HOW TO CONSTRUCT A SOUSLIN TREE THE RIGHT WAY

Abstract. An excerpt from the paper "A Microscopic approach to Souslin-tree constructions. Part I" by Brodsky and Rinot.

Throughout, let κ denote a regular uncountable cardinal.

For a set of ordinals, C, write $\mathrm{acc}(C) := \{\alpha \in C \mid \sup(C \cap \alpha) = \alpha > 0\}$ and $\mathrm{nacc}(C) := C \setminus \mathrm{acc}(C)$.

Definition 1 ([BR17])**.** $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ such that:

- for all $\alpha < \kappa$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$;
- for all $\alpha < \kappa$ and $\beta \in \mathrm{acc}(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$;
- for every cofinal $B \subseteq \kappa$, there exists an infinite ordinal $\alpha < \kappa$ such that $\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha$.

Clearly, $\diamondsuit(\omega_1) \implies \clubsuit(\omega_1) \implies \boxtimes^-(\omega_1)$. By [BR17], if $V = L$, then $\boxtimes^-(\kappa)$ holds for every (regular uncountable cardinal) κ that is not weakly compact.

Definition 2 ([BR17])**.** $\diamondsuit^-(H_\kappa)$ asserts the existence of a sequence $\langle S_\beta \mid \beta < \kappa\rangle$ such that for every parameter $p \in H_{\kappa^+}$ and subset $S \subseteq H_\kappa$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$ containing p such that $\kappa^{\mathcal{M}} := \mathcal{M} \cap \kappa$ is an ordinal $< \kappa$ and $S_{\kappa^{\mathcal{M}}} = \mathcal{M} \cap S$.

A proof of the easy fact that $\diamondsuit(\kappa)$ is equivalent to $\diamondsuit^-(H_\kappa)$ may be found as [BR17, Lemma 2.2].

Proposition 3 ([BR17, Proposition 2.3])**.** *If $\boxtimes^-(\kappa) + \diamondsuit(\kappa)$ holds, then there exists a κ-Souslin tree.*

Proof. Let $\langle C_\alpha \mid \alpha < \kappa\rangle$ be a witness to $\boxtimes^-(\kappa)$. Let $\langle S_\beta \mid \beta < \kappa\rangle$ be a witness to $\diamondsuit^-(H_\kappa)$. In addition, let $\lhd$ be some well-ordering of ${}^{<\kappa}2$.

We shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ such that $T := \bigcup_{\alpha<\kappa} T_\alpha$ ordered by $\subset$ will form a normal κ-Souslin tree whose α^{th}-level be T_α. Furthermore, we shall ensure that for all $\alpha < \kappa$, T_α be a subset of ${}^\alpha 2$ of size $\leq \max\{\aleph_0, |\alpha|\}$.

Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$, let $T_{\alpha+1} := \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle \mid t \in T_\alpha\}$.

Next, suppose that α is a nonzero limit ordinal, and that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been defined. Denote $T \restriction \alpha := \bigcup_{\beta<\alpha} T_\beta$. Constructing the level T_α involves deciding which branches through $(T \restriction \alpha, \subset)$ will have their limits placed into the tree. We need T_α to contain enough nodes to ensure that the tree is normal, so the idea is to attach to each node $x \in T \restriction C_\alpha := \bigcup_{\beta\in C_\alpha} T_\beta$ some node $\mathbf{b}^\alpha_x : \alpha \to 2$ above it, and then let

$$T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction C_\alpha\}.$$

Let $x \in T \restriction C_\alpha$ be arbitrary. As $\mathbf{b}^\alpha_x$ will be the limit of some branch through $\{y \in T \restriction \alpha \mid x \subseteq y\}$, and as $\sup(C_\alpha) = \alpha$, it makes sense to describe $\mathbf{b}^\alpha_x$ as the limit $\bigcup \mathrm{Im}(b^\alpha_x)$ of a sequence $b^\alpha_x \in \prod_{\beta \in C_\alpha \setminus \mathrm{dom}(x)} T_\beta$ such that:

- $b^\alpha_x(\mathrm{dom}(x)) = x$;
- $b^\alpha_x(\beta') \subset b^\alpha_x(\beta)$ for all $\beta' < \beta$ both from $C_\alpha \setminus \mathrm{dom}(x)$;
- $b^\alpha_x(\beta) = \bigcup \mathrm{Im}(b^\alpha_x \restriction \beta)$ for all $\beta \in \mathrm{acc}(C_\alpha \setminus \mathrm{dom}(x))$.

We do this by recursion:

Let $b^\alpha_x(\mathrm{dom}(x)) := x$. Next, suppose $\beta^- < \beta$ are successive points of $C_\alpha \setminus \mathrm{dom}(x)$, and $b^\alpha_x(\beta^-)$ has already been defined. In order to decide $b^\alpha_x(\beta)$, we advise with the following set:

$$Q^\alpha_{x,\beta} := \{t \in T_\beta \mid \exists s \in S_\beta\ (s \cup b^\alpha_x(\beta^-)) \subseteq t\}.$$

Now, consider the two possibilities:

- If $Q^\alpha_{x,\beta} \neq \emptyset$, then let $b^\alpha_x(\beta)$ be its $\lhd$-least element.
- Otherwise, let $b^\alpha_x(\beta)$ be the $\lhd$-least element of T_β that extends $b^\alpha_x(\beta^-)$. Such an element must exist, as the level T_β was constructed so as to preserve normality.

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Finally, suppose $\beta \in \mathrm{acc}(C_\alpha \setminus \mathrm{dom}(x))$ and $b^\alpha_x \restriction \beta$ has already been defined. As promised, we let $b^\alpha_x(\beta) := \bigcup \mathrm{Im}(b^\alpha_x \restriction \beta)$. It is clear that $b^\alpha_x(\beta) \in {}^\beta 2$, but we need more than that:

Claim 3.1. $b^\alpha_x(\beta) \in T_\beta$.

Proof. It suffices to prove that $b^\alpha_x \restriction \beta = b^\beta_x$, as this will imply that $b^\alpha_x(\beta) = \bigcup \mathrm{Im}(b^\beta_x) = \mathbf{b}^\beta_x \in T_\beta$.

First, note that since $\beta \in \mathrm{acc}(C_\alpha)$ and $\langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\boxtimes^-(\kappa)$-sequence, we have $\mathrm{dom}(b^\beta_x) = C_\beta \setminus \mathrm{dom}(x) = C_\alpha \cap \beta \setminus \mathrm{dom}(x) = \mathrm{dom}(b^\alpha_x) \cap \beta$. Call the latter by d. Now, we prove by induction that for every $\gamma \in d$, the value of $b^\beta_x(\gamma)$ was determined in exactly the same way as $b^\alpha_x(\gamma)$:

- Clearly, $b^\beta_x(\min(d)) = x = b^\alpha_x(\min(d))$.
- Suppose $\gamma^- < \gamma$ are successive points of d. Notice that the definition of $Q^\alpha_{x,\gamma}$ depends only on $b^\alpha_x(\gamma^-)$, S_γ, and T_γ, and so if $b^\alpha_x(\gamma^-) = b^\beta_x(\gamma^-)$, then $Q^\alpha_{x,\gamma} = Q^\beta_{x,\gamma}$, and hence $b^\alpha_x(\gamma) = b^\beta_x(\gamma)$.
- For $\gamma \in \mathrm{acc}(d)$: If the sequences are identical up to γ, then their limits must be identical. $\square$

This completes the definition of b^α_x for each $x \in T \restriction C_\alpha$, and hence of the level T_α.

