COVERING PROPERTIES: SEPARATING BETWEEN MENGER AND HUREWICZ

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Abstract. We present a simple construction of a topological space satisfying Menger's covering property, but not Hurewicz's property.

1. Introduction

1.1. **Background.** A topological space $\langle X, O\rangle$ is σ-compact iff $X = \bigcup_{n\in\mathbb{N}} K_n$, where K_n is a compact subspace for each $n \in \mathbb{N}$. $\mathcal{U}$ is an *open cover* of X iff $\mathcal{U} \subseteq O$ and $X \subseteq \bigcup \mathcal{U}$. $\mathcal{U}$ is a γ-*cover* iff $\mathcal{U}$ is infinite, and for each $x \in X$, $\{U \in \mathcal{U} \mid x \notin U\}$ is finite.

In [4], Menger conjectured that a space is σ-compact iff it satisfies *Menger's covering property*, that is, if for any countable sequence of open covers of X, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is an open cover of X. We denote this property by $S_{fin}(\mathcal{O}, \mathcal{O})$.

Hurewicz, who knew that a Luzin set is a consistent counter-example to Menger's conjecture, suggested his own property, *Hurewicz's covering property*, conjecturing in [3] that a space is σ-compact iff it satisfies $U_{fin}(\mathcal{O}, \Gamma)$, that is, if for any sequence of open covers of X, $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$, each do not contain a finite subcover, there exists some $\langle \mathcal{F}_n \in [\mathcal{U}_n]^{<\omega} \mid n \in \mathbb{N}\rangle$, such that $\{\bigcup \mathcal{F}_n \mid n \in \mathbb{N}\}$ forms a γ-cover of X.

It is not hard to see that σ-compact $\Rightarrow U_{fin}(\mathcal{O}, \Gamma) \Rightarrow S_{fin}(\mathcal{O}, \mathcal{O})$. In this paper, we prove that:

Theorem 1.1. $S_{fin}(\mathcal{O}, \mathcal{O}) \not\Rightarrow U_{fin}(\mathcal{O}, \Gamma)$.

This result was first established by Chaber and Pol in [1] using the topological "Michael technique" and a dichotomic argument.[1] Then, Tsaban and Zdomsky obtained in [6] more general results, among them, a combinatorial, direct and non-dichotomic, proof for this theorem. Our proof is dichotomic and focuses on obtaining exactly what is stated in Theorem 1.1, and hence it is the simplest of all.

The refutation of Menger's conjecture was first established by Fremlin and Miller in [2].

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[1]Distinguishing between the case $\mathfrak{b} = \mathfrak{d}$ and the case $\mathfrak{b} < \mathfrak{d}$ (See Definition 2.3).

1.2. **Notation.** We identify the set of natural numbers $\mathbb{N}$ with the ordinal ω, and each natural number n with its set of predecessors $\{k \in \mathbb{N} \mid k < n\}$. We sometime consider $2 = \{0, 1\}$ as a 2-points discrete metric space, ω as a countable discrete metric space, and $\omega + 1 := \omega \cup \{\omega\}$ as the one-point compactification of ω. For a set A and a cardinal μ, let $[A]^\mu := \{B \subseteq A \mid |B| = \mu\}$ and $[A]^{<\omega} := \{B \subseteq A \mid B \text{ is finite }\}$.

1.3. **Organization of this paper.** In section 2 we include all the relevant definitions and folklore facts needed to carry out the proof. In section 3 we prove the theorem mentioned in the abstract.

2. Basic facts

The *Baire space*, $\omega^\omega := \{f \mid f : \omega \to \omega\}$, is a product space $\prod_{n\in\omega} \omega$. Its topology is compatible with the complete metric $\rho(f, g) := 2^{-\Delta(f,g)}$ where $f, g \in \omega^\omega$ are distinct and $\Delta(f, g) := \min\{n \mid f(n) \neq g(n)\}$. Clearly, ω^ω is homeomorphic to its closed subspace $\omega^{\uparrow\omega} := \{f \in \omega^\omega \mid f \text{ is strictly-increasing }\}$.

