in \mathbb{N}$, $\langle A_m^n \mid n \in \mathbb{N}\rangle$ are disjoint.*
(3) *For different $m_1,...,m_k \in \mathbb{N}$, $\cap A_{m_1}^{n_1} \cdots \cap A_{m_k}^{n_k} \neq \emptyset$, for all $n_1,...,n_k \in \mathbb{N}$.*

Proof. Omitted. □

Now, for each $m \in \mathbb{N}$, let $\mathcal{U}_m := \{X \setminus A_n^m \mid n \in \mathbb{N}\}$. By property (1), members of $\mathcal{U}_m$ are open sets. Together with property (2), we get that $\mathcal{U}_m \in \Gamma$.

Finally, assume a sequence $\langle U_m \in \mathcal{U}_m \mid m \in \mathbb{N}\rangle$. For all $m \in \mathbb{N}$, there exists some $n_m \in \mathbb{N}$ such that $U_m = X \setminus A_m^{n_m}$. By property (3), $\mathcal{F} := \{A_m^{n_m} \mid m \in \mathbb{N}\}$ satisfies the finite intersection property. Together with with property (1), we obtain that $\bigcap \mathcal{F} \neq \emptyset$, and hence $\{U_m \mid m \in \mathbb{N}\} \notin \mathcal{O}$. □

Corollary 10.13. *For all $X \subseteq \mathbb{R}$, if X contains a perfect subset, then $X \not\models S_1(\Gamma,\mathcal{O})$.*

Proof. If X contains a perfect set, then it contains a closed subset which is homeomorphic to the cantor space. Now, it is easy to see that $S_1(\Gamma,\mathcal{O})$ is a closed-hereditary property. □

Corollary 10.14. *If $X \subseteq \mathbb{R}$ is an uncountable F_σ set, then $X \not\models S_1(\Gamma,\mathcal{O})$.*

Proof. Since any uncountable F_σ set contains a closed perfect subset. □

Theorem 10.15. *It is consistent that* $\mathfrak{b} = \mathsf{cov}(\mathcal{M})$*, while* $U_{fin}(\mathcal{O}, \Gamma) \neq S_1(\mathcal{O}, \mathcal{O})$*.*

First proof. By the arguments of Observation 5.23, if $\mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$, then there exists a set $X \subseteq \mathbb{N}^\mathbb{N}$ which is $\leq^*$-unbounded and $\mathsf{cov}(\mathcal{M})$-concentrated on its dense countable subset, by Corollary 7.24, $X \models S_1(\mathcal{O}, \mathcal{O})$, and by Theorem 5.19, $X \not\models U_{fin}(\mathcal{O}, \Gamma)$.

Finally, assuming CH, we indeed have $\mathfrak{b} = \mathsf{cov}(\mathcal{M}) = \mathsf{cof}(\mathcal{M})$. □

The essence of the preceding proof is Corollary 4.5 that implies that any Luzin subset of $\mathbb{N}^\mathbb{N}$ is $\leq^*$-unbounded. Also notice that since any Luzin set $L \subseteq \mathbb{R}$ satisfies $S_1(\mathcal{O}, \mathcal{O})$, and the latter implies SMZ, then there must exist some dense subset of $\mathbb{R}$ which is disjoint from L.

Observation 10.16. *A Sierpinski does not satisfy* $S_1(\mathcal{O}, \mathcal{O})$*.*

Proof. By Observation 7.14 and Proposition 7.3, if $S \models S_1(\mathcal{O}, \mathcal{O})$, then S is a null set. A Sierpinski set is an uncountable set have countable intersection with any null set, so it cannot be itself a null set. □

Lemma 10.17 (Scheepers-Just-Miller-Szeptycki)**.** *Any Sierpinski,* S*, satisfies* $S_1(\Gamma, \Gamma)$*.*