Put $T := \bigcup_{\alpha<\kappa} T_\alpha$. The definition of $T_{\alpha+1}$ for $\alpha < \kappa$ makes it clear that $(T, \subset)$ is splitting. Consequently, to prove that $(T, \subset)$ is κ-Souslin, it now suffices to show that it has no κ-sized antichains.

To this end, suppose that A is a maximal antichain in $(T, \subset)$. Put

$$B := \{\beta < \kappa \mid A \cap (T \restriction \beta) = S_\beta \text{ is a maximal antichain in } T \restriction \beta\}.$$

Claim 3.2. *B is cofinal in κ.*

Proof. Let $\epsilon < \kappa$ be arbitrary. We must show that $B \nsubseteq \epsilon$. Pick $\mathcal{M} \prec H_{\kappa^+}$ containing $p := \{A, T, \epsilon\}$ such that $S_{\kappa^\mathcal{M}} = \mathcal{M} \cap A$. Write $\beta := \kappa^\mathcal{M}$. By $\epsilon \in p \in \mathcal{M}$, we have $\beta > \epsilon$. We claim that $\beta \in B$.

For all $\alpha < \beta$, by $\alpha, T \in \mathcal{M}$, we have $T_\alpha \in \mathcal{M}$, and by $\mathcal{M} \models |T_\alpha| < \kappa$, we have $T_\alpha \subseteq \mathcal{M}$. So $T \restriction \beta \subseteq \mathcal{M}$. As $\mathrm{dom}(z) \in \mathcal{M}$ for all $z \in T \cap \mathcal{M}$, we conclude that $T \cap \mathcal{M} = T \restriction \beta$. Thus, $S_\beta = A \cap (T \restriction \beta)$.

Finally, since $H_{\kappa^+} \models A$ is a maximal antichain in T, it follows by elementarity that

$$\mathcal{M} \models A \text{ is a maximal antichain in } T.$$

Since $T \cap \mathcal{M} = T \restriction \beta$, we get that $A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$. Altogether, $\beta \in B \setminus \epsilon$. $\square$

Since B is cofinal in κ, let us fix an infinite ordinal $\alpha < \kappa$ for which $\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha$.

Claim 3.3. *$A \subseteq T \restriction \alpha$. In particular, $|A| < \kappa$*

Proof. Fix an arbitrary $z \in T$ with $\mathrm{dom}(z) \geq \alpha$. We shall show that $z \notin A$.

Since $y := z \restriction \alpha$ is in T_α, and α is a nonzero limit ordinal, the construction entails that $y = \mathbf{b}^\alpha_x = \bigcup_{\beta \in C_\alpha \setminus \mathrm{dom}(x)} b^\alpha_x(\beta)$ for some $x \in T \restriction C_\alpha$. Fix $\beta \in \mathrm{nacc}(C_\alpha) \cap B$ above $\mathrm{dom}(x)$. Denote $\beta^- := \sup(C_\alpha \cap \beta)$. Since $\beta \in B$, we know that $S_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$, and hence there is some $s \in S_\beta$ compatible with $b^\alpha_x(\beta^-)$, so that by normality of the tree, $Q^\alpha_{x,\beta} \neq \emptyset$. It follows that we chose $b^\alpha_x(\beta)$ to extend some $s \in S_\beta \subseteq A$. Altogether, $s \subseteq b^\alpha_x(\beta) \subset \mathbf{b}^\alpha_x = y \subseteq z$. As s is an element of the antichain A and z properly extends s, we have $z \notin A$. $\square$

This completes the proof. $\square$

Note that the preceding provides a uniform proof of the following.

Corollary 4 (Jensen, [J72]). *If $V = L$, then for every regular uncountable cardinal κ, TFAE:*

- *κ is not weakly compact;*
- *There exists a κ-Souslin tree.*

References

[BR17] Ari Meir Brodsky and Assaf Rinot. A microscopic approach to Souslin-tree constructions. Part I. *Ann. Pure Appl. Logic*, 168(11):1949–2007, 2017.

[J72] R. Björn Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308, 1972.

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