REVIEW ON TODD EISWORTH'S CHAPTER FOR THE HANDBOOK OF SET THEORY: "SUCCESSORS OF SINGULAR CARDINALS"

ASSAF RINOT

This chapter offers a comprehensive and lucid exposition of the questions and techniques involved in the study of combinatorics of successors of singular cardinals.

What is so special about successors of singular cardinals? They are successor cardinals, but are also similar to inaccessible cardinals, in the sense that there is no maximal regular cardinal below them, meaning for instance that there is no obvious obstruction for each of their stationary sets to reflect.[1] And indeed, Magidor proved [Mag82] that, modulo a large cardinal hypothesis, it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects (the proof may be found as Theoreom 2.15 of this chapter).

But even in the absence of large cardinals, singular cardinals and their successors are tightly related. Recall that by König's theorem [Kön05], the collection of all countable subsets of the (highly uncountable) cardinal $\aleph_\omega$ has cardinality at least the successor of $\aleph_\omega$. In fact, all reasonable finer measures (such as density [Koj15], and cofinality with respect to inclusion) of the collection $[\aleph_\omega]^\omega$ of all countable subsets of $\aleph_\omega$ measure it by at least $\aleph_{\omega+1}$. Arguably, a canonical witness to König's theorem is a *pcf scale*. A *pcf scale* is a pair $(\vec{\mu}, \vec{f})$, where $\vec{\mu} = \langle \mu_i : i < \sigma \rangle$ is an increasing sequence of regular cardinals, converging to some singular cardinal μ of cofinality σ, and $\vec{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$ is a sequence of functions such that:

(1) for all $\alpha < \mu^+$, $f_\alpha \in \prod_{i<\sigma} \mu_i$;
(2) for all $\alpha < \beta < \mu^+$, $D(f_\alpha, f_\beta) := \{i < \sigma \mid f_\alpha(i) \geq f_\beta(i)\}$ is bounded in σ;
(3) for all $f \in \prod_{i<\sigma} \mu_i$, there exists some $\beta < \mu^+$, such that $D(f, f_\beta)$ is bounded in σ.

By a theorem of Shelah [She94, p. 50], every singular cardinal μ admits such a pcf scale, and the proof may be found as Theorem 3.53 of this chapter.

Next, we briefly mention a few differences between successors of singular cardinals and successors of regular cardinals:

- By a theorem of Specker [Spe49], GCH entails that for every regular cardinal μ, there exists a μ^+-Aronszajn tree, that is, a tree of height μ^+ without chains or levels of size μ^+. On the other hand, Magidor and Shelah [MS96] proved that, modulo a large cardinal hypothesis, GCH is consistent with the nonexistence of an $\aleph_{\omega+1}$-Aronszajn tree (cf. [Sin12], [Nee14]).
- By a theorem of Shelah [She91], for every regular cardinal μ, the set $E^{\mu^+}_{\neq \mathrm{cf}(\mu)} := \{\alpha < \mu^+ \mid \mathrm{cf}(\alpha) \neq \mathrm{cf}(\mu)\}$ is in the approachability ideal $I[\mu^+]$ (Corollary 4.6 of this chapter), whereas by another theorem of Shelah [She79], modulo a large cardinal hypothesis, it is consistent that for some singular cardinal μ, $E^{\mu^+}_{\neq \mathrm{cf}(\mu)}$ is not in $I[\mu^+]$ (Theorem 3.20 of this chapter).

  The definition of the approachability ideal $I[\kappa]$ may be found in Definition 3.3 of this chapter, but we also include it in here: A subset $S \subseteq \kappa$ is in $I[\kappa]$ iff there exists a club $C \subseteq \kappa$ and a sequence $\langle a_i \mid i < \kappa \rangle$ satisfying the following. For every $\alpha \in S \cap C$, there exists a cofinal subset $A \subseteq \alpha$ of order-type $\mathrm{cf}(\alpha) < \alpha$ such that $\{A \cap \beta \mid \beta < \alpha\} \subseteq \{a_\beta \mid \beta < \alpha\}$.