Proof. Suppose $\langle \mathcal{U}_n \in \Gamma \mid n \in \mathbb{N} \rangle$ are given. By Observation 5.9, we may assume an enumeration $\mathcal{U}_n = \{U_n^k \mid k \in \mathbb{N}\}$ for all $n \in \mathbb{N}$. For all $x \in X$, define $f_x \in \mathbb{N}^\mathbb{N}$, by letting for all $n \in \mathbb{N}$:

$$f_x(n) := \min\{m \in \mathbb{N} \mid \forall k \geq m (x \in U_n^k)\}.$$

We claim that $x \overset{\psi}{\mapsto} f_x$ is a Borel map. Fix a finite function $\sigma : \{1, .., m\} \to \mathbb{N}$, we need to show that $A := \psi^{-1}[\sigma^\uparrow]$ is a Borel subset of S. Indeed, $A = \bigcap \{A_1^n, A_2^n \mid 1 \leq n \leq m\}$, where:

$$A_1^n = \{x \in S \mid \forall k \geq \sigma(n)(x \in U_n^k)\} = \bigcap_{k=\sigma(n)}^{\infty} U_n^k,$$

$$A_2^n = \{x \in S \mid \exists k < \sigma(n)(x \notin U_n^k)\} = \bigcup_{k<\sigma(n)} S \setminus U_n^k.$$

If follows Claim 5.29 that we may pick some $g \in \mathbb{N}^\mathbb{N}$ such that $\{f_x \mid x \in X\} \subseteq \underline{\{g\}}$. For all $n \in \mathbb{N}$, let $U_n := U_n^{g(n)}$. We claim that $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{O}$. To see this, fix $x \in X$.

Let $m \in \mathbb{N}$ be such that $f_x(n) \leq g(n)$ for all $m \geq n$, then, by definition of f_x, we have that $x \in U_n^{g(n)} = U_n$ for all $n \geq m$ and we are done. □

Corollary 10.18. *It is consistent that* $\mathfrak{b} = \mathsf{cov}(\mathcal{M})$*, while* $S_1(\Gamma, \Gamma) \neq S_1(\mathcal{O}, \mathcal{O})$*.*

Proof. By Corollary 3.8, assuming CH, there exists a Sierpinski set, S, and also $\mathfrak{b} = \mathsf{cov}(\mathcal{M})$. □

11. 26.01.06

Definition 11.1. An open cover $\mathcal{U}$ is an ω-cover of X iff:

- For every finite set $F \subseteq X$ there exist $U \in \mathcal{U}$ such that $F \subseteq U$.
- $X \notin \mathcal{U}$.

We denote the family all ω-covers of X by Ω.

Observation 11.2. *If $\mathcal{U}$ is an ω-cover of X, then for every finite subset $F \subseteq X$ there are infinitely many $U \in \mathcal{U}$ such that $F \subseteq U$. In particular, $\mathcal{U}$ is infinite.*

Proof. For all $U \in \mathcal{U}$, pick $x_U \in X \setminus U$ arbitrarily. Fix $F \in [X]^{<\omega}$.

We define an infinite family $\{U_n \mid n \in \mathbb{N}\}$ by induction. Let U_1 be such that $F \subseteq U_1$, and let U_{n+1} be such that $F \cup \{X_{U_1}, .., X_{U_n}\} \subseteq U_{n+1}$. □

We denote by $C(X)$ the set of all continuous functions from X to $\mathbb{R}$. We will consider this as a topological space, and the topology will be inherited from $\mathbb{R}^X \supseteq C(X)$. This topology is determined by pointwise convergence, that is, $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in X$. This topological space is not metrizable, thus the closure operator is not easy to figure.

Definition 11.3. A topological space X satisfies the Frèchet-Urysohn (FU) property iff for every $A \subset X$ and every $a \in \overline{A}$ there exist a sequence $\langle a_n | n \in \mathbb{N}\rangle$ such that $a_n \to a$.[31]

Definition 11.4. A topological space satisfies the property $\binom{\mathcal{A}}{\mathcal{B}}$ iff for every $\mathcal{U} \in \mathcal{A}$ there exist $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$.

For example, denote by Φ all finite open covers. The property $\binom{\mathcal{O}}{\Phi}$ is compactness.