The *Bartoszyński space*, $(\omega+1)^\omega$, is the product space $\prod_{n\in\omega}(\omega+1)$. It is homeomorphic to its subspace $(\omega+1)^{\uparrow\omega}$ of only strictly-increasing functions:

$$(\omega + 1)^{\uparrow\omega} := \left\{ f \in (\omega + 1)^\omega \middle| n < m \to \begin{pmatrix} f(n) < \omega \to f(n) < f(m) \\ f(n) = \omega \to f(m) = \omega \end{pmatrix} \right\}.$$

The *Cantor space*, $2^\omega := \{f \mid f : \omega \to 2\}$, is a product space $\prod_{n\in\omega} 2$. By Tychonoff's theorem, it is compact.

To each $A \subseteq \omega$, we attach a function $\chi_A \in 2^\omega$, letting $\chi_A(n) = 1$ iff $n \in A$. A moment's reflection makes it clear that $\psi : \omega^{\uparrow\omega} \to 2^\omega$ defined by letting $\psi(f) := \chi_{(\mathrm{Im}(f)\cap\omega)}$ for each $f \in \omega^{\uparrow\omega}$ is an homeomorphism.

Definition 2.1. For $A \subseteq \omega$, let $A^c := \omega \setminus A$.
For $f \in (\omega + 1)^{\uparrow\omega}$, let $f^c := \psi^{-1}(\chi_{\omega\setminus\mathrm{Im}(f)})$.

Evidently, $\chi_A \mapsto \chi_{A^c}$ (for all $A \subseteq \omega$) is an automorphism of the Cantor space, and hence *the complement operator* $f \mapsto f^c$ (for all $f \in (\omega+1)^{\uparrow\omega}$) is an automorphism of the Bartoszyński space.

Definition 2.2. For each $f, g \in \omega^{\uparrow\omega}$, let $f \leq^* g$ mean that there exists some $m \in \omega$ such that $f(n) \leq g(n)$ for all $n > m$. For $A \subseteq \omega^{\uparrow\omega}$, let the *downward closure of* A be $\underline{A} := \{g \in \omega^{\uparrow\omega} \mid \exists f \in A(g \leq^* f)\}$, and the *external cofinality of* A be $\mathrm{ecf}(A) := \min\{|B| \mid B \subseteq \omega^{\uparrow\omega}, A \subseteq \underline{B}\}$.

Definition 2.3. A subset $A \subseteq \omega^{\uparrow\omega}$ is said to be $\leq^*$-*bounded* iff $\mathrm{ecf}(A) \leq 1$, and *dominating* iff $\omega^{\uparrow\omega} \subseteq \underline{A}$. Let $\mathfrak{b} := \min\{|A| \mid A \subseteq \omega^{\uparrow\omega}, \mathrm{ecf}(A) > 1\}$, $\mathfrak{d} := \min\{|A| \mid A \subseteq \omega^{\uparrow\omega} \text{ is dominating }\}$, and $\mathfrak{c} := |\omega^{\uparrow\omega}|$.

It is not hard to see that $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c} = 2^{\aleph_0}$. However, the statement "$\mathfrak{b} < \mathfrak{d}$" is independent of ZFC, the usual axioms of set theory.

Lemma 2.4. *Suppose $Y \subseteq \omega^{\uparrow\omega}$ is a compact subspace, then $Y \subseteq \underline{\{g\}}$ for some $g \in \omega^{\uparrow\omega}$.*

Proof. For all $n \in \omega$, consider the projection $\pi_n : \omega^{\uparrow\omega} \to \omega$ such that $\pi_n(f) = f(n)$ for all $f \in \omega^{\uparrow\omega}$. By definition of the Baire space, each π_n is continuous and by the hypothesis, Y is compact and it follows that $\pi_n[Y]$ is compact in ω. Since any compact subspace of the discrete space ω is finite, we conclude that for all $n \in \omega$, there exists some $m_n \in \omega$ such that $\pi_n[Y] \subseteq \{0, .., m_m\}$. It other words, the function $g \in \omega^{\uparrow\omega}$ defined by letting $g(n) = n + \sum_{k=0}^{n} m_n$ for all $n \in \omega$ works. □

Lemma 2.5. *For all $g \in \omega^{\uparrow\omega}$, $D_g := \{f \in \omega^{\uparrow\omega} \mid \forall n \in \omega(f(n) \leq g(n))\}$ is a closed, nowhere-dense, subspace of $\omega^{\uparrow\omega}$.*

Proof. Fix $g \in \omega^{\uparrow\omega}$. Assume $h \in \omega^{\uparrow\omega} \setminus D_g$. Then there exists some $n \in \omega$ such that $h(n) > g(n)$. Then h is in the open set $U = \{f \in \omega^{\uparrow\omega} \mid f(n) = h(n)\}$ and $U \subseteq \omega^{\uparrow\omega} \setminus D_g$.