[1] Recall that a stationary set S is said to *reflect* if there exists some ordinal $\gamma < \sup(S)$ of uncountable cofinality for which $S \cap \gamma$ is stationary in γ.

- Recall that an algebra $\mathfrak{A}$ is said to be a *Jónsson algebra* if all of its proper subalgebras have smaller cardinality than that of $\mathfrak{A}$. For instance, the construction of a Jónsson group of cardinality $\aleph_1$ may be found in [She80].

  By a theorem of Tryba [Try84] and independently, Woodin, if κ is the successor of a regular cardinal, then there exists a Jónsson algebra of cardinality κ (Corollary 5.5 of this chapter). Whether the same statement holds true for successors of singular cardinals is a long-standing open problem (see [Eis12] for the best known partial answer).

Below, we shall review the content of each section, and mention a few relevant papers that appeared after the chapter was written.

▶ Section 1 provides some necessary background and motivation.

▶ Section 2 deals with some of the possible behaviors of stationary reflection. Specifically, Jensen's square principle $\square_\mu$ (Definition 2.1 of this chapter) is a strong anti-reflection principle while, on the other extreme, large cardinals (such as supercompact cardinals, see Definition 1.20) give rise to reflection, and even to combinatorial objects (such as indecomposable ultrafilters, see Definition 2.7) that entail reflection.

As mentioned earlier, in [Mag82] Magidor established the consistency of the statement "every stationary subset of $\aleph_{\omega+1}$ reflects". Magidor's original proof included a component of iterated club-shooting that was later eliminated by an idea of Shelah that had to do with the approachability ideal $I[\aleph_{\omega+1}]$. The proof given in [Mag82], as well as the one in this chapter, is Shelah's and serves as a first exposition to the approachability ideal. It is a curious fact that many years later, Magidor's original approach was found useful in getting finer models of reflection (see [CFM01]). A recent result concerning a fine form of reflection at successors of singulars may be found in [CLH16].

▶ Section 3 offers a thorough study of the approachability ideal $I[\lambda]$, and its weaker sibling $I[\lambda;\mu]$. Various equivalent characterizations of these ideals (stemming from [She79], [She93], [FM97]) are presented. The negative effects of large cardinals on the approachability ideal are discussed (Theorem 3.20). In the positive direction, a proof is given of Shelah's key theorem that for regular cardinals κ, σ, λ satisfying $\kappa^+ < \sigma < \lambda$, the ideal $I[\lambda]$ contains a stationary subset of E^λ_κ that reflects (also) at points of cofinality σ (Theorem 3.18). The proof relies on Shelah's *club guessing* principle, a concept that makes a second appearance in Section 5 of this chapter. Under the hypothesis that every stationary subset of $\aleph_{\omega+1}$ reflects, it is proved that if $\aleph_{\omega+1}$ is a strong limit, then $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ (Corollary 3.41). See [SV10] for a study of the effect of reflection principles at small regular cardinals on the approachability ideal at the successor of singular cardinals.

The second part of Section 3 deals with pcf scales, and the interrelation of sets in the approachability ideal with the sets of *good points* in such scales (see Theorem 3.55 of this chapter, as well as [CFM04], [GS08], [CF10]). A motivation for studying good points may be found, for instance, in [MS94] and [Cum97].

▶ Section 4 is dedicated to variations of Jensen's square principle $\square_\mu$, their effect on reflection of stationary sets, Jensen's diamond principle (Definition 4.22 of this chapter), and pcf scales. For more recent results, see [AKY09], [CM11], [Sak13], [TTP14], [Sak15].

The section offers a clear proof of an old and difficult theorem of Shelah from [She84] (Theorem 4.24 of this chapter). Incidentally, shortly before this Handbook went to print, Shelah found a simple (yet, ingenious) proof of a strong theorem concerning diamond [She10], and this was quickly adapted to the case of successors of singulars by Zeman [Zem10] (See also [Rin10], [Mat14], [Rin15]).