Theorem 11.5 (Gerlitz-Nagy)**.** *$C(X)$ satisfies the FU property iff $X \models \binom{\Omega}{\Gamma}$.*

The property $\binom{\Omega}{\Gamma}$ is also known as the γ-property and is equivalent to $S_1(\Omega, \Gamma)$:

Lemma 11.6. *$S_1(\Omega, \Gamma)$ implies $\binom{\Omega}{\Gamma}$.*

Proof. Suppose $\langle X, O\rangle$ is a topological space. $\Omega := \Omega_X, \Gamma := \Gamma_X$, and $X \models S_1(\Omega, \Gamma)$.

Fix $\mathcal{U} \in \Omega$. For all $n \in \mathbb{N}$, let $\mathcal{U}_n := \mathcal{U}$. It follows from $X \models S_1(\Omega, \Gamma)$ that there exists $\langle U_n \in \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ such that $\{U_n \mid n \in \mathbb{N}\} \in \Gamma$. Since $\{U_n \mid n \in \mathbb{N}\} \subseteq \mathcal{U}$, we are done. □

Theorem 11.7. $S_1(\Omega, \Gamma) = \binom{\Omega}{\Gamma}$.

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[31]In a general topological space, a sequence $\langle a_n \mid n \in \mathbb{N}\rangle$ converges to a iff every open set containing a, contains the tail of the sequence.

Definition 11.8. The *Rothberger space* is $[\mathbb{N}]^{\aleph_0} := \langle A \subseteq \mathbb{N} \mid |A| = \aleph_0 \rangle$.

- For $A, B \subseteq \mathbb{N}$: $A \subseteq^* B$ iff $|A \setminus B| < \omega$.
- $\mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0}$ is *centered* iff every $A_1, ..., A_k \in \mathcal{F}$ satisfies $\bigcap_{i \leq k} A_k$ is infinite.
- $A \in [\mathbb{N}]^{\aleph_0}$ is *almost intersection* of $\mathcal{F}$ iff for every $B \in \mathcal{F}$, $A \subseteq^* B$.
- $\mathfrak{p} := \min \left\{ |\mathcal{F}| \mid \mathcal{F} \subseteq [\mathbb{N}]^{\aleph_0} \text{ is centered and } \mathcal{F} \text{ does not have an almost intersection} \right\}$.

Lemma 11.9. $\mathfrak{p} > \aleph_0$.

Proof. Suppose $\mathcal{F} := \{B_n \in [\mathbb{N}]^{\aleph_0} \mid n \in \mathbb{N}\}$ is centered. For all $n \in \mathbb{N}$, put $A_n := B_1 \cap ... \cap B_n$, by the hypothesis on $\mathcal{F}$, $A_n \neq \emptyset$, so pick $x_n \in A_n \setminus \{x_1, .., x_{n-1}\}$. It is now obvious that $A = \{x_n \mid n \in \mathbb{N}\}$ is an almost intersection of $\mathcal{F}$. □

Lemma 11.10. *Suppose $\langle X, O \rangle$ is a topological space, and the product space X^k is Lindelöf for all $k \in \mathbb{N}$, then any ω-cover contains a countable ω-cover.*

Theorem 11.11. $\mathsf{non}(\binom{\Omega}{\Gamma}) = \mathfrak{p}$.

Proof. Suppose $X \in [\mathbb{R}]^{<\mathfrak{p}}$ and $\mathcal{U} \in \Omega$. By the preceding lemma, we may assume an enumeration $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\}$. For all $x \in X$, let $A_x := \{n \in \mathbb{N} \mid x \in U_n\}$. By Observation 11.2, $A_x \in [\mathbb{N}]^{\aleph_0}$ for all $x \in \mathbb{X}$ and $\mathcal{F} := \{A_x \mid x \in X\}$ is centered.

Since $|\mathcal{F}| \leq |X| < \mathfrak{p}$, we may pick an almost intersection $B \in [\mathbb{N}]^{\aleph_0}$.

We claim that $\{U_n \mid n \in B\} \in \Gamma$. Indeed, if $x \in X$ then $B \setminus A_x$ is finite, that is, $\{n \in B \mid x \notin U_n\}$ is finite.