To see that $\omega^{\uparrow\omega} \setminus D_g$ is dense, we fix an open set U, and show that $U \cap (\omega^{\uparrow\omega} \setminus D_g) \neq \emptyset$. Find $n \in \omega$, and $\sigma : \{0, ..n\} \to \omega$ such that $\{f \in \omega^{\uparrow\omega} \mid f \restriction \{0, .., n\} = \sigma\} \subseteq U$. Let $h \in \omega^{\uparrow\omega}$ be such that $h \restriction \{0, .., n\} = \sigma$ and $h(k) = g(k) + 1$ for all $k > n$. Clearly, $h \in U \setminus D_g$. □

Corollary 2.6. *For all $g \in \omega^{\uparrow\omega}$, $E_g := \{f \in \omega^{\uparrow\omega} \mid f \leq^* g\}$ is an F_σ meager subspace of $\omega^{\uparrow\omega}$.*

Proof. If σ is a finite sequence of natural numbers, we may consider $sw(\sigma, g) \in \omega^\omega$ such that $sw(\sigma, g)(n) = \sigma(n)$ if $n \in \mathrm{dom}(\sigma)$ and $sw(\sigma, g)(n) = g(n)$ otherwise. Then E_g is the countable union of closed, nowhere-dense, sets:

$$E_g = \bigcup\{D_{sw(\sigma,g)} \mid \sigma \text{ is a finite sequence of natural numbers}, sw(\sigma, g) \in \omega^{\uparrow\omega}\}. \quad \square$$

Finally, we would need the following auxiliary lemma:

Lemma 2.7 (Hurewicz)**.** *For a topological space $\langle X, O\rangle$ admitting a countable base of clopen sets, TFAE:*

(a.1) $X \models S_{fin}(\mathcal{O}, \mathcal{O})$.
(a.2) *Any continuous image of X into $\omega^{\uparrow\omega}$ is not dominating.*

and TFAE:

(b.1) $X \models U_{fin}(\mathcal{O}, \Gamma)$.
(b.2) *Any continuous image of X into $\omega^{\uparrow\omega}$ is $\leq^*$-bounded.*

Proof. See [5]. □

It is well-known that the Baire space and the Cantor space (and hence also the Bartoszyński spaces) are separable, and admits a countable base of clopen sets.

3. The simple construction

Theorem 3.1. *There exists $X \subseteq \omega^{\uparrow\omega}$ such that:*

(a) $X \models S_{fin}(\mathcal{O}, \mathcal{O})$,
(b) $X \not\models U_{fin}(\mathcal{O}, \Gamma)$ *(and in particular, X is not σ-compact).*

In particular, there exists a subspace of the Baire space which is a counter-example to Menger's conjecture.

Proof. If $\mathfrak{b} < \mathfrak{d}$, then pick a $\leq^*$-unbounded family $X \in [\omega^{\uparrow\omega}]^{\mathfrak{b}}$.

By Theorem 2.7, $X \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $X \not\models U_{fin}(\mathcal{O}, \Gamma)$.

Assume now $\mathfrak{b} = \mathfrak{d}$. Pick a dominating family $\{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^{\uparrow\omega}$.

Put $A := \{f \in \omega^{\uparrow\omega} \mid f^c \in \omega^{\uparrow\omega}\} = \{f \in \omega^{\uparrow\omega} \mid \omega \setminus \mathrm{Im}(f) \text{ is infinite }\}$.

We now define a sequence $\{g_\alpha \mid \alpha < \mathfrak{b}\} \subseteq A$ by induction on $\alpha < \mathfrak{b}$. Let $g_0 := f_0$, and assume $\{g_\beta \mid \beta < \alpha\}$ have already been defined.

Since $B := \{g_\beta, f_\beta, f_\beta^c \mid \beta < \alpha\} \subseteq \omega^{\uparrow\omega}$ is of cardinality $< \mathfrak{b}$, we may find some $h \in \omega^{\uparrow\omega}$ such that $B \subseteq \underline{\{h\}}$. Now, by Corollary 2.6, $C_1 := \{f \in \omega^{\uparrow\omega} \mid f \not\leq^* h\}$ is co-meager. It follows from the remark after Definition 2.1 that $C_2 := \{f^c \mid f \in \omega^{\uparrow\omega}, f \not\leq^* h\} \subseteq (\omega+1)^{\uparrow\omega}$ is also co-meager.