The second part of Section 4 deals with special types of pcf scales: *good*, *very good*, *better* and their applications to incompactness combinatorics such as the NPT and ADS principles (Definitions 4.44 and 4.71 of this chapter, respectively), simultaneous reflection of stationary sets, and reflection of generalized stationary sets (See also [Cum05], [She08], [FJS$^+$10]).

▶ Section 5 is a short section dedicated to results around the question of whether the successor of a singular cardinal must carry a Jónsson algebra.

Two theorems that allow one to infer the existence of a Jónsson algebra on some cardinal from the existence of Jónsson algebras on cardinals below it, are given. The second among these theorems motivates the study of Shelah's *club guessing* principle (see also [ES09], [Eis10], [Rin14c]). A related concept is the negative partition relation $\mu^+ \nrightarrow [\mu^+]^2_\theta$ that asserts the existence of a coloring of unordered pairs $c : [\mu^+]^2 \rightarrow \theta$ with the property that for every cofinal $A \subseteq \mu^+$ and every $i < \theta$, there exists some $\alpha < \beta$ from A for which $c(\alpha, \beta) = i$. A further generalization is the principle $\mathrm{Pr}_1(\mu^+, \theta, \sigma)$ that coincides with the former in the case $\sigma = 2$. The earliest demonstrations of the utility of the case $\sigma > 2$ appeared in [Roi78] and [Gal80].

Shelah proved that $\mathrm{Pr}_1(\mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ holds for every singular cardinal μ, and this may be found as Theorem 5.16 of this chapter. Extending a theorem of Moore [Moo06] (which extended a theorem of Todorcevic [Tod87]), Peng and Wu recently proved that $\mathrm{Pr}_1(\aleph_1, \aleph_1, n)$ holds for every positive integer n. For additional recently-published results in this direction, see [Eis12], [Rin12], [Eis13a], [Eis13b], [Rin14a], [Rin14b].

Upon the advice of the editor, we also include a short list of well-known open problems:

(1) Is it consistent that $\mu^+ \nrightarrow [\mu^+]^2_\mu$ fails for some singular cardinal μ?

(2) It is consistent that $E^{\aleph_{\omega+1}}_{\aleph_2} \notin I[\aleph_{\omega+1}]$ or $E^{\aleph_{\omega+1}}_{\aleph_3} \notin I[\aleph_{\omega+1}]$? What about the consistency of $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ or $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_3, \aleph_2)$?

(3) Is it consistent that GCH holds and $\diamondsuit(E^{\mu^+}_{\mathrm{cf}(\mu)})$ fails for some singular cardinal μ?

(4) Is it consistent that for some singular cardinal μ, and a reflecting stationary subset $S \subseteq \mu^+$, $\mathrm{NS}_{\mu^+} \restriction S$ is saturated?

(5) Suppose that $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$, and $2^{\aleph_\omega} > \aleph_{\omega+1}$.
Must there exist an $\aleph_{\omega+1}$-Aronszajn tree?

(6) Suppose that μ is a singular cardinal and there exists a special μ^+-Aronszajn tree.
Does GCH entail the existence of a μ^+-Souslin tree?

(7) Suppose that μ is a singular cardinal of countable cofinality.
Must there exist a matrix $\langle \xi_{\alpha,n} \mid \alpha < \mu^+, n < \omega \rangle$ such that for every club C in μ^+, there exists some $\alpha < \mu^+$ such that the following two conditions hold simultaneously?
   (a) $\langle \xi_{\alpha,n} \mid n < \omega \rangle$ is a strictly increasing sequence of ordinals from C, converging to α;
   (b) $\langle \mathrm{cf}(\xi_{\alpha,n}) \mid n < \omega \rangle$ is a strictly increasing sequence of cardinals, converging to μ.

In summary, this chapter is an invaluable introduction and invitation to the subject.

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Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel.
URL: `http://www.assafrinot.com`