We shall now introduce a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality $\mathfrak{p}$ with $X \not\models \binom{\Omega}{\Gamma}$.

By definition of $\mathfrak{p}$, there exists a centered family $X \subseteq [\mathbb{N}]^{\aleph_0}$ of cardinality $\mathfrak{p}$ with no almost intersection. For each $n \in \mathbb{N}$, let $U_n := \{A \in [\mathbb{N}]^{\aleph_0} \mid n \in A\}$, this is an open set and $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\} \in \Omega_X$, because if $F \subseteq X$ is finite, then centeredness of X implies that $I = \bigcap F$ is infinite, and hence $I \subseteq \{n \in \mathbb{N} \mid F \subseteq U_n\}$.

Finally, suppose there exists a strictly increasing function $k : \mathbb{N} \to \mathbb{N}$ such that $\{U_{k(n)} \mid n \in \mathbb{N}\} \in \Gamma_X$. We claim that $B := \mathrm{Im}(k)$ is an almost-intersection of X which is a contradiction. Indeed, for $A \in X$, if $\{n \in \mathbb{N} \mid A \notin U_{k(n)}\}$ is finite, then $B \setminus A$ is finite. □

12. 02.02.06

Definition 12.1. For families $\mathcal{A}, \mathcal{B}$, let $G_1(\mathcal{A}, \mathcal{B})$ denote the game of length ω, where at round $n \in \mathbb{N}$, player **I** picks $\mathcal{U}_n \in \mathcal{A}$ and player **II** responds with picking $U_n \in \mathcal{U}_n$.

$$\begin{array}{lccccccc}
\mathbf{I}: & \mathcal{U}_1 \in \mathcal{A} & & & \mathcal{U}_2 \in \mathcal{A} & & & \ldots \\
 & & \searrow & \nearrow & & \searrow & & \\
\mathbf{II}: & & U_1 \in \mathcal{U}_1 & & & & U_2 \in \mathcal{U}_2 & \ldots
\end{array}$$

Player **II** *wins this game* if $\{U_n \mid n \in \mathbb{N}\} \in \mathcal{B}$, otherwise, player **I** wins the game.

Let $\mathrm{Seq}(A)$ denote the family of finite sequences (including the empty sequence) with elements from a given set A. If $s = \langle x_1, .., x_n \rangle \in \mathrm{Seq}(A)$ and $x \in A$, then $s^\frown x := \langle x_1, .., x_n, x \rangle$.

Definition 12.2. A function $\sigma : \mathrm{Seq}(\bigcup \mathcal{A}) \to \mathcal{A}$ is a *winning strategy* for player **I** in $G_1(\mathcal{A}, \mathcal{B})$ iff player **I** plays according to this strategy, then he wins the game:

$$\begin{array}{lcccccccc}
\mathbf{I}: & \mathcal{U}_1 := \sigma(\emptyset) & & & \mathcal{U}_2 := \sigma\langle U_1 \rangle & & & \mathcal{U}_3 := \sigma\langle U_1, U_2 \rangle & \ldots \\
 & & \searrow & \nearrow & & \searrow & \nearrow & & \\
\mathbf{II}: & & U_1 \in \mathcal{U}_1 & & & U_2 \in \mathcal{U}_2 & & & \ldots
\end{array}$$

Definition 12.3. A function $\tau : \mathrm{Seq}(\mathcal{A}) \setminus \{\emptyset\} \to \bigcup \mathcal{A}$ is a *winning strategy* for player **II** in $G_1(\mathcal{A}, \mathcal{B})$ iff player **II** plays according to this strategy, then he wins the game:

$$\begin{array}{lccccccc}
\mathbf{I}: & \mathcal{U}_1 \in \mathcal{A} & & & \mathcal{U}_2 \in \mathcal{A} & & & \ldots \\
 & & \searrow & \nearrow & & \searrow & & \\
\mathbf{II}: & & U_1 := \tau\langle \mathcal{U}_1 \rangle & & & & U_2 := \tau\langle \mathcal{U}_1, \mathcal{U}_2 \rangle & \ldots
\end{array}$$

Definition 12.4. For a given families $\mathcal{A}, \mathcal{B}$, we write $\mathbf{I} \uparrow G_1(\mathcal{A}, \mathcal{B})$ to denote that player **I** has winning strategy in $G_1(\mathcal{A}, \mathcal{B})$. We define $\mathbf{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B}), \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B}), \mathbf{II} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$ in the obvious fashion.