Let $Q := (\omega+1)^{\uparrow\omega} \setminus \omega^{\uparrow\omega}$. Since Q is countable, it is meager. Thus, by Baire's category theorem, we may pick $g_\alpha \in C_1 \cap C_2 \setminus Q = \{f \in \omega^{\uparrow\omega} \mid f^c \in \omega^{\uparrow\omega}, f \not\leq^* h, f^c \not\leq^* h\}$. End of the construction.

Claim 3.2. *For all $f \in \omega^{\uparrow\omega}$:*

(1) $|\{g_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{f\}}| < \mathfrak{b}$
(2) $|\{g^c_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{f\}}| < \mathfrak{b}$

Proof. Pick $f \in \omega^{\uparrow\omega}$. By the choice of our dominating family, there exists some $\delta < \mathfrak{b}$ such that $f \leq^* f_\delta$. Assume $\delta < \alpha < \mathfrak{b}$, then by the choice of g_α, $\{n < \omega \mid f_\delta(n) \leq g_\alpha(n)\}$ and $\{n < \omega \mid f_\delta(n) \leq g^c_\alpha(n)\}$ are both infinite. In particular, $g_\alpha \not\leq^* f$ and $g^c_\alpha \not\leq^* h$, thus:

$$\max\left\{\left|\{g_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{f\}}\right|, \left|\{g_\alpha \mid \alpha < \mathfrak{b}\} \cap \underline{\{f\}}\right|\right\} \leq |\delta| < \mathfrak{b}. \quad \square$$

Put $Y := \{g_\alpha \mid \alpha < \mathfrak{b}\} \cup Q$ and let X be the image of Y under the complement operator. It is obvious that $X \subseteq \omega^{\uparrow\omega}$. Since X and Y are homeomorphic, it suffices to show that $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$ and $X \not\models U_{fin}(\mathcal{O}, \Gamma)$.

However, $X \not\models U_{fin}(\mathcal{O}, \Gamma)$ follows directly from Claim 3.2.2 and Theorem 2.7.b, and so, we are left with showing that $Y \models S_{fin}(\mathcal{O}, \mathcal{O})$.

Indeed, the standard argument of Tsaban and Zdmosky works here. Suppose $\langle \mathcal{U}_n \mid n \in \mathbb{N}\rangle$ is a family of open covers. Let $\{q_n \mid n \in \mathbb{N}\}$ enumerate Q. For each $n \in \mathbb{N}$, pick $U_{2n} \in \mathcal{U}_{2n}$ such that $q_n \in U_{2n}$. Let $U := \bigcup_{n\in\mathbb{N}} U_{2n}$. Since U is open, we get that $(\omega+1)^{\uparrow\omega} \setminus U$ is a compact subspace of $\omega^{\uparrow\omega}$, thus, by applying to Lemma 2.4, we may find $f \in \omega^{\uparrow\omega}$ with $Y \setminus Q \subseteq \underline{\{f\}}$.

It follows from Claim 3.2.1 that $|Y \setminus Q| < \mathfrak{b}$, and hence, by Theorem 2.7.a, there exists $\langle \mathcal{F}_{2n+1} \in [\mathcal{U}_{2n+1}]^{<\omega} \mid n \in \mathbb{N}\rangle$ such that $Y \setminus Q \subseteq \bigcup\bigcup_{n\in\mathbb{N}} \mathcal{F}_{2n+1}$. Letting $\mathcal{F}_{2n} := \{U_{2n}\}$ for all $n \in \mathbb{N}$, we get that $\langle \mathcal{F}_n \mid n \in \mathbb{N}\rangle$ works. $\square$

Corollary 3.3. *There exists $B \subseteq \omega^\omega$ which is $\leq^*$-unbounded but not dominating. Further more, B satisfies:*

(a) *For all $f \in \omega^\omega$, $|B \cap \underline{\{f\}}| < \mathfrak{b}$.*
(b) *For any continuous function $\varphi : \omega^\omega \to \omega^\omega$, $\varphi[B]$ is not $\leq^*$-dominating.*

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