The game $G_1(\mathcal{A}, \mathcal{B})$ is said to be *determined* iff $\mathbf{I} \uparrow G_1(\mathcal{A}, \mathcal{B}) \vee \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$.

Note that both players cannot have a winning strategy in the same game.

Observation 12.5. *Suppose $\langle X, O \rangle$ is a topological space, and $\mathcal{A}, \mathcal{B}$ are given families.*
Then $X \models \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ implies $X \models \mathbf{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$ implies $X \models S_1(\mathcal{A}, \mathcal{B})$.

Notice that if $G_1(\mathcal{A}, \mathcal{B})$ is determined, then $X \models \mathbf{II} \uparrow G_1(\mathcal{A}, \mathcal{B})$ iff $X \models \mathbf{I} \not\uparrow G_1(\mathcal{A}, \mathcal{B})$.

Lemma 12.6. *Suppose $\langle X, O \rangle$ is a topological space.*
We define the following cardinal function invariant:

$$\delta(X) := \min\{\kappa + \aleph_0 \mid \exists \mathcal{F} \in [[O]^{\leq \kappa}]^{\leq \kappa}. \forall \mathcal{U} \in \mathcal{F}\Big(\bigcup \mathcal{U} = X\Big) \wedge \forall \phi \in \prod \mathcal{F}\Big(|\bigcap \mathrm{Im}(\phi)| \leq \kappa\Big)\}.$$

Lemma 12.7. *If $\langle X, d \rangle$ is a metric space, then $\delta(X) \leq d(X)$.*

Proof. Put $\kappa := \delta(X)$. Let $D := \{x_i \mid i < \kappa\}$ enumerate a dense subset of X.

Let $\mathcal{F} := \{\mathcal{U}_n \mid n \in \mathbb{N}\}$, where $\mathcal{U}_n := \{\mathsf{B}_r(x_i) \mid i < \kappa, r \in \mathbb{Q} \cap (0, \frac{1}{n+1})\}$ for all $n \in \mathbb{N}$.

Since $|\mathcal{U}_n| \leq \aleph_0 \cdot \kappa = \kappa$ for all $n \in \mathbb{N}$, we have that $\mathcal{F} \in [[O]^{\leq\kappa}]^{\aleph_0}$, where O denotes the family of all open sets in this metric space. In particular, $\mathcal{F} \in [[O]^{\leq\kappa}]^{\leq\kappa}$.

Since D is a dense subset, we also have that $\bigcup \mathcal{U}_n = X$ for all $n \in \mathbb{N}$.

Finally, if $\phi \in \prod \mathcal{F}$ is a choice function, then letting $U_n := \phi(\mathcal{U}_n)$ for all $n \in \mathbb{N}$, we get that $\lim_{n\to\infty} \mathrm{Diam}(U_n) = 0$, and hence $\bigcap \mathrm{Im}(\phi) \leq 1 \leq \kappa$. □

Theorem 12.8. *Suppose $\langle X, O\rangle$ is a topological space, $\mathcal{O} := \mathcal{O}_X$, and $X \models \mathbf{II} \uparrow G_1(\mathcal{O}, \mathcal{O})$.*

Then $|X| \leq \delta(X)$.

Proof. Let $\mathcal{F} \in [[O]^{\leq\kappa}]^{\leq\kappa}$ be a witness to the value of $\kappa := \delta(X)$. We shall examine the outcome of the game $G_1(\mathcal{O}, \mathcal{O})$ when player **II** plays with a winning strategy, τ, against members of $\mathcal{F}$. For any sequence $s \in \mathrm{Seq}(\mathcal{F})$, let $\mathcal{A}_s := \{\tau(s^\frown\mathcal{U}) \mid \mathcal{U} \in \mathcal{F}\}$. Since $\mathcal{U} \mapsto \tau(s^\frown\mathcal{U})$ defines a choice function on $\mathcal{F}$, we know that $|\mathcal{A}_s| \leq \kappa$.

Claim 12.9. *$A := \bigcup_{s\in\mathrm{Seq}(\mathcal{F})} \bigcap \mathcal{A}_s$ is of cardinality $\leq \kappa$.*

Proof. $|\mathcal{F}| \leq \kappa$, and the latter is an infinite cardinal number, thus $|\mathrm{Seq}(\mathcal{F})| \leq \kappa$.

It follows that A is the union of length at most κ of sets of at most cardinality κ. □

Claim 12.10. *$A = X$.*

Proof. Suppose not and pick $x \in X \setminus A$. It follows that for all $s \in \mathrm{Seq}(\mathcal{F})$, there exists some $\mathcal{U} \in \mathcal{F}$ such that $x \notin \tau(s^\frown\mathcal{U})$. This implies that we may define inductively, a sequence $\langle \mathcal{U}_n \in \mathcal{F} \mid n \in \mathbb{N}\rangle$ such that $x \notin \tau\langle \mathcal{U}_1, .., \mathcal{U}_n\rangle$ for all $n \in \mathbb{N}$. In particular $\bigcup_{n\in\mathbb{N}} \tau\langle \mathcal{U}_1, .., \mathcal{U}_n\rangle \neq X$, a contradiction to the assumption that τ is a winning strategy for **II** in $G_1(\mathcal{O}, \mathcal{O})$. □

It follows that $|X| = |A| \leq \kappa$. □

Corollary 12.11 (Telgarski). *If $\langle X, d\rangle$ is a separable metric space, then $X \models \mathbf{II} \uparrow G_1(\mathcal{O}, \mathcal{O})$ iff X is countable.*

Proof. If X is countable, then it is easy to introduce a winning strategy for **II** in this game. For the other direction, we apply to Theorem 12.8 and Lemma 12.7 to conclude $|X| \leq \delta(X) \leq (X) = \aleph_0$. □

Define the game $G_{fin}(\mathcal{A}, \mathcal{B})$ in the obvious fashion, then:

Theorem 12.12 (Telgarski). *For all $X \subseteq \mathbb{R}$, $X \models \mathbf{II} \uparrow G_{fin}(\mathcal{O}, \mathcal{O})$ iff X is σ-compact.*

Proof. Essentially the same as in the proof of 12.8. □

Theorem 12.13 (Pavlikowsky). *For all $X \subseteq \mathbb{R}$, $X \models \mathbf{I} \not\uparrow G_1(\mathcal{O}, \mathcal{O})$ iff $X \models S_1(\mathcal{O}, \mathcal{O})$.*

Corollary 12.14. *It is consistent that the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is determined for all $X \subseteq \mathbb{R}$.*

Proof. Assume the Borel conjecture 7.5 (Recall that BC is consistent). Fix $X \subseteq \mathbb{R}$.

If $X \models S_1(\mathcal{O}, \mathcal{O})$, then by Observation 8.1, $|X| \leq \aleph_0$, together with Corollary 12.11, we conclude that $\mathbf{II} \uparrow G_1(\mathcal{O}_x, \mathcal{O}_X)$.

Suppose $X \not\models S_1(\mathcal{O}, \mathcal{O})$, then by Theorem 12.13, we have $\mathbf{I} \uparrow G_1(\mathcal{O}_x, \mathcal{O}_X)$. $\square$

Corollary 12.15 (Recław)**.** *It is consistent to have some $X \subseteq \mathbb{R}$ such that the game $G_1(\mathcal{O}_X, \mathcal{O}_X)$ is not determined.*

Proof. Let $L \subseteq \mathbb{R}$ be a Luzin set. $L \models S_1(\mathcal{O}, \mathcal{O})$, thus by Theorem 12.13, $L \models \mathbf{I} \not\uparrow G_1(\mathcal{O}, \mathcal{O})$.

L is uncountable, thus by Corollary 12.11, $L \models \mathbf{II} \not\uparrow G_1(\mathcal{O}, \mathcal{O})$. $\square$